Why Unary and Binary Operations in Logic: General Result Motivated by Interval-Valued Logics

Hung T. Nguyen
Mathem. Sciences, New Mexico State Univ.
Las Cruces, NM 88003, USA
hunguyen@nmsu.edu

Vladik Kreinovich
Comp. Science, Univ. of Texas at El Paso
El Paso, TX 79968, USA
vladik@cs.utep.edu

I.R. Goodman
Code D42215, Space & Naval Warfare Systems Center
San Diego, CA 92152, USA
goodman@spawar.navy.mil

Abstract

Traditionally, in logic, only unary and binary operations are used as basic ones – e.g., “not”, “and”, “or” – while the only ternary (and higher order) operations are the operations which come from a combination of unary and binary ones. For the classical logic, with the binary set of truth values $\{0, 1\}$, the possibility to express an arbitrary operation in terms of unary and binary ones is well known: it follows, e.g., from the well known possibility to express an arbitrary operation in DNF form. A similar representation result for $[0, 1]$-based logic was proven in our previous paper. In this paper, we expand this result to finite logics (more general than classical logic) and to multi-D analogues of the fuzzy logic – both motivated by interval-valued fuzzy logics.

1. Introduction

Traditionally, in logic, only unary and binary operations are used as basic ones – e.g., “not”, “and”, “or” – while the only ternary (and higher order) operations are the operations which come from a combination of unary and binary ones.

A natural question is: are such combinations sufficient? I.e., to be more precise, *can an arbitrary logical operation be represented as a combination of unary and binary ones?*

For the classical logic, with the binary set of truth values $V = \{0, 1\}$ (=$\{\text{false}, \text{true}\}$), the positive answer to this question is well known. Indeed, it is known that an arbitrary logical operation $f : V^n \to V$ can be represented, e.g., in DNF form and thus, it can indeed be represented as a combination of unary (“not”) and binary (“and” and “or”) operations.

We are interested in explaining why unary and binary logical operations are the only basic ones. If we assume that the logic of human reasoning is the two-valued (classical) logic, then the possibility to transform every logical function into a DNF form explains this empirical fact.

However, classical logic is not a perfect description of human reasoning: for example, it does not take into consideration *fuzziness* and uncertainty of human reasoning. This uncertainty is taken into consideration in *fuzzy logic* [9, 25, 29]. In the traditional fuzzy logic, the set of truth values is the entire interval $V = [0, 1]$. This interval has a natural notion of continuity, so it is natural to restrict ourselves to *continuous* unary and binary operations.

With this restriction in place, a natural question is: *can an arbitrary continuous function $f : [0, 1]^n \to [0, 1]$ be represented as a composition of continuous unary and binary operations?* The positive answer to this question was obtained in our papers [19, 22].

In $[0, 1]$-based fuzzy logic, an arbitrary logical operation can be represented as a composition of unary and binary ones. However, the $[0, 1]$-based fuzzy logic is, by itself, only an approximation to the actual human reasoning about uncertainty.

Indeed, how can we describe the expert’s degree of confidence $d(S)$ in a certain statement S? A natural way to determine this degree is, e.g., to ask an expert to estimate his degree of confidence on a scale from 0 to 10. If he selects 8, then we take $d(S) = 8/10$.

To get a more accurate result, we can then ask the same expert to estimate his degree of confidence on a finer scale, e.g., from 0 to 100, etc. For example, if an expert selects 81, we will take $d(S) = 81/100 = 0.81$. If we want an even more accurate estimate, we can ask the expert to estimate his degree of confidence on an even finer scale, etc.

The problem with this approach is that experts cannot describe their degrees of too fine scales. For example, an expert can point to 8 on a scale from 0 to 10, but this same expert will hardly be able to pinpoint a value on a scale from 0 to 100.

So, to attain a more adequate description of human reasoning, we must modify the traditional $[0,1]$-based fuzzy logic. Two types of modifications have been proposed.

One possibility is to take the finest (finite) scale which an expert can still use, and take the values on this scale as the desired degrees of confidence. This approach leads to a *finite-valued* fuzzy logic, in which the set of truth values V is finite.

This approach has been successfully used in practice; see, e.g., [1, 5, 20, 26]. It is therefore desirable to check whether in a finite logic, every operation can be represented as a composition of unary and binary operations.

The problem with finite-valued logics is that the set V of resulting truth values depends on which scale we use.

Instead of fixing a finite set, we can describe the expert's degree of confidence by an *interval* from $[0,1]$. For example, if an expert estimates his degree of confidence by a value 8 on a 0 to 10 scale, then the only thing that we know about the expert's degree of confidence is that it is closer to 0.8 (8/10) than to 0.7 or to 0.9, i.e., that it belongs to the interval $[0.75, 0.85]$.

So, a natural way of describing degrees of confidence more adequately is to use *intervals* $\mathbf{a} = [a^-, a^+]$ instead of real numbers. In this representation, real numbers can be viewed as particular – degenerate – cases of intervals $[a, a]$. The idea of using intervals have been originally proposed by Zadeh himself and further developed by Bandler and Kohout [2], Türkşen [27], and others; for a recent survey, see, e.g., [24].

In interval-valued fuzzy approach, to describe each degree of confidence, we must describe two real numbers: the lower endpoint and the upper endpoint of the corresponding "confidence interval".

We can go one step further and take into consideration that the endpoints of the corresponding interval are also not precisely known. Thus, each of these endpoints is, in actuality, an interval itself. So, to describe a degree of confidence, we now need *four* real numbers: two to describe the lower endpoint, and two to describe the upper one.

In general, we get a *multi-D* fuzzy logic. A natural question is: can every (continuous) operation on a multi-D fuzzy logic be represented as a composition of (continuous) unary and and binary operations?

Uncertainty of expert estimates is only one reason why we may want to go beyond the traditional $[0,1]$-valued logic; there are also other reasons:

- A 1-D value is a reqsonable way of describing the uncertainty of a single expert. However, the confidence strongly depends on the it consensus between different experts. We may want to use additional dimensions to describe how many expert share the original expert's opinion, and to what degree; see, e.g., [13, 23].
- Different experts may strongly disagree. To describe the dgeree of this disagreement, we also need additional numerical characteristics, which make the resulting logic multi-D; see, e.g., [21].

In all these cases, we need a multi-D logic to adequately describe expert's degree of confidence.

In this paper, we show that both for finite-valued logics and for multi-D logics, every logical operation can be represented as a composition of unary and binary operations. Thus, we give a general explanation for the above empirical fact.

2. Finite-Valued Logics

2.1. What Was Known Before

In the Introduction, we have already mentioned that for the 2-element set of truth values $V = \{0,1\}$, an arbitrary logical operation $f : V^n \to V$ can be represented as a composition of unary and binary operations. Specifically, in this case, an arbitrary logical operation can be represented as a composition of negation $a \to a'$, conjunction $\wedge$, and disjunction $\vee$.

In [6], we proved that the same is true for the case when V is a finite *Boolean algebra*. Specifically, we prove that for such sets V, an arbitrary logical operation can be represented as a composition of negation, conjunction ("intersection"), disjunction ("union"), constants, and a special unary operation called *absolute truth:*

Definition 1. *For an arbitrary Boolean algebra B, we define an absolute truth operation $t(a)$ as follows: $t(\mathbf{1}) = \mathbf{1}$ and $t(a) = \mathbf{0}$ for all $a \neq \mathbf{1}$.*

The function t is similar to the delta-function $\delta(x)$ (see, e.g., [30]), which is defined, crudely speaking, as a function which is different from 0 only at one point $x = 0$. It is even more similar to Kronecker's "delta" $\delta_{i,j}$ which is defined as $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ when $i \neq j$: we can easily see that $t(a) = \delta_{a,\mathbf{0}}$.

Proposition [6]. *For every finite Boolean algebra B, and for every positive integer n, every n-ary operation $f : B^n \to B$ can be represented as a composition of* $\vee$, $\wedge$, a', t, *and constants.*

This representation has the form

$$\bigvee \left[\mathrm{eq}(x_1, a_1) \wedge \ldots \wedge \mathrm{eq}(x_n, a_n) \wedge f(a_1, \ldots, a_n)\right],$$

where $\vee$ is taken over all tuples $(a_1, \ldots, a_n) \in B^n$, and $\mathrm{eq}(x, a)$ is defined as

$$\mathrm{eq}(x, a) \stackrel{\text{def}}{=} t((x \wedge a) \vee (x' \wedge a'))$$

and is equal to $\mathrm{eq}(x, a) = \mathbf{1}$ when $x = a$, and to $\mathrm{eq}(x, a) = \mathbf{0}$ when $x \neq a$.

This representation is similar to the known possibility to represent an arbitrary function as a linear combination of delta-functions, so-called "sifting property" ([30], pp. 11 and ff.):

$$f(x_1, \ldots, x_n) =$$

$$\int_{a_1} \ldots \int_{a_n} E(x_1, \ldots, x_n, a_1, \ldots, a_n)\, \mathrm{d}a_1 \ldots \mathrm{d}a_n,$$

where

$$E \stackrel{\text{def}}{=} \delta(x_1 - a_1) \cdot \ldots \cdot \delta(x_n - a_n) \cdot f(a_1, \ldots, a_n).$$

For the case when each variable x_i only takes integer value, we can have another analogue of the same result, with Kronecker's "delta" instead of a delta-function:

$$f(x_1, \ldots, x_n) =$$

$$\sum_{a_1} \ldots \sum_{a_n} \delta_{x_1, a_1} \cdot \ldots \cdot \delta_{x_n, a_n} \cdot f(a_1, \ldots, a_n).$$

2.2. New Result

Let us show that a similar representation can be used to prove this result for an arbitrary finite set V.

Theorem 1. *For every finite set V, and for every positive integer n, every n-ary operation $f : V^n \to V$ can be represented as a composition of unary and binary operations.*

(For reader's convenience, all the proofs are placed in the special Proofs section.)

3. Multi-D Logics

3.1. What Was Known Before

In the Introduction, we have already mentioned that for $V = [0, 1]$, an arbitrary continuous logical operation $f : V^n \to V$ can be represented as a composition of unary and binary operations. This result is based on the following known result:

Theorem (Kolmogorov). *Every continuous function of three or more variables can be represented as a composition of continuous functions of one or two variables.*

This result was proven by A. Kolmogorov [10] as a solution to the conjecture of Hilbert, formulated as the thirteenth problem [8]: one of 22 problems that Hilbert has proposed in 1900 as a challenge to the XX century mathematics.

This problem can be traced to the Babylonians, who found (see, e.g., [3]) that the solutions x of quadratic equations $ax^2 + bx + c = 0$ (viewed as function of three variables a, b, and c) can be represented as superpositions of functions of one and two variables, namely, arithmetic operations and square roots. Much later, similar results were obtained for functions of five variables a, b, c, d, e, that represent the solution of quartic equations $ax^4 + bx^3 + cx^2 + dx + e = 0$. But then, Galois proved in 1830 that for higher order equations, we cannot have such a representation. This negative result has caused Hilbert to conjecture that not all functions of several variables can be represented by functions of two or fewer variables. Hilbert's conjecture was refuted by Kolmogorov (see, e.g., [14], Chapter 11) and his student V. Arnold.

It is worth mentioning that Kolmogorov's result is not only of theoretical value: it was used to speed up actual computations (see, e.g., [4, 7, 11, 12, 16, 17]).

3.2. New Result

Let us show that one can generalize Kolmogorov's theorem and prove that a similar representation holds for multi-D logics as well.

Let m be a positive integer, and let V be a closure of a simply connected bounded open set in R^m (e.g., of a convex set). Such a set V will be called a *multi-D set of truth values*. For example, for interval-valued fuzzy sets,

$$V = \{(a, b) \mid 0 \leq a \leq b \leq 1\}.$$

Theorem 2. *For every multi-D set of truth values V, and for every positive integer n, every continuous n-ary operation $f : V^n \to V$ can be represented as a composition of continuous unary and binary operations.*

4. Proofs

Proof of Theorem 1. To describe the desired representation, let us pick two different elements from the set

V and denote them $\mathbf{0}$ and $\mathbf{1}$. We will then define three binary operations:

- $\text{eq}(a,b)$ is defined as $\text{eq}(a,b) \stackrel{\text{def}}{=} \mathbf{1}$ when $a = b$, and $\text{eq}(a,b) \stackrel{\text{def}}{=} \mathbf{0}$ when $a \neq b$.
- $\vee$ is defined in such a way that $\mathbf{0} \vee a = a \vee \mathbf{0} = a$ for all $a \in V$; when $a \neq \mathbf{0}$ and $b \neq \mathbf{0}$, we can take arbitrary values for $a \vee b$, e.g., we can assume that in this case, $a \vee b = \mathbf{1}$.
- $\wedge$ is defined in such a way that $\mathbf{1} \wedge a = a \wedge \mathbf{1} = a$ for all $a \in V$; when $a \neq \mathbf{1}$ and $b \neq \mathbf{1}$, we can take arbitrary values for $a \wedge b$, e.g., we can assume that in this case, $a \wedge b = \mathbf{0}$.

Now, it can easily checked that an arbitrary operation $f(x_1, \ldots, x_n)$ from V^n to V can be represented as

$$\bigvee \left[\text{eq}(x_1, a_1) \wedge \ldots \wedge \text{eq}(x_n, a_n) \wedge f(a_1, \ldots, a_n)\right],$$

where $\vee$ is taken over all tuples $(a_1, \ldots, a_n) \in V^n$. Q.E.D.

Proof of Theorem 2. This proof is similar to the one presented in [28].

First, since the set V is bounded, we can embed it into a box $A = [-\Delta, \Delta] \times \ldots \times [-\Delta, \Delta]$, i.e., a set of all points $s = (s_1, \ldots, s_m)$ for which $|s_1| \leq \Delta$, ..., $s_m \leq |\Delta|$. An arbitrary continuous function on a compact set V^n can be extended to the entire box A^n, so we can assume that f maps A^n into R^m.

Let us prove that every continuous function $f : A^n \to R^m$ of three or more variables can be represented as a composition of continuous unary and binary functions $g : A \to R^m$ and $h : A^2 \to R^m$.

Kolmogorov theorem proves that such a representation is possible for $m = 1$. Based on the case $m = 1$, we can now prove the theorem for all m, by using the following argument (its idea is similar to the one presented in [18]):

We have a function $s = f(s^{(1)}, \ldots, s^{(n)})$ of n variables

$$s^{(1)} = (s^{(1)}_1, \ldots, s^{(1)}_m) \in A,$$

$$\ldots,$$

$$s^{(n)} = (s^{(n}_1, \ldots, s^{(n}_m) \in A.$$

For each input $(s^{(1)}, \ldots, s^{(n)})$, the value

$$s = f(s^{(1)}, \ldots, s^{(n)})$$

of this function is a point

$$f(s^{(1)}, \ldots, s^{(n)}) =$$

$$(f_1(s^{(1)}, \ldots, s^{(n)}), \ldots, f_m(s^{(1)}, \ldots, s^{(n)}))$$

in the m-dimensional space, where by

$$f_i(s^{(1)}, \ldots, s^{(n)}),$$

we denoted the i-th component of the point $s = f(s^{(1)}, \ldots, s^{(n)})$. Therefore, each R^m-valued function $f : A^n \to S = R^m$ can be represented as m real-valued functions $f_i : A^n \to R$, $1 \leq i \leq m$.

Each of these functions $f_i : A^n \to R$ maps n elements from A (i.e., $n \times m$ components) into a real number. Therefore, each of these functions can be represented as a real-valued function of $n \times m$ *real* variables $s^{(1)}_1, \ldots, s^{(1)}_m, \ldots, s^{(n)}_1, \ldots, s^{(n)}_m$. Each of these m functions f_i can be (due to Kolmogorov's theorem) represented as a composition of functions of one and two variables. So, to represent the original function of n variables from A as a composition of functions of one or two variables from A, we can do the following:

First, we apply, to each input $s^{(j)} = (s^{(j)}_1, \ldots, s^{(j)}_m)$, m functions $\pi_1(s), \ldots, \pi_m(s)$ of one A-valued variable which transform an element $s = (s_1, \ldots, s_m) \in A$ into corresponding "degenerate" elements

$$\pi_1(s) \stackrel{\text{def}}{=} (s_1, \ldots, s_1), \ \ldots,$$

$$\pi_i(s) \stackrel{\text{def}}{=} (s_i, \ldots, s_i), \ \ldots,$$

$$\pi_m(s) \stackrel{\text{def}}{=} (s_m, \ldots, s_m).$$

When we apply these m functions to n input elements, we get $m \times n$ degenerate elements $\pi_i(s^{(j)}) = (s^{(j)}_i, \ldots, s^{(j)}_i)$, for all i from 1 to m and for all j from 1 to n.

Next, we follow the operations from Kolmogorov's theorem with these degenerate elements, and get the "degenerate"-valued functions

$$F_1(s^{(1)}, \ldots, s^{(n)}) \stackrel{\text{def}}{=}$$

$$(f_1(s^{(1)}, \ldots, s^{(n)}), \ldots, f_1(s^{(1)}, \ldots, s^{(n)})),$$

$$\ldots,$$

$$F_m(s^{(1)}, \ldots, s^{(n)}) \stackrel{\text{def}}{=}$$

$$(f_m(s^{(1)}, \ldots, s^{(n)}), \ldots, f_m(s^{(1)}, \ldots, s^{(n)})),$$

as the desired compositions of R^m-valued functions of one or two R^m-valued variables.

Finally, we use *combination functions* $C_2(s, s')$, ..., $C_m(s, s')$ to combine the functions $F_1, \ldots, F_m$ into a single function f. Namely, these functions work as follows:

$$C_2((s_1, \ldots), (s'_1, s'_2, \ldots, s'_m)) \stackrel{\text{def}}{=}$$

$$(s_1, s'_2, \ldots, s'_m),$$

$$\ldots,$$

$$C_j((s_1,\ldots,s_{j-1},s_j,\ldots),(s'_1,\ldots,s'_{j-1},s'_j,\ldots)) \stackrel{\rm def}{=} (s_1,\ldots,s_{j-1},s'_j,\ldots,s'_m),$$

$$\ldots,$$

$$C_m((s_1,\ldots,s_{m-1},s_m),(s'_1,\ldots,s'_{m-1},s'_m)) \stackrel{\rm def}{=} (s_1,\ldots,s_{m-1},s'_m).$$

We apply these combination functions to the values produced by the functions $F_1,\ldots,F_m$, to get the results $I_2 = C_2(F_1,F_2)$, $I_3 = C_3(F_2,I_2)$, …, $I_j = C_j(F_{j-1},I_j)$, … As a result, we get:

$$I_2 = C_2(F_1,F_2) = (f_1,f_2,\ldots,f_2),$$

$$I_3 = C_3(I_2,F_3) = (f_1,f_2,f_3,\ldots,f_3),$$

$$\ldots,$$

$$I_j = C_3(I_{j-1},F_j) = (f_1,\ldots,f_j,\ldots,f_j),$$

and finally,

$$I_m = C_m(I_{m-1},F_m) = (f_1,\ldots,f_m) = f(s^{(1)},\ldots,s^{(n)}).$$

Thus, the function $f(s^{(1)},\ldots,s^{(n)})$ has been represented as a composition of functions of one or two R^m-valued variables. The statement is proven.

This representation is not exactly what we want, because the corresponding unary and binary functions $g : A \to R^m$ and $h : A^2 \to R^m$ can take values outside the original set of truth values V. To complete the proof, we must therefore "compress" the corresponding functions – e.g., by applying appropriate linear transformation to each coordinate in their ranges. After we compressed all these functions, we get not the original function f, but the "compressed" one, so, to get f, we must apply "un-compression" (inverse linear transformation).

By definition, this "un-compression" U is a unary operation which transforms the original set V into a larger set, so to need to make U into a unary logical operation. Since the set V is simply connected, it is a *retract* of R^m (see, e.g., [15], Ch. 8), i.e., there exists a continuous mapping $r : R^m \to V$ for which $r(s) = s$ for all $s \in V$. Thus, as the desired unary logical operation, we can take $\widetilde{U}(s) \stackrel{\rm def}{=} r(U(s))$: the use of r does not change the result f which was already in V, but brings all the values outside V back into the set V. Q.E.D.

5. Conclusion

Traditionally, in logic, only unary and binary operations are used as basic ones. In traditional (2-valued) logic, the use of only unary and binary operations is justified by the known possibility to represent an arbitrary n-ary logical operation as a composition of unary and binary ones. A similar representation result is true for the $[0,1]$-based fuzzy logic. However, the $[0,1]$-based fuzzy logic is only an approximation to the actual human reasoning about uncertainty. A more accurate description of human reasoning requires that we take into consideration the uncertainty with which we know the values from the interval $[0,1]$. This additional uncertainty leads to two modifications of the $[0,1]$-based fuzzy logic: finite-valued logic and multi-D logic.

We show that for both modifications, an arbitrary n-ary logical operation can be represented as a composition of unary and binary ones. Thus, the above justification for using only unary and binary logical operation as basic ones is still valid if we take interval uncertainty into consideration.

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