INTERVAL METHODS IN REMOTE SENSING: RELIABLE SUB-DIVISION OF GEOLOGICAL AREAS

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Abstract An appropriate subdivision of a geophysical area into segments enables us to extrapolate the results obtained in some locations within the segment (where extensive research was done) to other locations within the same segment, and thus, get a good understanding of the locations which weren't thoroughly analyzed.

Often, different evidence and different experts' intuition support different subdivisions schemes. For example, in our area – Rio Grande rift zone – there is some geochemical evidence that this zone is divided into three segments, but, in the viewpoint of many researchers, this evidence is not yet sufficiently convincing.

We show that if we use topographical information (this information, e.g., comes from satellite photos), then interval methods lead to a reliable justification for the tripartite subdivision of the Rio Grande rift zone.

1. FORMULATION OF THE PROBLEM

In geophysics, appropriate subdivision of an area into segments is extremely important, because it enables us to extrapolate the results obtained in some locations within the segment (where extensive research was done) to other locations within the same segment, and thus, get a good understanding of the locations which weren't that thoroughly analyzed.

The subdivision of a geological zone into segments is often a controversial issue, with different evidence and different experts' intuition supporting different subdivisions.

For example, in our area – Rio Grande rift zone – there is some geochemical evidence that this zone is divided into three segments [9]:

- the southern segment which is located, approximately, between the latitudes $y = 29^\circ$ and $y = 34^\circ$;
- the central segment – from $y = 34.5^\circ$ to $y = 38^\circ$; and
- the northern segment – from $y = 38^\circ$ to $y = 41^\circ$.

However, in the viewpoint of many researchers, this evidence is not yet sufficiently convincing.

It is therefore desirable to develop new techniques for zone subdivision, techniques which would be in the least possible way dependent on the (subjective) expert opinion and would, thus, be maximally reliable.

2. MAIN IDEA: USING TOPOGRAPHIC INFORMATION

One reason for subjectivity of the geological subdivision is the fact that the existing subdivision is often based on the chemical and physical analysis of several samples collected throughout the area, and often, we do not have a statistically sufficient amount of thoroughly analyzed geological samples to make the conclusion about the subdivision statistically convincing.

To make this conclusion more reliable, we can use, instead of the more rare *geological* samples, a more abundant *topographical* information (this information, e.g., comes from satellite photos). We can characterize each part of the divided zone by its topography.

3. PRELIMINARY PHYSICAL ANALYSIS OF TOPOGRAPHIC INFORMATION

General idea of symmetry. A topographical information is a shape of the Earth landscape. This landscape is caused by many factors whose contribution is often only known on a qualitative level; so, we can represents this landscape as a realization of a random field.

When viewed in geological time, the geophysical forces which form this landscape are largely the same all over the Earth. In more precise terms, this means that these forces are invariant with respect to all possible rotations of the Earth's sphere. Therefore, as a reasonable first approximation to the actual landscape, we can consider a random field which is invariant with respect to all these rotations, i.e., a homogeneous isotropic random field. Such a field is indeed a very good approximation to the actual landscape, actively used in computer simulations (see, e.g., [8]). This approximation, however, treats all Earth areas as similar and does not bring us any closer to solving our problem – of sub-dividing geological zones. To solve this problem, we must therefore use a more accurate description of the landscape.

This need for a more accurate description is consistent with the more detailed analysis of the geophysical data, an analysis which shows that the actual landscape is not exactly homogeneous and isotropic. From the physical viewpoint, the resulting anisotropy can be explained by the fact that highly symmetric distributions are known to be *unstable*; arbitrarily small perturbations cause drastic changes in the distribution (in physical terms, we get a *spontaneous symmetry violation*).

In principle, it is possible to have a perturbation that changes the initial highly symmetric state into a state with no symmetries at all, but statistical physics teaches us that it is much more probable to have a gradual symmetry violation: first, some of the symmetries are violated, while some still remain; then, some other symmetries are violated, etc. Similarly, a (highly organized) solid body normally goes through a (somewhat organized) liquid phase before it reaches a (completely disorganized) gas phase.

Thus, it is reasonable to expect that the resulting random field is not complete asymmetric, but is invariant w.r.t. some subgroup G' of the initial group G. The original 3-dimensional symmetry group G consists of all rotations around the center. The only non-trivial continuous subgroups of this groups are subgroups formed by rotations around a line. Thus, we can conclude that the random field corresponding to a landscape should be invariant with respect to rotations around a certain axis. In a 2-D map representation, rotations around an axis correspond to shifts in a corresponding direction. Thus, we can conclude that the statistical characteristics of the map should be invariant with respect to shift.

This conclusion works well for the Rio Grande rift zone, where the empirical statistical characteristics drastically change with latitude y and change much smaller with longitude x. So, as a good description for the actual landscape, we can consider a random field all characteristics of which are invariant with respect to longitude x.

The desired sub-division must be invariant with respect to the same symmetry, so for each latitude y, all the points located on this latitude should belong to the same zone. So, to get the desired classification, we must decide, for each horizontal line $y = \text{const}$, whether this line belongs to a certain region or not.

Comment. Similar symmetry ideas has been successfully used to describe other geometric shapes:

- shapes of *celestial objects* [2, 3, 4];
- shapes in *fracture theory*: for a symmetric body, each fault (crack, etc.) is a spontaneous symmetry violation [16]; this fact not only *explains* the shapes of the faults [16], it enables us to describe the best sensor locations for *detecting* these faults [12, 13, 14], etc.

(A general *symmetry* approach, with possibly non-geometric symmetries, enables us to explain the empirical optimality of different heuristic techniques such as fuzzy, neural, genetic, etc. [10].)

From general symmetry to spectrum. Within each line $y = \text{const}$, the landscape $x \to h(x, y)$ is described by a shift-invariant random process; in probability theory, random functions of one variable are usually functions of time, so shift-invariant processes are also called *stationary*. Since the exact shape of a landscape is the result of many independent factors contribution of each of which is relatively small, we can apply the central limit theorem and conclude that this process can be reasonably described as Gaussian; see, e.g., [15, 18]. (The empirical analysis of topographical data shows that we indeed have a Gaussian distribution.)

It is known that a stationary Gaussian process can be uniquely characterized by its average and its *spectrum*, i.e., the absolute value $|H(\omega, y)|$ of its Fourier transform. For the landscape, the average practically does not change with y, so we should only consider the spectrum.

Since we are interested only in the large-scale classification, it makes sense to only use the spectrum values corresponding to relatively large spatial wavelengths, i.e., wavelengths L for which $L \geq L_0$ for some appropriate value L_0. In particular, for the sub-division of the Rio Grande rift, it makes sense to use only wavelengths of $L_0 = 1000$ km or larger.

Since, for the Rio Grande Rift, we are interested in the classification of horizontal zones, it makes sense to do the following:

- divide the Rio Grande Rift into small (e.g., 1°) zones $[y^-, y^+]$ (with y from $y^- = 30$ to $y^+ = 31$, from $y^- = 31$ to $y^+ = 32$, $\ldots$, from $y^- = 40$ to $y^+ = 41$);
- for each of these zones, take the topographic data, i.e., the height $h(x, y)$ described as a function of longitude x and latitude y;

- for each zone and for each y, compute the Fourier transform $H(\omega, y)$ with respect to x;
- for each zone, combine all the spectral values which correspond to large wavelength (i.e., for which $\omega \le 1/L_0$), and compute the resulting spectral value

$$S(y^-) = \int_{y=y^-}^{y^+} \int_{\omega=0}^{1/L_0} |H(\omega, y)|^2 \, \mathrm{d}\omega \, \mathrm{d}y.$$

We are interested in comparing the spectral values $S(y)$ corresponding to different latitudes y, so we are not interested in the *absolute* values of $S(y)$, only in *relative* values. Thus, to simplify the data, we can *normalize* them by, e.g., dividing each value $S(y^-)$ by the largest $S_{\max}$ of these values. In particular, for the Rio Grande rift, the resulting values of $y^- = y_1, y_2, \ldots$ and $s_i = S(y_i)/S_{\max}$ are as follows:

y_i	29	30	31	32	33	34
s_i	0.28	0.24	0.21	0.16	0.20	0.29

35	36	37	38	39	40	41
0.31	0.35	0.46	1.00	0.80	0.96	0.74

Based only on these spectral values s_i, we will try to classify locations into several clusters ("segments").

Traditional clustering techniques do not lead to reliable classification. Most existing clustering methods (see, e.g., [6]) assume that we have some information about the *probability* distributions of the measurement errors. In real-life topographical analysis, we do not have enough data to uniquely determine these distributions, and different possible distributions lead to drastically different conclusions about the subdivision of a geological zone.

Similarly, *fuzzy* clustering techniques (see, e.g., [5]) depend on the subjective fuzzy estimates, and different expert estimates often lead to drastically different subdivisions.

Therefore, we need a new, less subjective clustering method.

4. INTERVAL APPROACH

Segments as monotonicity regions. How can we detect the segments based on these values s_i? In order to answer this question, let us first plot the dependence of s_i on y_i and see if there is any visible feature of this plot which can be associated with a subdivision of the area into geophysical segments.

If we plot the dependence of s_i on y_i, we will see that at first, the function $s(y)$ is (approximately) decreasing, then it is (approximately) increasing, and then it is (approximately) decreasing again. Interestingly, these "monotonicity regions" seem to be in good accordance with the empirical subdivision of the rift into segments [9]. In view of this observation, we will identify geophysical segments with the monotonicity regions of the (unknown) function $s(y)$.

Let's use intervals. The heights are measured pretty accurately, so the only errors in the values s_i come from discretization. In other words, we would like to know the values of the function $s(y) = S(y)/S_{\max}$ for all y, but we only know the values $s_1 = s(y_1)$, $\ldots$, $s_n = s(y_n)$ of this function for the points $y_1, \ldots, y_n$. For each y which is different from y_i, it is reasonable to estimate $s(y)$ as the value $s_i = s(y_i)$ at the point y_i which is the closest to y (and, ideally, which belongs to the same segment as y_i). For each point y_i, what is the largest possible error Δ_i of the corresponding approximation?

When $y > y_i$, the point y_i is still the closest until we reach the midpoint $y_{\rm mid} = (y_i + y_{i+1})/2$ between y_i and y_{i+1}. It is reasonable to assume that the largest possible approximation error $|s(y) - s_i|$ for such points is attained when the distance between y and y_i is the largest, i.e., when y is this midpoint; in this case, the approximation error is equal to $|s(y_{\rm mid}) - s_i|$.

If the points y_i and y_{i+1} belong to *the same* segment, then the dependence of $s(y)$ on y should be reasonably smooth for $y \in [y_i, y_{i+1}]$. Therefore, on a narrow interval $[y_i, y_{i+1}]$, we can, with reasonable accuracy, ignore quadratic and higher terms in the expansion of $s(y_i + \Delta y)$ and thus, approximate $s(y)$ by a linear function. For a linear function $s(y)$, the difference $s(y_{\rm mid}) - s(y_i)$ is equal to the half of the difference $s(y_{i+1}) - s(y_i) = s_{i+1} - s_i$; thus, for $y > y_i$, the approximation error is bounded by $0.5 \cdot |s_{i+1} - s_i|$.

If the points y_i and y_{i+1} belong to *different* segments, then the dependence $s(y)$ should exhibit some non-smoothness, and it is reasonable to expect that the difference $|s_{i+1} - s_i|$ is much higher than the approximation error.

In both cases, the approximation error is bounded by $0.5 \cdot |s_{i+1} - s_i|$.

Similarly, for $y < y_i$, the approximation error is bounded by $0.5 \cdot |s_i - s_{i-1}|$ if the points y_i and y_{i-1} belong to the same segment, and is much smaller if they don't. In both cases, the approximation error is bounded by $0.5 \cdot |s_i - s_{i-1}|$.

We have two bounds on the approximation error and we can therefore conclude that the approximation error cannot exceed the smallest Δ_i of these two bounds, i.e., the value

$$\Delta_i = 0.5 \cdot \min(|s_i - s_{i_1}|, |s_{i+1} - s_i|).$$

As a result, instead of the *exact* values s_i, for each i, we get the *interval* $\mathbf{s}_i = [s_i^-, s_i^+]$ of possible values of $s(y)$, where $s_i^- = s_i - \Delta_i$ and $s_i^+ = s_i + \Delta_i$. In particular, for the Rio Grande rift, the corresponding intervals are as follows:

y_i	$y_1 = 29$	$y_2 = 30$	$y_3 = 31$
$\mathbf{s}_i = [s_i^-, s_i^+]$	[0.26,0.30]	[0.225,0.255]	[0.195,0.225]
$y_4 = 32$	$y_5 = 33$	$y_6 = 34$	$y_7 = 35$
[0.14,0.18]	[0.18,0.22]	[0.28,0.30]	[0.30,0.32]
$y_8 = 36$	$y_9 = 37$	$y_{10} = 38$	$y_{11} = 39$
[0.33,0.37]	[0.405,0.515]	[0.80,1.10]	[0.72,0.88]
$y_{12} = 40$	$y_{13} = 41$		
[0.88,1.04]	[0.63,0.85]		

How to find monotonicity regions of a function defined with interval uncertainty: idea. We want to find regions of uncertainty of a function $s(y)$, but we do not know the exact form of this function; all we know is that for every i, $s(y_i) \in \mathbf{s}_i$ for known intervals $\mathbf{s}_i$. How can we find the monotonicity regions in the situation with such interval uncertainty?

Of course, since we only know the values of the function $s(y)$ in finitely many points y_i, this function can have as many monotonicity regions between y_i and y_{i+1} as possible. What we are interested in is funding the subdivision into monotonicity regions which can be deduced from the data. The first natural question is: can we explain the data by assuming that the dependence $s(y)$ is monotonic? If not, then we can ask for the possibility of having a function $s(y)$ with exactly two monotonicity regions:

- if such a function is possible, then we are interested in possible locations of such regions;
- if such a function is not possible, then we will try to find a function $s(y)$ which is consisted with our interval data and which has three monotonicity regions, etc.

This problem was first formalized and solved in [17]. The corresponding algorithm is based on the following idea.

If the function $s(y)$ is non-decreasing, then, for $i < j$, we have $s(y_i) \le s(y_j)$; therefore, $s_i^- \le s(y_i) \le s(y_j) \le s_j^+$ and $s_i^- \le s_j^+$. It turns out that, vice versa, if the inequality $s_i^- \le s_j^+$ hold for every $i < j$, then there is a non-decreasing function $s(y)$ for which $s(y_i) \in \mathbf{s}_i$ for all i (e.g., we can take $s(y_i) = \max(s_1^-, \ldots, s_i^-)$ for all i and use linear interpolation to define the values $s(y)$ for $y \ne y_i$.) Thus, to check monotonicity, it is sufficient to check these inequalities.

If we have already checked these inequalities for the intervals $\mathbf{s}_1, \ldots, \mathbf{s}_k$, and we add the new interval $\mathbf{s}_{k+1}$, then, to confirm that it is still possible for a function $s(z)$ to be non-decreasing, it is sufficient to check that $s_i^- \le s_{k+1}^+$ for all $i = 1, \ldots, k$. These k inequalities are equivalent to a single inequality $m_k \le s_{k+1}^+$, where we denoted $m_k = \max(s_1^-, \ldots, s_k^-)$.

Similarly, in order to check that a function $s(y)$ can be non-increasing, it is sufficient to check that $s_i^+ \ge s_j^-$ for all pairs $i > j$. If we have already checked these inequalities for the intervals $\mathbf{s}_1, \ldots, \mathbf{s}_k$, and we add the new interval $\mathbf{s}_{k+1}$, then, to confirm that it is still possible for a function $s(z)$ to be non-increasing, it is sufficient to check that $M_k \ge s_{k+1}^-$, where we denoted $M_k = \min(s_1^+, \ldots, s_k^+)$.

The values m_k and M_k needed for these comparisons do not to be re-computed for every k; if we have already computed $M_{k-1} = \min(s_1^+, \ldots, s_{k-1}^+)$, then we can compute M_k as $M_k = \min(M_{k-1}, s_k^+)$ (and m_k as $m_k = \max(m_{k-1}, s_k^-)$).

How to find monotonicity regions of a function defined with interval uncertainty: algorithm. Thus, to find the monotonicity segments, we can use the following algorithm. In this algorithm, we process the intervals $\mathbf{s}_1, \ldots, \mathbf{s}_n$ one by one.

When we have the 1-st interval, then the only information that we have about the function $s(y)$ is that $s(y_1) \in \mathbf{s}_1$. This information is consistent with the function $s(y)$ being a constant, i.e., both non-decreasing and non-increasing. Thus, we are still consistent with monotonicity. To start the process of computing m_k and M_k, we assign $m_1 := s_1^-$ and $M_1 := s_1^+$.

If the first k intervals $\mathbf{s}_k$ are consistent with the assumption that the function $s(y)$ is non-decreasing, then when we get the new interval $\mathbf{s}_{k+1}$, we check whether $m_k \le s_{k+1}^+$. Then:

- If $m_k \le s_{k+1}^+$, then the new point y_{k+1} is still within the same monotonicity region. To prepare for the next interval, we compute $m_{k+1} = \min(m_k, s_{k+1}^-)$.

- If $m_k > s^+_{k+1}$, this means that the new point y_{k+1} cannot be within the same monotonicity region, and so the monotonicity region must end before y_{k+1}. The point y_{k+1} itself belongs to a different monotonicity region which may expand both to the previous values y_k, $y_{k-1}, \ldots$, and to the following values $y_{k+2}, \ldots$; this new region can be traced in the same manner.

Similarly, if the first k intervals $\mathbf{s}_k$ are consistent with the assumption that the function $s(y)$ is non-increasing, then when we get the new interval $\mathbf{s}_{k+1}$, we check whether $M_k \geq s^-_{k+1}$. Then:

- If $M_k \geq s^-_{k+1}$, then the new point y_{k+1} is still within the same monotonicity region. To prepare for the next interval, we compute $M_{k+1} = \max(M_k, s^+_{k+1})$.
- If $M_k < s^-_{k+1}$, this means that the new point y_{k+1} cannot be within the same monotonicity region, and so the monotonicity region must end before y_{k+1}. The point y_{k+1} itself belongs to a different monotonicity region which may expand both to the previous values y_k, $y_{k-1}, \ldots$, and to the following values $y_{k+2}, \ldots$; this new region can be traced in the same manner.

How to find monotonicity regions of a function defined with interval uncertainty: example. Let us illustrate the above algorithm on the Rio Grande rift example.

We start with the 1-st interval, for which $m_1 := s^-_1 = 0.26$ and $M_1 := s^+_1 = 0.30$.

For the 2-nd interval, we check the inequalities $0.225 = s^-_2 \leq M_1 = 0.30$ and $0.255 = s^+_2 \geq m_1 = 0.26$. The first inequality holds, but the second does not hold, so we are in a non-increasing region. Thus, we compute $M_2 = \min(M_1, s^+_2) = \min(0.30, 0.255) = 0.255$.

For the 3-rd interval, the inequality $0.195 = s^-_3 \leq M_2 = 0.255$ still holds, so we are still in the non-increasing region. To prepare for the next interval, we compute $M_3 = \min(M_2, s^+_3) = \min(0.255, 0.225) = 0.225$.

For the 4-th interval, the inequality $0.14 = s^-_4 \leq M_3 = 0.225$ still holds, so we compute $M_4 = \min(M_3, s^+_4) = \min(0.225, 0.18) = 0.18$.

For the 5-th interval, the inequality $0.18 = s^-_4 \leq M_4 = 0.18$ still holds, so we compute $M_5 = \min(M_4, s^+_5) = \min(0.18, 0.22) = 0.18$.

For the 6-th interval, the inequality $0.28 = s^-_5 \leq M_5 = 0.18$ no longer holds, so the first monotonicity region cannot continue past $y_6 = 34$. Thus, the first monotonicity region must be within $[29, 34]$.

The point y_6 must belong to the new monotonicity region, where the function $s(y)$ is non-decreasing instead of non-increasing. Before we go forward and start checking on the points y_7, etc., we must go back and check which points y_5, y_4, ..., can belong to this new region. If we have already checked that $y_6, \ldots, y_k$ belong to this region, this means that $s_i^- \le s_j^+$ for all such $i < j$. To check whether y_{k-1} belongs to this same region, we must check whether $s_{k-1}^- \le s_i^+$ for all such i, i.e., whether $s_{k-1}^- \le \widetilde{M}_k = \min(s_6^+, \ldots, s_k^+)$.

We start with $\widetilde{M}_6 = s_6^+ = 0.30$. Since the inequality $0.16 = s_5^- \le \widetilde{M}_6 = 0.30$ holds, y_5 also belongs to this region, so we compute $\widetilde{M}_5 = \min(\widetilde{M}_6, s_5^+) = \min(0.30, 0.22) = 0.22$.

For the 4-th value, the inequality $0.14 = s_4^- \le \widetilde{M}_5 = 0.22$ holds, so y_4 also belongs to this region, and we compute $\widetilde{M}_4 = \min(\widetilde{M}_5, s_4^+) = \min(0.22, 0.18) = 0.18$.

For the 3-rd value, the inequality $0.195 = s_3^- \le \widetilde{M}_4 = 0.18$ is not true, so the new region must stop before $y_3 = 31$. Thus, the new region start at 31.

Now, we can move forward and check whether points y_7, ..., belong to the new monotonicity region.

We start with $m_6 := s_6^- = 0.28$. On the next step, we check whether $0.32 = s_7^+ \ge m_6 = 0.28$, and since this inequality holds, we compute $m_7 = \max(m_6, s_7^-) = \max(0.28, 0.30) = 0.30$. Similarly, we compute $m_8 = 0.33$, $m_9 = 0.405$, $m_{10} = 0.80$, $m_{11} = 0.80$, and $m_{12} = 0.88$. For the 13-th interval, the inequality $0.85 = s_{13}^+ \ge m_{12} = 0.88$ is no longer true, so y_{13} cannot belong to the second (non-decreasing) monotonicity region. Thus, the second region must stop before $y_{13} = 41$, and the point y_{13} must belong to the *third* monotonicity region.

To find out which other points belong to this third region, we must go back and check which points y_{12}, y_{11}, ..., can belong to this new region. If we have already checked that $y_{13}, \ldots, y_k$ belong to this region, this means that $s_i^+ \ge s_j^-$ for all such $i < j$. To check whether y_{k-1} belongs to this same region, we must check whether $s_{k-1}^+ \ge s_i^-$ for all such i, i.e., whether $s_{k-1}^+ \ge \widetilde{m}_k = \max(s_{13}^-, \ldots, s_k^-)$.

We start with $\widetilde{m}_{13} = s_{13}^- = 0.63$. Since the inequality $1.04 = s_{12}^+ \ge \widetilde{m}_{13} = 0.63$ holds, y_{12} also belongs to this region, so we compute $\widetilde{m}_{12} = \max(\widetilde{m}_{13}, s_{12}^-) = \max(0.63, 0.88) = 0.88$.

For the 11-th value, the inequality $0.88 = s_{11}^+ \ge \widetilde{m}_{12} = 0.88$ holds, so y_{11} also belongs to this region, and we compute $\widetilde{m}_{11} = \max(\widetilde{m}_{12}, s_{11}^-) = \max(0.88, 0.72) = 0.88$.

For the 10-th value, the inequality $1.10 = s_{10}^+ \ge \widetilde{m}_{11} = 0.88$ holds, so y_{10} also belongs to this region, and we compute $\widetilde{m}_{10} = \max(\widetilde{m}_{11}, s_{10}^-) = \max(0.88, 0.80) = 0.88$.

For the 9-th value, the inequality $0.515 = s_9^+ \geq \widetilde{m}_{10} = 0.88$ is not true, so the new region must stop before $y_9 = 37$. Thus, the new region starts at 37.

Conclusion. Thus, we have three monotonicity regions: $[29, 34]$, $[31, 41]$, and $[37, 41]$. The fact that we have discovered exactly three monotonicity regions is in good accordance with the geochemical data from [9].

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