Optimal Finite Characterization of Linear Problems with Inexact Data

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Abstract

For many linear problems, in order to check whether a certain property is true for all matrices A from an interval matrix $\mathbf{A}$, it is sufficient to check this property for finitely many "vertex" matrices $A \in \mathbf{A}$. J. Rohn has discovered that we do not need to use all 2^{n^2} vertex matrices, it is sufficient to only check these properties for $2^{2n-1} \ll 2^{n^2}$ vertex matrices of a special type A_{yz}. In this paper, we show that a further reduction is impossible: without checking all 2^{2n-1} matrices A_{yz}, we cannot guarantee that the desired property holds for all $A \in \mathbf{A}$. Thus, these special vertex matrices provide an *optimal* finite characterization of linear problems with inexact data.

1 Introduction

Many practical problems are described by systems of linear equations and/or inequalities, i.e., as *linear problems.* The components A_{ij} of the corresponding matrices A are often not exactly known; for each of these components, we only know the interval $[\underline{A}_{ij}, \overline{A}_{ij}]$ of possible values. The class of all matrices A which are consistent with this information is called an *interval matrix*

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A : \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}$ is a matrix with components $\underline{A}_{ij}$, $\overline{A}$ is a matrix with components $\overline{A}_{ij}$, and $A \le B$ means that $A_{ij} \le B_{ij}$ for all i and j. In practice, all the elements of the matrices are rational numbers (it is worth mentioning that our results hold for real numbers as well).

We say that an interval matrix $\mathbf{A}$ *satisfies* a property $\mathcal{P}$ (e.g., is non-singular or positive definite) if all matrices $A \in \mathbf{A}$ satisfy this property. It is known that for many such properties, an interval matrix satisfies the property $\mathcal{P}$ if and only if all its *vertex matrices*, i.e., matrices for which $A_{ij} \in \{\underline{A}_{ij}, \overline{A}_{ij}\}$ for all i and j, satisfy this property. Thus, in order to check whether a given interval matrix satisfies the property $\mathcal{P}$, it is sufficient to check this property for a finite set of vertex matrices.

This set is finite but huge: e.g., for $n \times n$ square matrices, we have 2^{n^2} possible vertex matrices; as a result, for large n, checking all such matrices requires an unrealistic amount of computation time.

In [3, 6], it was shown that for many properties $\mathcal{P}$, we do not need to check all these matrices: it is sufficient to use vertex matrices from the following special class. Namely, let us define $e \stackrel{\text{def}}{=} (1, \ldots, 1)^T$,

$$Y \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : |y| = e\} = \text{ the set of all } \pm 1 - \text{vectors.}$$

For every $y, z \in Y$, we can can define a matrix A_{yz} if we set, for every i and j,

- $(A_{yz})_{ij} \stackrel{\text{def}}{=} \overline{A}_{ij}$ if $y_i \cdot z_j = -1$, and
- $(A_{yz})_{ij} \stackrel{\text{def}}{=} \underline{A}_{ij}$ if $y_i \cdot z_j = 1$.

(these matrices were first introduced in [3], p. 43). Each such matrix is a vertex matrix, but there are only 2^{2n-1} matrices A_{yz} compared to 2^{n^2} vertex matrices ($2n-1$ since $A_{yz} = A_{-y,-z}$). For some problems, it is sufficient to check only some of such matrices, e.g., only matrices A_{yy} or only matrices $A_{y,-y}$ (in both cases, we need only 2^{n-1} vertex matrices).

For such problems, a natural question is: can we further decrease the set of checked matrices? In this paper, we show that for most problems described in [3, 6], further decrease is impossible: all 2^{2n-1} (corr., 2^{n-1}) vertex matrices A_{yz} (corr., A_{yy}) are needed. To be more precise: there exist cases when the property $\mathcal{P}$ holds for all but one of these matrices and still does not hold for the corresponding interval matrix $\mathbf{A}$. In this sense, finite characterizations presented in [3, 6] are optimal.

These results are in good accordance with the fact that many of the corresponding problems are NP-hard (see, e.g., [2]) and therefore, less than exponential finite characterizations are not to be expected.

Comment. The fact that a exponential $\approx 2^n$ finite characterization cannot be decreased is not as pessimistic as it may seem:

- First, NP-hardness means that we cannot expect less than exponential-time algorithms for solving the corresponding problems. Of course, this does not necessarily mean that the algorithms based on checking all 2^{n-1} vertex matrices are necessarily optimal; we may have faster – although still exponential-time – algorithms based on different ideas.

- Second, the fact that we need to check all 2^{n-1} matrices does not necessarily mean that the computation time of the corresponding algorithm for checking the property $\mathcal{P}$ for an interval matrix is 2^{n-1} times larger than the computation time t of checking this property for a single matrix. For some properties, it was shown that many of these 2^{n-1} checkings contain the exact same computational steps; so, when we need to check all these matrices, we can perform the common steps only once. As a result, the total computational time for all the checkings is much smaller than $2^{n-1} \times t$ [7].

2 Regularity

Definition 2.1. *A square interval matrix* $\mathbf{A}$ *is called regular if each* $A \in \mathbf{A}$ *is regular.*

The problem of checking whether a given interval matrix is regular is known to be NP-hard (see, e.g., [2]).

Theorem 2.1. [1, 3] $\mathbf{A}$ *is regular if and only if for all the matrices* A_{yz}*, the determinant* det A_{yz} *has the same sign.*

The following result shows that all 2^{2n-1} different matrices A_{yz} are needed for this characterization:

Theorem 2.2. *For every* n*, and for every pair* $\langle \widetilde{y}, \widetilde{z} \rangle$*,* $\widetilde{y}, \widetilde{z} \in Y$*, there exists an interval matrix* $\mathbf{A}$*, for which*

- *for all pairs* $\langle y, z \rangle \neq \langle \widetilde{y}, \widetilde{z} \rangle, \langle -\widetilde{y}, -\widetilde{z} \rangle$*, all the values* det A_{yz} *have the same sign;*
- $\mathbf{A}$ *is not regular.*

Proof. Let δ_{ij} denote components of a unit matrix I ($\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$). Let us consider the interval matrix with

$$\underline{A}_{ij} = 2n \cdot \delta_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j - \widetilde{y}_i \cdot \widetilde{z}_j - e_i \cdot e_j; \tag{2.1}$$

$$\overline{A}_{ij} = 2n \cdot \delta_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j - \widetilde{y}_i \cdot \widetilde{z}_j + e_i \cdot e_j. \tag{2.2}$$

For this interval matrix, for every $y, z \in Y$, we have

$$(A_{yz})_{ij} = 2n \cdot \delta_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j - \widetilde{y}_i \cdot \widetilde{z}_j - y_i \cdot z_j. \tag{2.3}$$

Let us show that for all pairs $\langle y, z \rangle \neq \langle \widetilde{y}, \widetilde{z} \rangle$ and $\langle y, z \rangle \neq \langle -\widetilde{y}, -\widetilde{z} \rangle$, the determinants of the matrices A_{yz} have exactly the same sign, and that det $A_{\widetilde{y}\widetilde{z}} = 0$.

To prove this, let us first slightly simplify the computations by noticing that for every $y, z \in Y$, the matrix A_{yz} can be represented as

$$(A_{yz})_{ij} = (B_{pq})_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j, \tag{2.4}$$

where

$$(B_{pq})_{ij} = 2n \cdot \delta_{ij} - e_i \cdot e_j - p_i \cdot q_j; \tag{2.5}$$

$$p_i = y_i \cdot \widetilde{y}_i; \quad q_i = z_i \cdot \widetilde{z}_i. \tag{2.6}$$

In short, to get from B_{pq} to A_{yz}, we multiply each i-th row by $\widetilde{y}_i$, and each j-th column by $\widetilde{z}_j$. In particular, the matrix $A_{\widetilde{y}\widetilde{z}}$ corresponds to $p = q = e$.

By definition, the determinant of an $n \times n$ matrix is a linear combination of the n-factor products, each of which contain exactly one component from each row and exactly one component from each column. Thus, when we substitute the expression (2.4) into the formula for det A_{yz}, we conclude that

$$\det A_{yz} = \det B_{pq} \cdot \widetilde{Y} \cdot \widetilde{Z}, \tag{2.6}$$

where

$$\widetilde{Y} = \prod_i \widetilde{y}_i; \quad \widetilde{Z} = \prod_j \widetilde{z}_j. \tag{2.7}$$

The values $\widetilde{Y}$ and $\widetilde{Z}$ are the products of $+/-1$'s, so each of them is equal to ± 1. Hence, to prove that all the matrices A_{yz}, $\langle y, z\rangle \neq \langle \widetilde{y}, \widetilde{z}\rangle, \langle -\widetilde{y}, -\widetilde{z}\rangle$, have the determinants of the same sign, it is sufficient to prove that all the matrices B_{pq}, $\langle p, q\rangle \neq \langle e, e\rangle, \langle -e, -e\rangle$, have the determinants of the same sign.

We will show that all these matrices B_{pq} are positive definite and therefore, they all have positive determinants. By definition, positive definiteness means that if $x = (x_1, \ldots, x_n) \neq 0$, then

$$Q \stackrel{\text{def}}{=} \sum_{ij} (B_{pq})_{ij} \cdot x_i \cdot x_j > 0. \tag{2.8}$$

Indeed, by definition (2.5) of the matrix B_{pq}, we have

$$Q = 2n \cdot \sum_i (x_i)^2 - \left(\sum_i e_i \cdot x_i\right) \cdot \left(\sum_j e_j \cdot x_j\right) - \left(\sum_i p_i \cdot x_i\right) \cdot \left(\sum_j q_j \cdot x_j\right) =$$

$$2n \cdot \|x\|^2 - (e, x)^2 - (p, x) \cdot (q, x), \tag{2.9}$$

where (a, b) denotes a scalar (dot) product of the two vectors. For the scalar product, we have a known inequality $|(e, x)| \leq \|e\| \cdot \|x\|$, in which the equality is possible only if vectors e and x are collinear: $e \parallel x$. Here, $e = (1, \ldots, 1)^T$, so $\|e\| = \sqrt{n}$, $|(e, x)| \leq \sqrt{n} \cdot \|x\|$, and

$$(e, x)^2 \leq n \cdot \|x\|^2, \tag{2.10}$$

and the equality is possible only if $x \parallel e$.

Similarly, $|(p, x)| \leq \sqrt{n} \cdot \|x\|$ and $|(q, x)| \leq \sqrt{n} \cdot \|x\|$. Hence,

$$(p, x) \cdot (q, x) \leq n \cdot \|x\|^2, \tag{2.11}$$

and the equality is possible only if $p \parallel x$, $q \parallel x$, and both p and q are on the same side as x (else we would have $(p,x)\cdot(q,x) = -n\cdot\|x\|^2$). Substituting (2.10) and (2.11) into (2.9), we conclude that

$$Q \ge 2n\|x\|^2 - n\|x\|^2 - n\|x\|^2 = 0, \tag{2.12}$$

and the equality is possible only when $x \parallel e$, $x \parallel p$, and $x \parallel q$ (hence $p \parallel e$ and $q \parallel e$), and p and q are on the same side of e. Since $p, q \in Y$, the only possibility for equality is, hence, when either $p = q = e$, or $p = q = -e$. So, for all other pairs, the equality is impossible, and the matrix B_{pq} is indeed positive definite.

To complete the proof, we will show that $\det A_{\tilde{y}\tilde{z}} = 0$. As we have mentioned, this is equivalent to showing that $\det B_{ee} = 0$. Indeed, due to formula (2.9), we have

$$Q \stackrel{\text{def}}{=} \sum_{ij} (B_{ee})_{ij} \cdot x_i \cdot x_j = 2n \cdot \|x\|^2 - 2(e,x)^2. \tag{2.13}$$

If we select an orthonormal basis in which $e^{(1)} = e/\|e\| = e/\sqrt{n}$, then, in this basis, we have $(e,x) = \sqrt{n}\cdot x_1$, hence, the formula (2.13) leads to

$$Q = 2n\cdot x_1^2 + 2n\cdot x_2^2 + \ldots + 2n\cdot x_n^2 - 2n\cdot x_1^2 = 2n\cdot x_2^2 + \ldots + 2n\cdot x_n^2. \tag{2.14}$$

In other words, in this basis, the symmetric matrix B_{ee} becomes diagonal, with one of the eigenvalues 0, hence its determinant is 0. Thus, due to Theorem 2.1, $\mathbf{A}$ is not a regular matrix. Q.E.D.

3 Positive (semi)definiteness

Definition 3.1. *A square interval matrix* $\mathbf{A}$ *is called positive (semi)definite if each* $A \in \mathbf{A}$ *is positive (semi)definite.*

The problems of checking whether a given interval matrix is positive definite or positive semidefinite are known to be NP-hard (see, e.g., [2]).

Theorem 3.1. [5] $\mathbf{A}$ *is positive (semi)definite if and only if* $(A_{yy} + A_{yy}^T)/2$ *is positive (semi)definite for each* $y \in Y$.

The following result shows that all 2^{n-1} different matrices A_{yy} are needed for this characterization:

Theorem 3.2. *For every* n, *and for every* $\tilde{y} \in Y$, *there exists an interval matrix* $\mathbf{A}$, *for which*

- *the matrix* $(A_{yy} + A_{yy}^T)/2$ *is positive (semi)definite for all* $y \ne \tilde{y}, -\tilde{y}$, *and*
- $\mathbf{A}$ *is not positive (semi)definite.*

Proof. Let us first prove this result for positive definiteness. For this, we will consider the following interval matrix:

$$\underline{A}_{ij} = 2n \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j - e_i \cdot e_j; \tag{3.1}$$

$$\overline{A}_{ij} = 2n \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j + e_i \cdot e_j. \tag{3.2}$$

For this interval matrix, for every $y \in Y$, we have

$$(A_{yy})_{ij} = 2n \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j - y_i \cdot y_j. \tag{3.3}$$

This is a symmetric matrix, so $A_{yy} = A_{yy}^T$ and $(A_{yy} + A_{yy}^T)/2 = A_{yy}$.

Similarly to the proof of positive definiteness of a matrix B_{pq} in the proof of Theorem 2.1, we can show that for all $y \neq \widetilde{y}, -\widetilde{y}$, the matrix A_{yy} is positive definite, while for $y = \widetilde{y}$, it is only positive semi-definite and not positive definite. Thus, for positive definiteness, the theorem is proven.

To prove a similar result for positive semi-definiteness, we consider an interval matrix

$$\underline{B}_{ij} = (2n - \varepsilon) \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j - e_i \cdot e_j; \tag{3.4}$$

$$\overline{B}_{ij} = (2n - \varepsilon) \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j + e_i \cdot e_j, \tag{3.5}$$

for some small $\varepsilon > 0$. For this interval matrix, for every $y \in Y$, we have

$$(B_{yy})_{ij} = (2n - \varepsilon) \cdot \delta_{ij} - \widetilde{y}_i \cdot \widetilde{y}_j - y_i \cdot y_j. \tag{3.3}$$

Since all the matrices A_{yy} for $y \neq \widetilde{y}, -\widetilde{y}$ were positive definite, for sufficiently small ε, the new matrices $B_{yy} = A_{yy} - \varepsilon \cdot I$ are still positive definite. On the other hand, since the matrix $A_{\widetilde{y}\widetilde{y}}$ was positive semi-definite, with one of the eigenvalues 0, the new matrix $B_{\widetilde{y}\widetilde{y}} = A_{\widetilde{y}\widetilde{y}}$ has a negative eigenvalue $-\varepsilon$ and hence, is not positive semi-definite. So, for positive semi-definiteness, the theorem is also proven. Q.E.D.

4 Stability

Definition 4.1. *A square symmetric interval matrix* $\mathbf{A}$ *(i.e., both* $\underline{A}$*,* $\overline{A}$ *symmetric) is called stable if each* $A \in \mathbf{A}$ *is stable, i.e.,* $\mathrm{Re}\,\lambda < 0$ *for each eigenvalue* λ *of* A*.*

The problem of checking whether a given interval matrix is stable is known to be NP-hard (see, e.g., [2]).

Theorem 4.1. [5] $\mathbf{A}$ *is stable if and only if* $A_{y,-y}$ *is stable for each* $y \in Y$*.*

The following result shows that all 2^{n-1} different matrices $A_{y,-y}$ are needed for this characterization:

Theorem 4.2. *For every n, and for every $\widetilde{y} \in Y$, there exists an interval matrix* $\mathbf{A}$*, for which*

- *the matrix $A_{y,-y}$ is stable for all $y \neq \widetilde{y}, -\widetilde{y}$, and*
- $\mathbf{A}$ *is not stable.*

Proof. As the desired interval matrix, let us take the interval matrix which is equal to minus the interval matrix (3.1), (3.2), i.e., the matrix

$$\underline{A}_{ij} = -2n \cdot \delta_{ij} + \widetilde{y}_i \cdot \widetilde{y}_j - e_i \cdot e_j; \tag{4.1}$$

$$\overline{A}_{ij} = -2n \cdot \delta_{ij} + \widetilde{y}_i \cdot \widetilde{y}_j + e_i \cdot e_j. \tag{4.2}$$

For this interval matrix, for every $y \in Y$, we have

$$(A_{y,-y})_{ij} = -2n \cdot \delta_{ij} + \widetilde{y}_i \cdot \widetilde{y}_j + y_i \cdot y_j. \tag{4.3}$$

Similarly to the proof of positive definiteness of a matrix B_{pq} in the proof of Theorem 2.1, we can show that:

- for all $y \neq \widetilde{y}, -\widetilde{y}$, the symmetric matrix $A_{y,-y}$ is negative definite, hence stable, while
- for $y = \widetilde{y}$, the corresponding matrix has a 0 eigenvalue and is, hence, not stable.

Q.E.D.

5 Linear interval equations

Definition 5.1. *For an interval matrix* $\mathbf{A}$ *and an interval vector* $\mathbf{b}$*, we define* $[\underline{x}, \overline{x}]$ *as the interval hull of the solution set*

$$X = \{x : Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}.$$

The problem of computing this interval hull is known to be NP-hard (see, e.g., [2]).

This interval hull can be characterized in terms of the matrices A_{yz} and vectors b_y, which are defined, for every $y \in Y$, as follows:

- $(b_y)_i = \overline{b}_i$ if $y_i = 1$, and
- $(b_y)_i = \underline{b}_i$ if $y_i = -1$.

Theorem 5.1. [3] *If* $\mathbf{A}$ *is regular, then we have:*

$$\underline{x} = \min_{y,z\in Y} A_{yz}^{-1} b_y; \quad \overline{x} = \max_{y,z\in Y} A_{yz}^{-1} b_y.$$

The following result shows that all 2^{2n} different pairs $\langle y, z\rangle$ are needed for this characterization:

Theorem 5.2. *For every* n*, and for every pair* $\langle \widetilde{y}, \widetilde{z}\rangle$*,* $\widetilde{y}, \widetilde{z} \in Y$*, there exists a regular interval matrix* $\mathbf{A}$ *and an interval vector* $\mathbf{b}$*, for which either*

$$\underline{x} \neq \min_{\langle y,z\rangle \neq \langle \tilde{y},\tilde{z}\rangle} A_{yz}^{-1} b_y$$

or

$$\overline{x} \neq \max_{\langle y,z\rangle \neq \langle \tilde{y},\tilde{z}\rangle} A_{yz}^{-1} b_y.$$

Proof. Let us first show that such a pair exists for $\widetilde{y} = \widetilde{z} = e$. Indeed, in this case, we can pick a positive number $\varepsilon > 0$ and take the following interval matrix:

$$\underline{A}_{ij} = (2n+\varepsilon)\cdot\delta_{ij} - e_i\cdot e_j - e_i\cdot e_j; \tag{5.1}$$

$$\overline{A}_{ij} = (2n+\varepsilon)\cdot\delta_{ij} - e_i\cdot e_j + e_i\cdot e_j, \tag{5.2}$$

and the interval vector

$$\underline{b}_i = -e_i, \quad \overline{b}_i = e_i. \tag{5.3}$$

For this choice, for every $y, z \in Y$, we have $b_y = y$ and

$$(A_{yz})_{ij} = (2n+\varepsilon)\cdot\delta_{ij} - e_i\cdot e_j - y_i\cdot z_j. \tag{5.4}$$

In the proof of Theorem 2.1, we have shown that for $\varepsilon = 0$, this interval matrix is semi-definite, hence, when we add $\varepsilon \cdot I$, we get a positive definite interval matrix – which is thus regular.

For $y = z = e$, the vector $x = A_{ee}^{-1} b_e$ is a solution to the linear system $A_{ee}x = e$, i.e., to the system:

$$(2n+\varepsilon)\cdot x_i - 2(x,e) = 1, \tag{5.5}$$

where $(x,e) = \sum x_i \cdot e_i = \sum x_i$. Moving the term $2(x,e)$ to the right-hand side and dividing both sides by $2n+\varepsilon$, we conclude that

$$x_i = \frac{1+2(x,e)}{2n+\varepsilon}. \tag{5.6}$$

The right-hand side of this formula does not depend on i, so $x_1 = \ldots = x_n =$ const. Thus, $(x,e) = n\cdot x_i$, and the equation (5.5) leads to

$$(2n+\varepsilon)\cdot x_i - 2n\cdot x_i = \varepsilon\cdot x_i = 1, \tag{5.7}$$

i.e., to

$$x_i = \frac{1}{\varepsilon}. \tag{5.8}$$

Let us show that for every pair $\langle y, z\rangle \neq \langle e, e\rangle$, the vector $x = A_{yz}^{-1} b_y$ has smaller component values. Indeed, this vector is a solution to the linear system $A_{yz}x = b_y = y$, i.e., to the system:

$$(2n+\varepsilon)\cdot x_i - (x,e) - (x,z)\cdot y_i = y_i, \tag{5.9}$$

Moving the term $2(x,e)$ to the right-hand side and dividing both sides by $2n+\varepsilon$, we conclude that

$$x_i = \frac{y_i + (x,e) + (x,z)\cdot y_i}{2n+\varepsilon}. \tag{5.10}$$

By definition, $(x,e) = \sum x_i$, hence, $|(x,e)| \le \sum |x_i|$; the equality is attained only in two cases:

- if every component of x_i is non-negative (i.e., has the same sign as e_i), or
- if every component of x_i is non-positive (i.e., has the same sign as $-e_i$).

Similarly, $|(x,z)| \le \sum |x_i|$, and the equality happens only is attained only in two cases:

- if every component of x_i has the same sign as z_i, or
- if every component of x_i has the same sign as $-z_i$.

Thus,

$$|y_i + (x,e) + y_i\cdot(x,z)| \le 1 + 2\cdot\sum |x_i|, \tag{5.11}$$

with the equality possible only if all the values y_i have the same sign, same as (x,e), and all the values (x,z) are positive (hence, all the components of x_i and e_i have the same sign, and so do x_i and z_i). Applying the inequality (5.11) to the formula (5.10), we conclude that

$$|x_i| \le \frac{1+2\cdot\sum|x_i|}{2n+\varepsilon.} \tag{5.12}$$

Adding these inequalities for $i = 1, \ldots, n$, we conclude that

$$\sum |x_i| \le \frac{n}{2n+\varepsilon}\cdot\left(1+2\cdot\sum|x_i|\right) = \frac{n}{2n+\varepsilon} + \frac{2n}{2n+\varepsilon}\cdot\sum|x_i|, \tag{5.13}$$

hence

$$\left(1 - \frac{2n}{2n+\varepsilon}\right)\cdot\sum|x_i| \le \frac{n}{2n+\varepsilon}, \tag{5.14}$$

$$\frac{\varepsilon}{2n+\varepsilon}\cdot\sum|x_i| \le \frac{n}{2n+\varepsilon} \tag{5.15}$$

and

$$\sum |x_i| \le \frac{n}{\varepsilon}. \tag{5.16}$$

From (5.12) and (5.16), we can now conclude that

$$|x_i| \le \frac{1 + 2n/\varepsilon}{2n + \varepsilon} = \frac{1}{\varepsilon}, \tag{5.17}$$

hence

$$x_i \le \frac{1}{\varepsilon}, \tag{5.18}$$

and the equality is only possible if all the components of the vectors e, y, and z have the same signs, i.e., if $e = y = z$.

Thus, the maximum in x_i is attained only for $\langle y, z\rangle = \langle e, e\rangle$, and so, if we omit this pair, we do not get the correct interval hull of the solution of the system of linear equations. Thus, for the case when $\widetilde{y} = \widetilde{z} = e$, the theorem is proven.

In the general case, we can repeat the same proof for

$$\underline{A}_{ij} = (2n + \varepsilon) \cdot \delta_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j - \widetilde{y}_i \cdot \widetilde{z}_j - e_i \cdot e_j; \tag{5.19}$$

$$\overline{A}_{ij} = (2n + \varepsilon) \cdot \delta_{ij} \cdot \widetilde{y}_i \cdot \widetilde{z}_j - \widetilde{y}_i \cdot \widetilde{z}_j + e_i \cdot e_j, \tag{5.20}$$

and the same interval vector (5.3). Q.E.D.

6 Inverse interval matrix

Definition 6.1. *For a regular* $\mathbf{A}$*, we define* $[\underline{B}, \overline{B}]$ *as the interval hull of the set* $\{A^{-1} : A \in \mathbf{A}\}$.

The problem of computing this interval hull is known to be NP-hard (see, e.g., [2]).

Theorem 6.1. [4] *For a regular* $\mathbf{A}$*, we have*

$$\underline{B} = \min_{y,z \in Y} A_{yz}^{-1}; \quad \overline{B} = \max_{y,z \in Y} A_{yz}^{-1}.$$

The following result shows that all 2^{2n-1} different matrices A_{yz} are needed for this characterization:

Theorem 6.2. *For every n, and for every pair $\langle\widetilde{y},\widetilde{z}\rangle$, $\widetilde{y},\widetilde{z}\in Y$, there exist:*

- *a regular interval matrix* $\mathbf{A}$ *for which*

$$\underline{B}\neq \min_{\langle y,z\rangle\neq\langle\tilde{y},\tilde{z}\rangle,\langle-\tilde{y},-\tilde{z}\rangle} A_{yz}^{-1};$$

- *a regular interval matrix* $\mathbf{A}$ *for which*

$$\overline{B}\neq \max_{\langle y,z\rangle\neq\langle\tilde{y},\tilde{z}\rangle,\langle-\tilde{y},-\tilde{z}\rangle} A_{yz}^{-1}.$$

Proof. In this proof, we can take the same interval matrix (5.1), (5.2) (corr., (5.19), (5.20)) as in the proof of Theorem 5.2. For $\widetilde{y}=\widetilde{z}=e$, the inverse matrix A_{ee}^{-1} to $(A_{ee})_{ij}=(2n+\varepsilon)\cdot\delta_{ij}-2e_i\cdot e_j$ is easy to compute: due to symmetry, it also has to have a similar form $C_1\cdot\delta_{jk}+C_2\cdot e_j\cdot e_k$; multiplying the two matrices and equating the result with the unit matrix, we conclude that $C_1=\dfrac{1}{2n+\varepsilon}$ and $C_2=\frac{2}{\varepsilon}\cdot C_1$, hence:

$$(A_{ee}^{-1})_{jk}=\frac{1}{2n+\varepsilon}\delta_{jk}+\frac{2}{\varepsilon\cdot(2n+\varepsilon)}\cdot e_j\cdot e_k, \tag{6.1}$$

i.e.,

$$(A_{ee}^{-1})_{jj}=\frac{2+\varepsilon}{\varepsilon\cdot(2n+\varepsilon)};\quad (A_{ee}^{-1})_{jk}=\frac{2}{\varepsilon\cdot(2n+\varepsilon)}\text{ for } j\neq k. \tag{6.2}$$

It can be shown that such high values cannot be achieved for any other matrix A_{yz}. Indeed, e.g., the first row of the inverse matrix A_{yz}^{-1} is a solution x to the linear system

$$\sum_j((2n+\varepsilon)\cdot\delta_{ij}-e_i\cdot e_j-y_i\cdot z_j)\cdot x_j=\delta_{1i}, \tag{6.3}$$

i.e.,

$$\begin{aligned}(2n+\varepsilon)\cdot x_1-(e\cdot x)-y_1\cdot(x,z)&=1;\\ (2n+\varepsilon)\cdot x_2-(e\cdot x)-y_2\cdot(x,z)&=0;\\ &\dots\\ (2n+\varepsilon)\cdot x_n-(e\cdot x)-y_n\cdot(x,z)&=0.\end{aligned} \tag{6.4}$$

From these equations, we can get (similarly to the proof of Theorem 5.2) estimates on x_i, hence on $\sum x_i=(e,x)$ etc., and thus show that these components cannot be as high as (6.1), (6.2).

For $\widetilde{y}\neq e$ and $\widetilde{z}\neq e$, the proof is similar. Q.E.D.

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References

[1] M. Baumann, "A regularity criterion for interval matrices", In: J. Garloff et al., eds., *Collection of Scientific Papers Honoring Prof. Dr. Karl Nickel on Occasion of his 60th Birthday, Part I*, Frieburg University, Freiburg, 1984, pp. 45–50.

[2] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational complexity and feasibility of data processing and interval computations*, Kluwer, Dordrecht, 1998.

[3] J. Rohn, "Systems of linear interval equations", *Linear Algebra and Its Applications*, 1989, Vol. 126, pp. 39–78.

[4] J. Rohn, "Inverse interval matrix", *SIAM Journal on Numerical Analysis*, 1993, Vol. 30, pp. 864–870.

[5] J. Rohn, "Positive definiteness and stability of interval matrices", *SIAM Journal on Matrix Algebra and Applications*, 1994, Vol. 15, pp. 175–184.

[6] J. Rohn, "Finite Characterization of Some Linear Problems with Inexact Data", *Abstracts of SCAN'2000/Interval'2000*, Karlsruhe, Germany, September 19–22, 2000, p. 32.

[7] M. Tsatsomeros and L. Li, "A Recursive Test for P-Matrices", *BIT Numerical Mathematics*, 2000, Vol. 40, No. 2, pp. 404–408.