A New Universal Approximation Result For Fuzzy Systems, Which Reflects CNF–DNF Duality

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Abstract

There are two main fuzzy system methodologies for translating expert rules into a logical formula: In Mamdani's methodology, we get a DNF formula (disjunction of conjunctions), and in a methodology which uses logical implications, we get, in effect, a CNF formula (conjunction of disjunctions). For both methodologies, universal approximation results have been proven which produce, for each approximated function $f(x)$, two different approximating relations $R_{\rm DNF}(x,y)$ and $R_{\rm CNF}(x,y)$. Since in fuzzy logic, there is a known relation $F_{\rm CNF}(x) \le F_{\rm DNF}(x)$ between CNF and DNF forms of a propositional formula F, it is reasonable to expect that we would be able to prove the existence of approximations for which a similar relation $R_{\rm CNF}(x,y) \le R_{\rm DNF}(x,y)$ holds. Such existence is proved in our paper.

1 Introduction

1.1 Fuzzy control: in brief

Fuzzy control (see, e.g., [9]) is a methodology that translates the expert's if-then rules of the type

$$\text{if } A_i(x) \text{ then } B_i(y), \quad 1 \le i \le N, \tag{1}$$

or

$$\text{if } A_{i1}(x_1) \text{ and} \ldots \text{and } A_{in}(x_n) \text{ then } B_i(y), \tag{2}$$

in which the properties $A_i(x)$ and $B_j(x)$ are described by using words from natural languages (such as "x is small"), into a *control strategy*, i.e., into a function $f : X \to Y$ describing what exactly control we should apply for a given input $x \in X$. This methodology consists of three major steps:

- first, we *formalize* each "linguistic" property $A_i(x)$ or $B_i(y)$ as a *fuzzy set*, i.e., as a function $A_i : X \to [0,1]$ which describes, for each object $x \in X$, to what extent this property holds for this x (e.g., to what extent x is small);
- then, we *combine* these fuzzy sets into a *fuzzy relation*, i.e. a function $R(x,y) : X \times Y \to [0,1]$ which describes, for each input $x \in X$ and for each possible output $y \in Y$, to what extent this particular outputs satisfies the expert's rules;
- finally, we apply some *defuzzification procedure* to the *fuzzy relation* $R(x,y)$, and get the desired control strategy, as a function $\widetilde{f} : X \to Y$.

1.2 Mamdani's (DNF) approach

In most practical application of fuzzy control, *Mamdani's* approach is used in the combination (second) step. In this approach, the fuzzy relation $R(x,y)$ is represented by a logical formula

$$(A_1(x)\&B_1(y)) \vee \ldots \vee (A_N(x)\&B_N(y)), \tag{3}$$

or as

$$(A_{11}(x)\& \ldots \&A_{1n}(x_n)\&B_1(y)) \vee \ldots \vee (A_{N1}(x_1)\& \ldots \&A_{Nn}(x_n)\&B_N(y)) \tag{4}$$

where '$\&$' and '$\vee$' stand for connectives of conjunction and disjunction respectively. In logical terms, we have a disjunction of conjunctions $A_i(x)\&B_i(y)$, i.e., a formula in a Disjunctive Normal Form – DNF.

Then, we select an interpretation: a t-norm $f_\& : [0,1] \times [0,1] \to [0,1]$ for conjunction and a t-conorm $f_\vee : [0,1] \times [0,1] \to [0,1]$ for disjunction (see, e.g.,

[4, 10]), and use these operations in the corresponding formulas (3) and (4), resulting in:

$$R_{\mathrm{DNF}}(x,y) = f_{\vee}(f_{\&}(A_1(x),B_1(y)),\ldots,f_{\&}(A_N(x),B_N(y)), \tag{5}$$

$$\begin{aligned} R_{\mathrm{DNF}}(x,y) = f_{\vee}[&f_{\&}(A_{11}(x),\ldots,A_{1n}(x_n),B_1(y)),\ldots,\\ &f_{\&}(A_{N1}(x_1),\ldots,A_{Nn}(x_n),B_N(y))]. \end{aligned} \tag{6}$$

1.3 Logical implication (CNF) approach

From the logical viewpoint, it is somewhat more natural to represent the fuzzy relation $R(x,y)$ as a conjunction of implications:

$$(A_1(x) \to B_1(y)) \& \ldots \& (A_N(x) \to B_N(y)), \tag{7}$$

or

$$\begin{aligned} &((A_{11}(x) \& \ldots \& A_{1n}(x_n)) \to B_1(y)) \& \ldots \& \\ &((A_{N1}(x_1) \& \ldots \& A_{Nn}(x_n)) \to B_N(y)). \end{aligned} \tag{8}$$

In this case, we select the interpretation: $f_{\&} : [0,1] \times [0,1] \to [0,1]$ for conjunction and $f_{\to} : [0,1] \times [0,1] \to [0,1]$, for implication and use these operations in the corresponding formulas (7) and (8), resulting in:

$$R_{\to}(x,y) = f_{\&}(f_{\to}(A_1(x),B_1(y)),\ldots,f_{\to}(A_N(x),B_N(y))), \tag{9}$$

$$\begin{aligned} R_{\to}(x,y) = f_{\&}[&f_{\to}(f_{\&}(A_{11}(x),\ldots,A_{1n}(x_n)),B_1(y)),\ldots,\\ &f_{\to}(f_{\&}(A_{N1}(x_1),\ldots,A_{Nn}(x_n)),B_N(y))]. \end{aligned} \tag{10}$$

In particular, since in classical logic $A \to B$ is equivalent to $\neg A \vee B$, and $(A_1(x) \& \ldots \& A_n) \to B$ to $\neg A_1 \vee \ldots \neg A_n \vee B$, it makes sense to consider representations of formulas (7) and (8) in the following form

$$(\neg A_1(x) \vee B_1(y)) \& \ldots \& (\neg A_N(x) \vee B_N(y)), \tag{11}$$

or

$$\begin{aligned} &(\neg A_{11}(x) \vee \ldots \vee \neg A_{1n}(x_n) \vee B_1(y)) \& \ldots \\ &\&(\neg A_{N1}(x_1) \vee \ldots \vee \neg A_{Nn}(x_n) \vee B_N(y)). \end{aligned} \tag{12}$$

In logical terms, we have a conjunction of disjunctions $A_i(x) \& B_i(y)$, i.e., a formula in a Conjunctive Normal Form – CNF.

In DNF, we have outside disjunction and inside conjunctions; in CNF, the roles of disjunction and conjunction are reversed: we have outside conjunction and inside disjunctions. In logic, conjunction and disjunction are often called *dual* logical operations; in view of this terminology, CNF and DNF are also often called *dual* forms.

Then, we select the interpretation: a t-norm $f_\&$: $[0,1] \times [0,1] \to [0,1]$ for conjunction, a t-conorm $f_\vee$: $[0,1] \times [0,1] \to [0,1]$ for disjunction and a *fuzzy negation* $f_\neg$; $[0,1] \to [0,1]$, and use these operations in the corresponding formulas (11) and (12), resulting in:

$$R_{\rm CNF}(x,y) = f_\&(f_\vee(f_\neg(A_1(x)), B_1(y)), \ldots, f_\vee(f_\neg(A_N(x)), B_N(y))), \qquad (13)$$

$$R_{\rm CNF}(x,y) = f_\&[f_\vee(f_\neg(A_{11}(x)), \ldots, f_\neg(A_{1n}(x_n)), B_1(y)), \ldots,$$
$$f_\vee(f_\neg(A_{N1}(x_1)), \ldots, f_\neg(A_{Nn}(x_n)), B_N(y))]. \qquad (14)$$

1.4 Relation between DNF and CNF approaches

For each of these two methodologies, it is desirable to check that this methodology is *universal*, i.e., that if we use this methodology, then, for an arbitrary control function $f : X \to Y$, and for an arbitrary accuracy, there exist appropriate if-then rules for which the resulting control strategy represented by $\widetilde{f}(x)$ approximates the original control function $f(x)$ within a given accuracy.

There exists many universal approximation results for approximations which are derived from Mamdani-style DNF formulas; first such results were formulated and proved, almost simultaneously, in 1990–92 papers by J. Buckley, Z. Cao, E. Czogala, D. Dubois, M. Grabisch, J. Han, Y. Hayashi, C.-C. Jou, A. Kandel, B. Kosko, J. Mendel, H. Prade, and L.-X. Wang; for a recent survey, see, e.g., [6] and references therein. There also exist several universal approximation results for implication-style CNF formulas [1, 2, 3, 11, 13].

These results are usually proved separately and provide two different (seemingly unrelated) approximations. In logic, however, CNF and DNF forms are related. In classical (2-valued) logic, every propositional formula F can be represented in both DNF and CNF forms $F_{\rm DNF}$ and $F_{\rm CNF}$; for every input x, these forms lead to exactly the same truth value: $F_{\rm CNF}(x) = F(x) = F_{\rm DNF}(x)$. In fuzzy logic, each propositional formula can also be transformed (generally, non-equivalently, see [11]) into CNF and DNF forms, so that using $f_\& = \min$, $f_\vee = \max$, and $f_\neg(a) = 1 - a$, we get $F_{\rm CNF}(x) \le F_{\rm DNF}(x)$ (to be more precise, $F_{\rm CNF}(x) \le F(x) \le F_{\rm DNF}(x)$; see, e.g., [12, 14]). In view of this relation, it is desirable to have a universal approximation result for CNF and DNF formulas which is consistent with this "fuzzy duality", i.e., in which there is a similar relation between the fuzzy relations $R_{\rm DNF}(x,y)$ and $R_{\rm CNF}(x,y)$ which approximate the desired function f. Such a result is presented in this paper.

In proving this duality-related result, we also somewhat generalize the known CNF and DNF universal approximation theorems.

2 General Case: Functions Defined on an Arbitrary Compact Set

Definition 1. *Let $k = 1$ or $k = 2$, and let $\oplus$ be a propositional k-ary operation in classical (2-valued) logic (e.g., $\&$, $\vee$, $\neg$, $\rightarrow$). We say that an operation $f_\oplus : [0,1]^k \rightarrow [0,1]$ is consistent with classical logic if it coincides with $\oplus$ when all its inputs are 0's and 1's (corresponding to "false" and "true").*

Please note that we did not require that $f_\oplus$ is continuous (as a function), or that $f_\&(a,b)$ is commutative or associative, etc.

Definition 2. *Let X be a compact metric space with a metric d_X, Y be a complete metric space with a metric d_Y, $f : X \rightarrow Y$ be a continuous function from X to Y, and $\varepsilon > 0$ be a real number. We say that a fuzzy relation $R : X \times Y \rightarrow [0,1]$ ε-approximates a function $f : X \rightarrow Y$ if the following two conditions hold:*

- *for every $x \in X$, $R(x, f(x)) > 0$, and*
- *for every $x \in X$ and $y \in Y$, if $R(x,y) > 0$, then $d_Y(y, f(x)) \leq \varepsilon$.*

Theorem 1. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \rightarrow Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:*

- *both fuzzy relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using the interpretation determined by $f_\&$, $f_\vee$, and $f_\neg$) ε-approximate f, and*
- *$R_{\rm CNF}(x,y) \leq R_{\rm DNF}(x,y)$ for all x and y.*

Comment. For the convenience of the readers, all the proofs are placed in the special Proofs section.

A similar result holds for $R_\rightarrow$ instead of $R_{\rm CNF}$:

Theorem 1′. *Let operations $f_\&$, $f_\vee$, and $f_\rightarrow$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \rightarrow Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which:*

- *both fuzzy relations $R_\rightarrow$ and $R_{\rm DNF}$ (obtained using the interpretation determined by $f_\&$, $f_\vee$, and $f_\rightarrow$) ε-approximate f, and*
- *$R_\rightarrow(x,y) \leq R_{\rm DNF}(x,y)$ for all x and y.*

From the fact that a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, we can conclude that the result $\widetilde{f}(x) = D(\mu_x)$ of applying, for every $x \in X$,

a defuzzification procedure D (see below) to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$:

Definition 3. *By a defuzzification procedure, we mean a mapping $D : [0,1]^Y \to Y$ which maps every membership function $\mu : Y \to [0,1]$ (which is not identically zero) into an element $D(\mu) \in Y$ for which $\mu(D(\mu)) > 0$.*

Proposition 1. *If a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, then, for every defuzzification procedure D, the result $\widetilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$, i.e., $d_Y\left(\widetilde{f}(x), f(x)\right) \le \varepsilon$.*

Corollary 1. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \to Y$ into a complete metric space Y, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D, the results $\widetilde{f}_{\rm DNF}(x)$ and $\widetilde{f}_{\rm CNF}(x)$ of defuzzifying the relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\neg$) are ε-close to f.*

Corollary 1′. *Let operations $f_\&$, $f_\vee$, and $f_\to$ be consistent with classical logic. Then, for every compact metric space X, for every continuous function $f : X \to Y$ into a complete metric space Y, for every real number $\varepsilon > 0$, there exist fuzzy rules of type (1) for which, for each defuzzification procedure D, the results $\widetilde{f}_{\rm DNF}(x)$ and $\widetilde{f}_{\rm CNF}(x)$ of defuzzifying the relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\to$) are ε-close to f.*

3 Towards a More Realistic Situation: Case When $Y = \mathbb{R}$

In the case when Y is a real line ($Y = \mathbb{R}$), we can use a different class of possible "defuzzification procedures" and still get the same universal approximation result. Namely, we can use the following definition:

Definition 3′. ($Y = \mathbb{R}$) *By a defuzzification procedure, we mean a mapping D which maps every non-zero membership function $\mu : \mathbb{R} \to [0,1]$ into an real number $D(\mu)$ in such a way that for an arbitrary interval $[a,b]$, if a membership function $\mu(x)$ is equal to 0 for all values x outside an interval $[a,b]$, then $D(\mu) \in [a,b]$.*

Comment. Both centroid and center-of-maximum are defuzzification procedures in this sense.

Proposition 1′. ($Y = \mathbb{R}$) *If a fuzzy relation $R(x,y)$ ε-approximates a function $f(x)$, then, for every defuzzification procedure D, the result $\widetilde{f}(x) = D(\mu_x)$ of applying this defuzzification procedure D to the corresponding membership function $\mu_x(y) = R(x,y)$ is ε-close to $f(x)$, i.e., $d_Y\left(\widetilde{f}(x), f(x)\right) \le \varepsilon$.*

4 Realistic Case: Functions From $\mathbb{R}^n$ to $\mathbb{R}$

For the case when $X = \mathbb{R}^n$, we can prove similar results with rules of type (2):

Theorem 2. *Let operations $f_\&$, $f_\vee$, and $f_\neg$ be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \to \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- *both fuzzy relations $R_{\rm DNF}$ and $R_{\rm CNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\neg$) ε-approximate f, and*
- *$R_{\rm CNF}(x, y) \le R_{\rm DNF}(x, y)$ for all $x = (x_1, \ldots, x_n) \in X$ and y.*

Theorem 2′. *Let operations $f_\&$, $f_\vee$, and $f_\to$ be consistent with classical logic. Then, for every integer $n > 0$, for every compact set $X \subset \mathbb{R}^n$, for every continuous function $f : X \to \mathbb{R}$, and for every real number $\varepsilon > 0$, there exist fuzzy rules of type (2) for which:*

- *both fuzzy relations $R_\to$ and $R_{\rm DNF}$ (obtained using $f_\&$, $f_\vee$, and $f_\to$) ε-approximate f, and*
- *$R_\to(x, y) \le R_{\rm DNF}(x, y)$ for all $x(x_1, \ldots, x_n) \in X$ and y.*

In our universal approximation result, we prove that for every function $f : X \to Y$, we can select the rules and the membership functions for which we get the desired approximation property.

For Mamdani's (DNF) case, a stronger statement is true: that whatever "realistic" membership function $\mu_0(x)$ we choose, we can always find rules in which all the membership functions $A_{ik}(x_k)$ and $B_i(y)$ are *of the type* μ_0, i.e., they all have the form $\mu(x) = \mu_0(a \cdot x + b)$ for some real numbers $a \ne 0$ and b (see, e.g., [7, 8]).

From the proof of Theorems 2 and 2′, we see that all the membership functions used in the approximation have the same type, i.e., that there is a type μ_0 which provides a universal approximation property both for DNF and for CNF forms. We do not know whether a similar result is true for an arbitrary given type.

5 Proofs

5.1 Proof of Theorem 1

This proof is similar to the original Kosko's proof [5] of a universal approximation result for DNF (i.e., for Mamdani methodology), and to our own proofs from [7, 8, 11].

1°. Let us take $\varepsilon_1 = \varepsilon/2$. Since a function f is continuous on a compact set X, it is also uniformly continuous. Therefore, there exists $\delta > 0$ such that if $d_X(x, x') \le \delta$, then $d_Y(f(x), f(x')) \le \varepsilon_1$.

Since X is a compact metric space, there exists a finite δ-net for X, i.e., a finite set of elements $x^{(1)}, \ldots, x^{(N)} \in X$ for which, for every $x \in X$, there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$. For each of these elements $x^{(i)}$, we can find $y^{(i)} = f\left(x^{(i)}\right)$. We will show that Theorem 1 holds for N rules of type (1) where for every i,

- $A_i(x) = 1$ if $d_X\left(x, x^{(i)}\right) \le \delta$, and $A_i(x) = 0$ otherwise;
- $B_i(y) = 1$ if $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$, and $B_i(y) = 0$ otherwise.

All these fuzzy sets A_i and B_i are crisp: indeed, $A_i(x)$ is a characteristic function of the inequality $d_X\left(x, x^{(i)}\right) \le \delta$, and $B_i(y)$ is a characteristic function of the inequality $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$.

Since all the $f_\&$, $f_\vee$, and $f_\neg$ are consistent with classical logic and A_i, B_i are crisp, we conclude that the relations R_{CNF} and R_{DNF} can be obtained using classical logical connectives. Namely,

$$R_{\mathrm{DNF}}(x, y) = 1 \iff \exists i \left(d_X\left(x, x^{(i)}\right) \le \delta \,\&\, d_Y\left(y, y^{(i)}\right) \le \varepsilon_1\right); \tag{15}$$

$$R_{\mathrm{CNF}}(x, y) = 1 \iff \forall i \left(d_X\left(x, x^{(i)}\right) > \delta \vee d_Y\left(y, y^{(i)}\right) \le \varepsilon_1\right). \tag{16}$$

2°. Let us first show that the relation R_{DNF} ε-approximates the given function f.

2.1°. In accordance with the definition of ε-approximation, we first prove that for every $x \in X$, we have $R_{\mathrm{DNF}}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X. Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$. Due to our choice of δ, we conclude that $d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Thus, (15) is true, hence, $R_{\mathrm{DNF}}(x, f(x)) > 0$.

2.2°. Let us now prove that for every $x \in X$ and $y \in Y$, if $R_{\mathrm{DNF}}(x, y) > 0$, then $d_Y(y, f(x)) \le \varepsilon$.

Indeed, since R_{DNF} is a crisp relation, the only possibility for $R_{\mathrm{DNF}}(x, y) > 0$ is to have $R_{\mathrm{DNF}}(x, y) = 1$, i.e., $R_{\mathrm{DNF}}(x, y)$ to be true. This means that there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$ and $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. Due to our choice of δ, from $d_X\left(x, x^{(i)}\right) \le \delta$, we can conclude that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \le d_Y\left(y, y^{(i)}\right) + d_Y\left(y^{(i)}, f(x)\right) \le \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

3°. Let us now show that the relation R_{CNF} also ε-approximates the given function f.

3.1°. In accordance with the definition of ε-approximation, we first prove that for every $x \in X$, we have $R_{\mathrm{CNF}}(x, f(x)) > 0$.

Indeed, let x be an arbitrary element of the set X. For every i, we have $d_X\left(x, x^{(i)}\right) \le \delta$ or $d_X\left(x, x^{(i)}\right) > \delta$. By definition of δ, the inequality $d_X\left(x, x^{(i)}\right) \le \delta$ implies that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Therefore, we have $d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$ or or $d_X\left(x, x^{(i)}\right) > \delta$. Hence, (16) is true, and $R_{\mathrm{CNF}}(x, f(x)) > 0$.

3.2°. Let us show that for every $x \in X$ and $y \in Y$, if $R_{\mathrm{CNF}}(x, y) > 0$, then $d_Y(y, f(x)) \le \varepsilon$.

Indeed, since R_{CNF} is a crisp relation, the only possibility for $R_{\mathrm{CNF}}(x, y) > 0$ is to have $R_{\mathrm{CNF}}(x, y) = 1$, i.e., $R_{\mathrm{CNF}}(x, y)$ to be true. This means that for every i, either $d_X\left(x, x^{(i)}\right) > \delta$ or $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. Equivalently, this disjunction means that the inequality $d_X\left(x, x^{(i)}\right) \le \delta$ implies (in the logical sense, where implication is material!) that $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$.

Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for $d_X\left(x, x^{(i)}\right) \le \delta$. For this i, we thus $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. On the other hand, due to our choice of δ, from $d_X\left(x, x^{(i)}\right) \le \delta$, we can conclude that $d_Y\left(f(x), f\left(x^{(i)}\right)\right) = d_Y\left(f(x), y^{(i)}\right) \le \varepsilon_1$. Thus, from the triangle inequality, we conclude that $d_Y(y, f(x)) \le d_Y\left(y, y^{(i)}\right) + d_Y\left(y^{(i)}, f(x)\right) \le \varepsilon_1 + \varepsilon_1 = \varepsilon$. The statement is proven.

4°. To complete the proof of the theorem, we must now show that $R_{\mathrm{CNF}}(x, y) \le R_{\mathrm{DNF}}(x, y)$ for all x and y.

Since both relations $R_{\mathrm{CNF}}(x, y)$ and $R_{\mathrm{DNF}}(x, y)$ are crisp, the desired inequality is equivalent to saying that for every x and y, if $R_{\mathrm{CNF}}(x, y)$ is true, then $R_{\mathrm{DNF}}(x, y)$ should also be true. Indeed, let $R_{\mathrm{CNF}}(x, y)$ hold for some x and y. This means that for every i, the inequality $d_X\left(x, x^{(i)}\right) \le \delta$ implies (in the logical sense) that $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. Since $x^{(1)}, \ldots, x^{(N)}$ is a δ-net, there exists an i for which $d_X\left(x, x^{(i)}\right) \le \delta$. Thus, for this i, we have $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$. So, for this i, both inequalities $d_X\left(x, x^{(i)}\right) \le \delta$ and $d_Y\left(y, y^{(i)}\right) \le \varepsilon_1$ are true, hence the formula (16) is true. The statement is proven, and so is the theorem.

Comment. We have already mentioned that in crisp (2-valued) propositional logic, CNF and DNF forms represent the same function. It deserves mentioning that although in our proof, the relations $R_{\mathrm{CNF}}(x, y)$ and $R_{\mathrm{DNF}}(x, y)$ are both crisp, they do not necessarily the same. Indeed, let us consider the simplest case when $X = Y = [0, 1]$ with a normal metric $d_X(x, x') = d_Y(x, x') = |x - x'|$, and $f(x) = x$. Then, $\delta = \varepsilon_1 = \varepsilon/2$. As a δ-net, we can select the points $x^{(i)} = (2i - 1) \cdot \delta$, i.e., $x^{(1)} = \delta$, $x^{(2)} = 3\delta$, etc.; then, $y^{(i)} = x^{(i)}$. Here, for $x = 2\delta$ and $y = 0$, we have $d_X\left(x, x^{(1)}\right) \le \delta$ and $d_Y\left(y, y^{(1)}\right) \le \varepsilon_1$, so $R_{\mathrm{DNF}}(x, y)$

is true. However, the property $R_{\mathrm{CNF}}(x,y)$ is not true, because for $i=2$, we have $d_X\left(x,x^{(2)}\right)\le\delta$, but $d_Y\left(y,y^{(2)}\right)=3\delta=3\varepsilon_1>\varepsilon_1$.

5.2 Proof of Theorem 1′.

For this proof, we can use the exact same crisp sets A_i and B_i as in the proof of Theorem 1.

5.3 Proof of Proposition 1

By definition, $\widetilde{f}(x)=D(\mu_x)$, where the membership function $\mu_x: Y\to[0,1]$ is defined as $\mu_x(y)=R(x,y)$. By the definition of a defuzzification procedure, for every $x\in X$, we have $\mu_x(D(\mu_x))>0$, i.e., by definition of μ_x, $R\left(x,\widetilde{f}(x)\right)>0$. From the definition of ε-approximation, we can now conclude that $d_Y\left(\widetilde{f}(x),f(x)\right)\le\varepsilon$. The proposition is proven.

5.4 Proof of Proposition 1′

By definition, $\widetilde{f}(x)=D(\mu_x)$, where the membership function $\mu_x: Y\to[0,1]$ is defined as $\mu_x(y)=R(x,y)$. From the definition of ε-approximation, we conclude that if $R(x,y)>0$, then $d_Y(y,f(x))=|y-f(x)|\le\varepsilon$. Thus, if $|y-f(x)|>\varepsilon$, we have $R(x,y)=\mu_x(y)=0$. Hence, the function $\mu_x(y)$ is equal to 0 outside the interval $[f(x)-\varepsilon,f(x)+\varepsilon]$. By definition of a defuzzification procedure, we can now conclude that the result $\widetilde{f}(x)$ of its defuzzification also belongs to the same interval, i.e., that $\left|\widetilde{f}(x)-f(x)\right|\le\varepsilon$. The proposition is proven.

5.5 Proof of Theorems 2 and 2′

This proof is similar to the proofs of Theorems 1 and 1′. Indeed, for $X\subset\mathbb{R}^n$, continuity of a function $f: X\to\mathbb{R}$ with respect to a normal (Euclidean) metric is equivalent to its continuity with respect to the uniform metric $d_X(x,x')=\max_i|x_i-x'_i|$. Thus, from the proofs of Theorems 1 and 1′, we conclude that there exist appropriate rules of type (1), with $A_i(x)=1\iff d_X\left(x,x^{(i)}\right)\le\delta$ and $A_i(x)=0$ for all other x, and $B_i(y)=1\iff\left|y-y^{(i)}\right|\le\varepsilon_1$ and $B_i(y)=0$ for all other y. By the definition of the uniform metric d_X, the inequality $d_X\left(x,x^{(i)}\right)\le\delta$ is equivalent to $A_{i1}(x_1)\&\ldots\&A_{in}(x_n)=1$, where $A_{ik}(x_k)=1\iff\left|x_k-x_k^{(i)}\right|\le\delta$ and $A_{ik}(x_k)=0$ for all other x_k. Thus, rules of type (1) can be reformulated in the desired form (2). The theorems are thus proven.

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