On Representation and Approximation of Operations in Boolean Algebras

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Abstract

Several universal approximation and universal representation results are known for *non-Boolean* multi-valued logics such as fuzzy logics. In this paper, we show that similar results can be proven for multi-valued *Boolean* logics as well.

Introduction. The study of Boolean algebras started when G. Boole [1] considered the simplest Boolean algebra – the two-valued algebra $B_{(2)} = \{\mathbf{0}, \mathbf{1}\}$ (={false,true}). In this algebra, the operation $\vee$ ($\cup$), $\wedge$ ($\cap$), and negation a' ($\neg a$) have the direct logical meaning of "or", "and", and "not". It is known that in this Boolean algebra, an arbitrary *operation*, i.e., an arbitrary function $B \times \ldots \times B \to B$, can be represented as a superposition of these three basic logical operations: e.g., the implication $a \to b$ can be represented as $b \vee \neg a$, etc.

Logic is still one of the area of application of Boolean algebras, but, starting from the classical Kolmogorov's monograph [4], Boolean algebras – namely, algebras of events – became an important tool in another area: foundations of probability.

In contrast to the Boolean algebra $B_{(2)}$, in more complex Boolean algebras, there are functions $B \times \ldots \times B \to B$ which cannot be represented in terms of the three basic operations. Until recently, these functions were rarely used; traditionally, in probability applications, only the standard operations $\vee$, $\wedge$, and a' were used, or operations which can be explicitly described in terms of these three. Some complex events can be easily described by using these three operations, but some other complex events, like a hypothetic *conditional event* "a if b" whose probability is equal to the conditional probability $P(a|b)$, cannot be described via these three operations (this result is due to Lewis; for its detailed description, see, e.g., Chapter 1 of [2]). Some researchers even thought that since we cannot get an expression for a conditional event by using the three basic operations of Boolean algebra, we, therefore, cannot describe such events in Boolean algebra at all. It was recently shown, however (see, e.g., [2, 3]), that if we use a new operation, an operation which cannot be explicitly represented in terms of $\vee$, $\wedge$, and a', then it *is* possible to describe conditional events within the Boolean algebra formalism.

Namely, to describe conditional events "a if b", we can consider, instead of an individual event, a potentially infinite sequence of similar events, and interpret "a if b" as "a is true in the first moment of time in which b is true".

Comment. Let us describe this representation in more formal terms for those readers who are familiar with the main notions of mathematical probability theory; other readers can skip this comment. In more formal terms, instead of the original Boolean algebra B of all events (measurable subsets) of a σ-algebra Ω, we consider the set $\Omega^{\mathbb{N}} = \Omega \times \Omega \times \ldots \times \Omega \times \ldots$ of all infinite sequences $\omega = (\omega_1, \omega_2, \ldots)$ of events $\omega_i \in \Omega$ with a product measure (this construction is standard in probability theory where it is used to formulate and prove limit theorems), and the new Boolean algebra of all events on Ω^N. Then, the event "a if b" is interpreted as

$$(b(\omega_1) \wedge a(\omega_1)) \vee$$
$$(\neg b(\omega_1) \wedge b(\omega_2) \wedge a(\omega_2)) \vee$$
$$(\neg b(\omega_1) \wedge \neg b(\omega_2) \wedge b(\omega_3) \wedge a(\omega_3)) \vee \ldots$$

End of comment.

This successful use of a new operation raises natural important questions:

- for what Boolean algebras every operation can be described as a composition of the three basic ones?
- if for some Boolean algebra, not every operation can be described as a composition of the three basic ones, can we add some fourth operation (or finitely many operations) so that every operation will be representable as a composition of these four?
- if this is not possible, can we at least add a fourth operation so that every operation will be *approximable* by compositions of these four operations?

In this paper, we give answers to these three questions.

Definition 1. *Let B be a Boolean algebra, and let n be a positive integer. By an n-ary operation, we mean a function $f : B^n \to B$.*

Proposition 1. *For every Boolean algebra B which is different from $B_{(2)} = \{\mathbf{0}, \mathbf{1}\}$, and for every $n \geq 1$, then there exists a n-ary operation which cannot be represented as a composition of $\vee$, $\wedge$, a', and constants.*

The proof is based on the fact that for every Boolean algebra B with $E > 2$ elements, there are exactly E^{E^n} n-nary operations but only E^{2^n} operations which can be represented as a composition of $\vee$, $\wedge$, a', and constants, and $E^{2^n} < E^{E^n}$. If we do not allow constants, then we get an even smaller number of representable n-ary operations: 2^{2^n}. (For readers' convenience, all the proofs are placed at the end of the paper).

Comment. The fact that not every operation can be represented is known; moreover, there exist conditions (see, e.g., Ch. 13, Section 4, of [6]) which are necessary and sufficient for the existence of a representable operation f for which $f(a_{k1}, \ldots, a_{kn}) = b_k$ for known values a_{ki} and b_k, $1 \leq k \leq K$.

The standard two-valued "logical" algebra $B_{(2)} = \{\mathbf{0}, \mathbf{1}\}$ describes the standard (two-valued) logic. When a Boolean algebra is different from the two-valued "logical" algebra $B_{(2)} = \{\mathbf{0}, \mathbf{1}\}$, it means that this Boolean algebra has more than 2 elements in it, so it corresponds to a *multi-valued* logic. With this interpretation in mind, it is not surprising that some operations cannot be represented as a composition of the basic three logical ones: indeed, the situation is similar in another widely used multi-valued logic called *fuzzy logic* in which the set of possible truth values is the interval $[0, 1]$. For fuzzy logic, it is known [5] that we will be able to represent an arbitrary operation as a composition if we add, to the three basic logical operations, a new unary operation called *hedge*. It is therefore reasonable to conjecture that if we add an appropriate unary operation, we will be able to get a similar representation for a Boolean logic as well. This conjecture is indeed true (at least for finite Boolean logics):

Definition 2. *For an arbitrary Boolean algebra B, we define an absolute truth operation $t(a)$ as follows: $t(\mathbf{1}) = \mathbf{1}$ and $t(a) = \mathbf{0}$ for all $a \neq \mathbf{1}$.*

The function t is similar to the delta-function $\delta(x)$ (see, e.g., [10]), which is defined, crudely speaking, as a function which is different from 0 only at one point $x = 0$. It is even more similar to Kronecker's "delta" $\delta_{i,j}$ which is defined as $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ when $i \neq j$: we can easily see that $t(a) = \delta_{a,\mathbf{0}}$.

Proposition 2. *For every finite Boolean algebra, and for every positive integer n, every n-ary operation f can be represented as a composition of $\vee$, $\wedge$, a', t, and constants.*

This representation has the form

$$\bigvee_{(a_1, \ldots, a_n) \in B^n} [\mathrm{eq}(x_1, a_1) \wedge \ldots \wedge \mathrm{eq}(x_n, a_n) \wedge f(a_1, \ldots, a_n)], \qquad (1)$$

where $\mathrm{eq}(x, a)$ is defined as $\mathrm{eq}(x, a) \stackrel{\text{def}}{=} t((x \wedge a) \vee (x' \wedge a'))$ and is equal to $\mathrm{eq}(x, a) = \mathbf{1}$ when $x = a$, and to $\mathrm{eq}(x, a) = \mathbf{0}$ when $x \neq a$.

This representation is similar to the known possibility to represent an arbitrary function as a linear combination of delta-functions, so-called "sifting property" ([10], pp. 11 and ff.):

$$f(x_1,\ldots,x_n)=\int_{a_1}\ldots\int_{a_n}\delta(x_1-a_1)\cdot\ldots\cdot\delta(x_n-a_n)\cdot f(a_1,\ldots,a_n)\,\mathrm{d}a_1\ldots\mathrm{d}a_n.$$

For the case when each variable x_i only takes integer value, we can have another analogue of the same result, with Kronecker's "delta" instead of a delta-function:

$$f(x_1,\ldots,x_n)=\sum_{a_1}\ldots\sum_{a_n}\delta_{x_1,a_1}\cdot\ldots\cdot\delta_{x_n,a_n}\cdot f(a_1,\ldots,a_n).$$

The above formula still holds for infinite Boolean algebras, but we have to use an infinite union. If we only allow finite unions, then such a representation is no longer possible:

Proposition 3. *There exists an infinite Boolean algebra B for which, no matter what finite (or even countably infinite) class of operation we choose, there exists an unary operation which cannot be represented as a composition of constants and operation from this class.*

Since we cannot *represent* operations, maybe we can *approximate* them? In a reasonable framework in which we can naturally define the meaning of the word "approximate", this is indeed true. Indeed, a well-known Stone's representation theorem (see, e.g., [8]) states that every Boolean algebra is isomorphic to the class of all compact and open subsets of a compact topological space U. In view of this result, the following definition seems natural:

Definition 3. *For every compact metric space U with a metric d, by $B(U)$, let us denote a Boolean algebra generated by its compact subsets. On this Boolean algebra, we can define Hausdorff metric*

$$d_H(a,b)=\max\left\{\max_{u\in a}d(u,b),\max_{v\in b}d(v,b)\right\},$$

where

$$d(u,b)\stackrel{\text{def}}{=}\min_{v\in b}d(u,v).$$

For an arbitrary positive real number $\varepsilon>0$, two operations $f,g:B(U)^n\to B(U)$ will be called ε-close if for every $a_1,\ldots,a_n\in B(U)$, we have $d_H(f(a_1,\ldots,a_n),g(a_1,\ldots,a_n))\le\varepsilon$.

Proposition 4. *Let U be a compact metric space, let f be a continuous n-ary operation on $B(U)$, and let $\varepsilon>0$. Then, there exists a composition g of $\vee$, $\wedge$, a', t, and constants, which is ε-close to f.*

Proof of Proposition 1. Every Boolean algebra has elements $\mathbf{0}$ and $\mathbf{1}$. Thus, if a Boolean algebra is different from a 2-element set $\{\mathbf{0},\mathbf{1}\}$, it means that its number of elements E is 3 or greater.

An n-ary operation is simply a function from the set B^n with E^n elements to the set B of E elements. Thus, the total number of n-ary operations is $E^{(E^n)} = E^{E^n}$.

How many n-ary operations can be represented as a composition of $\vee$, $\wedge$, a', and constants? Using the general rules of Boolean algebra, we can reduce each such composition $f(x_1, \ldots, x_n)$ to a *disjunctive normal form*, i.e., to the form $C_1 \vee C_2 \vee \ldots \vee C_k$, where each conjunction C_i has the form

$$x_1^{\varepsilon_1} \wedge \ldots \wedge x_n^{\varepsilon_n} \wedge a_{\varepsilon_1 \ldots \varepsilon_n},$$

with $\varepsilon_i \in \{+, -\}$, $x^+ \stackrel{\text{def}}{=} x$, $x^- \stackrel{\text{def}}{=} x'$, and $a_{\varepsilon_1 \ldots \varepsilon_n}$ a constant. (This representation is known in Boolean algebra theory; see, e.g., [6].) Different expressions of this type can be described by assigning, to each of 2^n combinations $(\varepsilon_1, \ldots, \varepsilon_n) \in \{+, -\}^n$ of $+$'s and $-$'s, an element $a_{\varepsilon_1 \ldots \varepsilon_n} \in B$.

Different assignments lead to different functions: indeed, if we fix the n-ary operation f, i.e., if we fix the values $f(x_1, \ldots, x_n)$ for all possible combinations $x = (x_1, \ldots, x_n) \in B^n$, then, for each combination $(\varepsilon_1, \ldots, \varepsilon_n)$, we can reconstruct $a_{\varepsilon_1, \ldots, \varepsilon_n}$ as $f(x(\varepsilon_1), \ldots, x(\varepsilon_n))$, where $x(+) = \mathbf{1}$ and $x(-) = \mathbf{0}$.

Thus, representable n-ary operations are in 1-1 correspondence with assignments, i.e., with functions $a : \{+, -\}^n \to B$. The set $\{+, -\}^n$ has 2^n elements, the set B has E elements, so there are exactly $E^{(2^n)} = E^{2^n}$ such functions and hence, exactly E^{2^n} representable n-ary operations.

To complete the proof, we must show that for every $E > 2$, the number E^{2^n} of representable n-ary operations is smaller than the total number E^{E^n} of n-ary operations.

- For finite E, from $E > 2$, we conclude that $2^n < E^n$ and hence, that $E^{2^n} < E^{E^n}$.
- For infinite E, we have $E^{2^n} = E$ for every finite n, while $E^{E^n} \geq 2^E > E$ (see, e.g., [7, 9]), so, there also exist non-representable n-ary operations.

In both cases, the proposition is proven.

Proof of Proposition 2. Definition of $\mathrm{eq}(x, a)$ as $t((x \wedge a) \vee (x' \wedge a'))$ describes equality in terms of the four operations. Indeed, one can easily prove that this expression is equal to $\mathbf{1}$ if and only if x is equal to a, otherwise, this expression is equal to $\mathbf{0}$. After that, the proof is straightforward – one can easily check that an arbitrary function $f : B^n \to B$ can be represented in the form (1). The proposition is proven.

Proof of Proposition 3. As the desired Boolean algebra B, we will take the class of all possible finite and co-finite (= complement to finite) subsets of the set R of all real numbers. There are continuum many 1-element sets, continuum many 2-element sets, etc. The entire algebra can be thus represented as a countable union of sets of cardinality continuum, and therefore, the set B itself also has a cardinality continuum $\mathcal{C} = \aleph_1$. The total number of unary operations, i.e., functions from B to B is, therefore, equal to $\mathcal{C}^{\mathcal{C}} = \aleph_2$.

Then, there are countably many different formulas with blank places for constants. There are continuum many constants, so when we substitute these constants into the formulas, we get continuum many representable functions. So, out of $\mathcal{C}^{\mathcal{C}}$ unary operations, only $\mathcal{C}$ are representable. Hence, there exist unary operations which are not representable. The proposition is proven.

Proof of Proposition 4. Since an operation f is a continuous function on a compact set, it is uniformly continuous, and there exists a δ for which, if $d_H(a_i, b_i) \le \delta$ for all i from 1 to n, then $d_H(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \le \varepsilon$.

Since U is a compact metric set, it has a finite $(\delta/2)$-net, i.e., a finite list $u_1, \ldots, u_N \in U$ for which U is contained in the union of N closed balls $c_j = B_{\delta/2}(u_j) = \{x \mid d(x, u_j) \le \delta/2\}$ of radii $\delta/2$ with centers in u_j.

For every $J \subseteq \{1, \ldots, N\}$, by C_J, we will denote the set

$$\bigvee_{j \in J} c_j.$$

Let us now show that f is ε-close to the representable operation $g(a_1, \ldots, a_n)$ which is defined by the following formula:

$$\bigvee_{J_1, \ldots, J_n} \bigwedge_{i=1}^{n} \left[\left(\bigwedge_{j \in J_i} \neg \mathrm{eq}(a_i \wedge c_j, \mathbf{0}) \right) \wedge \left(\bigwedge_{j \notin J_i} \mathrm{eq}(a_i \wedge c_j, \mathbf{0}) \right) \wedge f(C_{J_1}, \ldots, C_{J_n}) \right].$$

According to this formula, if we are given a sequence $a_1, \ldots, a_n$, then for each element a_i, we construct a union C_{J_i} of all the balls c_j which have non-empty intersection with a_i, and take $f(C_{J_1}, \ldots, C_{J_n})$ as the value of $g(a_1, \ldots, a_n)$.

Let us show that for each i, $d_H(a_i, C_{J_i}) \le \delta$. Then, due to our choice of δ, we would be able to conclude that $g(a_1, \ldots, a_n) = f(C_{J_1}, \ldots, C_{J_n})$ is ε-close to $f(a_1, \ldots, a_n)$, i.e., that f and g are indeed ε-close.

To prove that $d_H(a_i, C_{J_i}) \le \delta$, we must prove that:

- every element $u \in a_i$ is δ-close to some element from C_{J_i}, and
- that every element $u \in C_{J_i}$ is δ-close to some element from a_i.

Let us first prove the first of these two statements. Since $u_1, \ldots, u_N$ form a $(\delta/2)$-net for U, there exists a j for which $d(u, u_j) \le \delta/2$. This means that $u \in B_{\delta/2}(u_j) = c_j$; since also $u \in a_i$, we conclude that $a_i \cap c_j \ne \emptyset$, i.e., in Boolean algebra notations, that $a_i \wedge c_j \ne \mathbf{0}$ and therefore (by definition of C_{J_i}), that $c_j \subseteq C_{J_i}$. From $u \in c_j \subseteq C_{J_i}$, we conclude that $u \in C_{J_i}$, so u is $\delta/2$-close to some element from C_{J_i}: namely, to itself.

Let us now prove the second of the two statements. Let $u \in C_{J_i}$. Since C_{J_i} is defined as a union of all the balls c_j which have non-zero intersection with a_i, we can thus conclude that $u \in c_j$ for some ball $c_j = B_{\delta/2}(u_j)$ for which $c_j \cup a_i \neq \emptyset$, i.e., for which there exist an element $z \in c_j \cup a_i$. Let us show that $z \in a_i$ is the desired element from a_i which is δ-close to u. Indeed, from $u \in c_j = B_{\delta/2}(u_j)$, we conclude that $d(u, u_j) \leq \delta/2$. Similarly, from $z \in c_j = B_{\delta/2}(u_j)$, we conclude that $d(u_j, z) \leq \delta/2$. Thus, $d(u, z) \leq d(u, u_j) + d(u_j, z) \leq \delta/2 + \delta/2 = \delta$.

Both statements are proven, and so is the proposition.

Acknowledgments. This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants No. DUE-9750858 and CDA-9522207, by United Space Alliance grant No. NAS 9-20000 (PWO C0C67713A6), and by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-95-1-0518, and by the National Security Agency under Grant No. MDA904-98-1-0561.

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