# Towards More Realistic (e.g., Non-Associative) "and"- and "or"-Operations in Fuzzy Logic

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**Abstract**

How is fuzzy logic usually formalized? There are many seemingly reasonable requirements that a logic should satisfy: e.g., since $A\&B$ and $B\&A$ are the same, the corresponding and-operation should be commutative. Similarly, since $A\&A$ means the same as $A$, we should expect that the and-operation should also satisfy this property, etc. It turns out to be impossible to satisfy all these seemingly natural requirements, so usually, some requirements are picked as absolutely true (like commutativity or associativity), and others are ignored if they contradict to the picked ones.

This idea leads to a neat mathematical theory, but the analysis of real-life expert reasoning shows that all the requirements are only approximately satisfied. we should require all of these requirements to be satisfied to some extent. In this paper, we show the preliminary results of analyzing such operations. In particular, we show that non-associative operations explain the empirical $7 \pm 2$ law in psychology according to which a person can normally distinguish between no more than 7 plus minus 2 classes.

**Keywords:** Fuzzy logic, Non-associative operations

## 1 Introduction

In many application areas, there are tasks which take a lot of expert's time; example: interpreting the satellite photos. It is desirable to automate these time-consuming tasks.

One of the main obstacles to automating expert activity is the fact that experts often cannot express their activity in precise terms, they use vague (fuzzy) terms from natural language to describe it. For example, in satellite photo interpretation, an expert may follow a rule like "if an object is very small, it is probably a speckle unless a similar object appears on different photos of the same area"; here, "very small" and "similar" are examples of fuzzy terms from natural language.

To describe such fuzzy words, L. Zadeh proposed to use a special generalization of 2-valued logic called fuzzy logic, in which a statement, in addition to being absolutely true and absolutely false, can also take additional truth values corresponding to uncertainty. How is fuzzy logic usually formalized [5, 10])? There are many seemingly reasonable requirements that a logic should satisfy: e.g., since $A\&B$ and $B\&A$ are the same, the corresponding and-operation should be commutative. Similarly, since $A\&A$ means the same as $A$, we should expect that the and-operation should also satisfy this property, etc. It turns out to be impossible to satisfy all these seemingly natural requirements, so usually, some requirements are picked as absolutely true (like commutativity or associativity), and others are ignored if they contradict to the picked ones.

This idea leads to a neat mathematical theory, but the analysis of real-life expert reasoning shows that all the requirements are only approximately satisfied. Therefore, to achieve a more adequate representation of expert reasoning, instead of fixing some requirements as absolute and ignoring the others, we should require all of these requirements to be satisfied to some extent. In this paper, we show the preliminary results of analyzing such operations.

In particular, we show that non-associative operations explain the empirical $7 \pm 2$ law in psychology according to which a person can normally distinguish between no more than 7 plus minus 2 classes.

## 2 First Approach

If we know the degrees of certainty (subjective probabilities) $p(S_1)$ and $p(S_2)$ in two statements $S_1$ and $S_2$, then possible values of $p(S_1 \,\&\, S_2)$ form an interval

$$p = [\max(p_1 + p_2 - 1, 0), \min(p_1, p_2)].$$

As a numerical estimate, it is natural to use a midpoint of this interval:

$$p_1 \,\&\, p_2 \stackrel{\rm def}{=} \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2); \qquad (1)$$

Similar, for the "or"-operation, we can take the midpoint of the corresponding interval $[\max(p_1, p_2), \min(p_1 + p_2, 1)]$:

$$p_1 \vee p_2 \stackrel{\rm def}{=} \frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1). \qquad (2)$$

There is a problem with these operations. Indeed, any "and" operation $p_1 \,\&\, p_2$ enables us to produce an estimate for $P(S_1 \,\&\, S_2)$ provided that we know estimates $p_1$ for $p(S_1)$ and $p_2$ for $p(S_2)$. If we are interested in estimating the degree of belief in a conjunction of three statements $S_1 \,\&\, S_2 \,\&\, S_3$, then we can use the same operation twice:

- first, we apply the "and" operation to $p_1$ and $p_2$ and get an estimate $p_1 \,\&\, p_2$ for the probability of $S_1 \,\&\, S_2$;

- then, we apply the "and" operation to this estimate $p_1 \,\&\, p_2$ and $p_3$, and get an estimate $(p_1 \,\&\, p_2) \,\&\, p_3$ for the probability of $(S_1 \,\&\, S_2) \,\&\, S_3$.

Alternatively, we can get start by combining $S_2$ and $S_3$, and get an estimate $p_1 \,\&\, (p_2 \,\&\, p_3)$ for the same probability $p(S_1 \,\&\, S_2 \,\&\, S_3)$. Intuitively, we would expect these two estimates to coincide: $(p_1 \,\&\, p_2) \,\&\, p_3 = p_1 \,\&\, (p_2 \,\&\, p_3)$, i.e., in algebraic terms, we expect the operation $\&$ to be associative. Unfortunately, midpoint operations are *not* associative: e.g.,

$$(0.4 \,\&\, 0.6) \,\&\, 0.8 = 0.2 \,\&\, 0.8 = 0.1,$$

while

$$0.4 \,\&\, (0.6 \,\&\, 0.8) = 0.4 \,\&\, 0.5 = 0.2 \neq 0.1.$$

By itself, a small non-associativity may not be so bad:

- associativity comes from the requirement that our reasoning be rational, while
- it is well known that our actual handling of uncertainty is not exactly following rationality requirements; see, e.g., [12].

So, it is desirable to find out how non-associative can these operations be.

To be more precise, we know that the midpoint operations are non-associative, i.e., that sometimes, $(a \,\&\, b) \,\&\, c \neq a \,\&\, (b \,\&\, c)$. We want to know how big can the difference $(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)$ can be.

**Theorem 1** [3, 13]**.**

$$\max_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = \frac{1}{9}.$$

**Theorem 2** [3, 13]**.**

$$\max_{a,b,c} |(a \vee b) \vee c - a \vee (b \vee c)| = \frac{1}{9}.$$

Human experts do not use all the numbers from the interval $[0,1]$ to describe their possible degrees of belief; they use a few words like "very probable", "mildly probable", etc. Each of words is a "granule" covering the entire sub-interval of values. Since the largest possible non-associativity degree $|(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)|$ is equal to $1/9$, this non-associativity is negligible if the corresponding realistic "granular" degree of belief have granules of width $\geq 1/9$. One can fit no more than 9 granules of such width in the interval $[0,1]$. This may explain why humans are most comfortable with $\leq 9$ items to choose from – the famous "7 plus minus 2" law; see, e.g., [7, 8].

This general psychological law has also been confirmed in our specific area of formalizing expert knowledge: namely, in [2], it was shown that this law explains why in intelligent control, experts normally use $\leq 9$ different degrees (such as "small", "medium", etc.) to describe the value of each characteristic.

Instead of selecting a midpoint, we can make a more general selection of a value in the interval $\mathbf{p}$. By a *choice function*, we mean a function $s$ that maps every interval $\mathbf{u} = [u^-, u^+]$ into a point $s(\mathbf{u}) \in \mathbf{u}$ so that for every $c$ and $\lambda > 0$:

- $s([u^- + c, u^+ + c]) = s([u^-, u^+]) + c$ (*shift-invariance*);
- $s([\lambda \cdot u^-, \lambda \cdot u^+]) = \lambda \cdot s([u^-, u^+])$ (*unit-invariance*).

**Proposition 1** [9]. *Every choice function has the form* $s([u^-, u^+]) = \alpha \cdot u^- + (1-\alpha) \cdot u^+$ *for some* $\alpha \in [0, 1]$.

The combination $p = \alpha \cdot p^- + (1-\alpha) \cdot p^+$ (first proposed by Hurwicz [4]) has been successfully used in areas ranging from submarine detection to petroleum engineering [9]; in [16, 17], this approach is applied to second-order probabilities.

With this approach, we get the following formulas which generalize the above definitions:

$$p_1 \,\&\, p_2 \stackrel{\text{def}}{=} \alpha \cdot \max(p_1 + p_2 - 1, 0) + (1-\alpha) \cdot \min(p_1, p_2); \tag{3}$$

$$p_1 \vee p_2 \stackrel{\text{def}}{=} \alpha \cdot \max(p_1, p_2) + (1-\alpha) \cdot \min(p_1 + p_2, 1). \tag{4}$$

**Theorem 3** [3, 13].

$$\max_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = \frac{\alpha \cdot (1-\alpha)}{2 + \alpha \cdot (1-\alpha)};$$

$$\max_{a,b,c} |(a \vee b) \vee c - a \vee (b \vee c)| = \frac{\alpha \cdot (1-\alpha)}{2 + \alpha \cdot (1-\alpha)}.$$

*Comment.* This non-associativity degree is the smallest ($= 0$) when $\alpha = 0$ or $\alpha = 1$, and the largest ($= 1/9$) for midpoint operations ($\alpha = 0.5$).

*Comment.* In our proof, it was useful to first show that the new operations have *some* properties of associativity: namely, it turns out that for every $\alpha$, both operations are *semi-associative* in the sense that $a \leq b \leq c$ implies that $a \,\&\, (b \,\&\, c) \geq b \,\&\, (a \,\&\, c) \geq c \,\&\, (a \,\&\, b)$.

## 3 Second Approach

A t-norm $a \,\&\, b$ describes the degree to which two conditions $A$ and $B$ are both satisfied if we know that the first condition $A$ is satisfied with a degree $a$, and the second condition $B$ is satisfied with a degree $b$.

In effect, t-norms describe the situations when both conditions are absolutely necessary, so that if one of the conditions is not satisfied, we completely reject the corresponding alternative. There are many such situations, but there are also many other situations, in which, although we say that we want the first condition to be satisfied *and* the second condition to be satisfied, etc., but if one of these conditions is not satisfied, we may still consider the corresponding alternative.

For example, a computer science department may be looking for a person who is a brilliant researcher *and* a very good lecturer *and* is knowledgeable in all the areas of computer science, i.e., in data structures *and* in operating systems *and* in software engineering etc. Ideally, all these conditions should be met. However, if a brilliant researcher with a reputation of a good lecturer applies for a position, then, even if he does not know anything about operating systems, a department would most probably not definitely reject him.

In short, in many real-life situations, even if one of the conditions $A$, $B$ is not satisfied at all, e.g., if $a = 0$, we may still have some non-zero degree of belief in the conjunction $A\&B$ – in direct contrast to the fact that for a t-norm, in this case, $0 \,\&\, b = 0$. This difference between the formal notion of a t-norm and the human use of "and" was noticed several decades ago, in the experiments of H.-J. Zimmermann and P. Zysno described in [18]. To get a more adequate description of human "and"-operations, the authors of [18] propose to use, instead of t-norm, a *combination* (e.g., linear combination) of a t-norm and a t-conorm, e.g., to use a combination

$$p_1 \,\&\, p_2 \stackrel{\text{def}}{=} \alpha \cdot \min(p_1, p_2) + (1 - \alpha) \cdot \max(p_1, p_2). \tag{5}$$

Such a combination is also not associative. How non-associative can it be? To answer this question, we prove that it is *semi-associative*:

**Proposition 2.** *If $a \geq b \geq c$, then*

$$a \,\&\, (b \,\&\, c) \geq b \,\&\, (a \,\&\, c) \geq c \,\&\, (a \,\&\, b).$$

**Theorem 4.**

$$\max_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = \alpha \cdot (1 - \alpha).$$

*Comment.* These results were previously announced in [1, 6].

**Proof of Proposition 2 and Theorem 4.** Let $a \geq b \geq c$.

$1^\circ$. Let us first prove that

$$a \,\&\, (b \,\&\, c) \geq b \,\&\, (a \,\&\, c).$$

Indeed, in this case, $b \,\&\, c = (1 - \alpha) \cdot b + \alpha \cdot c$. Since $b \leq a$ and $c \leq a$, we can conclude that $b \,\&\, c \leq a$. Therefore,

$$a \,\&\, (b \,\&\, c) = (1 - \alpha) \cdot a + \alpha \cdot (b \,\&\, c) =$$

$$(1 - \alpha) \cdot a + \alpha \cdot (1 - \alpha) \cdot b + \alpha^2 \cdot c. \tag{6}$$

Similarly, $a \,\&\, c = (1-\alpha)\cdot a + \alpha \cdot c$. The expression for $b \,\&\, (a \,\&\, c)$ depends on whether $b \ge (a \,\&\, c)$ or not. Correspondingly, let us consider both cases.

1.1°. Let us first consider the case when

$$b \ge (a \,\&\, c).$$

In this case,

$$b \ge (1-\alpha)\cdot a + \alpha \cdot c, \tag{7}$$

hence

$$b \,\&\, (a \,\&\, c) = \alpha \cdot (a \,\&\, c) + (1-\alpha)\cdot b =$$
$$\alpha \cdot (1-\alpha)\cdot a + (1-\alpha)\cdot b + \alpha^2 \cdot c. \tag{8}$$

The difference between the expressions (6) and (8) is equal to $(1-\alpha)^2 \cdot (a-b)$, so this difference is non-negative. For this case, the desired inequality is proven.

1.2°. Let us now consider the case when

$$b < (a \,\&\, c).$$

In this case,

$$b < (1-\alpha)\cdot a + \alpha \cdot c, \tag{9}$$

hence

$$b \,\&\, (a \,\&\, c) = (1-\alpha) \cdot (a \,\&\, c) + \alpha \cdot b =$$
$$(1-\alpha)^2 \cdot a + \alpha \cdot b + \alpha \cdot (1-\alpha) \cdot c. \tag{10}$$

The difference between the expressions (6) and (10) is equal to:

$$\alpha \cdot (1-\alpha)\cdot a - \alpha^2 \cdot b + \alpha \cdot (2\alpha - 1)\cdot c = \alpha \cdot \sigma,$$

where by $\sigma$, we denoted the expression

$$(1-\alpha)\cdot a - \alpha \cdot b + (2\alpha-1)\cdot c.$$

Due to (9), we have

$$\alpha \cdot b < \alpha \cdot (1-\alpha)\cdot a + \alpha^2 \cdot c,$$

hence

$$\sigma > (1-\alpha)\cdot a - \alpha \cdot (1-\alpha)\cdot a - \alpha^2 \cdot c + (2\alpha - 1)\cdot c =$$
$$(1-\alpha)^2 \cdot a - (1-\alpha)^2 \cdot c = (1-\alpha)^2 \cdot (a-c).$$

Since $a \ge c$, we conclude that $\sigma \ge 0$, hence the difference between (6) and (10) is also non-negative. So, for this second case, the desired inequality is also proven.

2°. Let us now prove that

$$b \,\&\, (a \,\&\, c) \ge c \,\&\, (a \,\&\, b).$$

Since $a \geq b$, we have $a \,\&\, b = (1-\alpha)\cdot a + \alpha \cdot b$. From $a \geq c$ and $b \geq c$, we conclude that

$$a \,\&\, b = (1-\alpha)\cdot a + \alpha \cdot b \geq c.$$

Thus,

$$c \,\&\, (a \,\&\, b) = (1-\alpha)\cdot (a \,\&\, b) + \alpha \cdot c = \\ (1-\alpha)^2 \cdot a + \alpha \cdot (1-\alpha) \cdot b + \alpha \cdot c. \tag{11}$$

To prove the desired inequality, we consider the same two cases as in Part 1 of this proof.

2.1°. Let us first consider the case when

$$b \geq (a \,\&\, c).$$

In this case, $b \,\&\, (a \,\&\, c)$ is described by the expression (8). The difference between the expressions (8) and (11) is equal to

$$(1-\alpha)\cdot(2\alpha-1)\cdot a + (1-\alpha)^2 \cdot b - \alpha \cdot (1-\alpha) \cdot c = (1-\alpha)\cdot \sigma,$$

where by $\sigma$, we denoted the expression:

$$(2\alpha - 1)\cdot a + (1-\alpha)\cdot b - \alpha \cdot c.$$

Due to (7), we have

$$(1-\alpha)\cdot b \geq (1-\alpha)^2 \cdot a + \alpha \cdot (1-\alpha)\cdot c,$$

hence

$$\sigma \geq (2\alpha-1)\cdot a + (1-\alpha)^2 \cdot a + \alpha \cdot (1-\alpha)\cdot c - \alpha \cdot c = \\ \alpha^2 \cdot (a - c).$$

Since $a \geq c$, we conclude that $\sigma \geq 0$, hence the difference between (8) and (11) is also non-negative. So, for this case, the desired inequality is proven.

2.2°. Let us first consider the case when

$$b < (a \,\&\, c).$$

In this case, $b \,\&\, (a \,\&\, c)$ is described by the expression (10). The difference between the expressions (10) and (11) is equal to $\alpha^2 \cdot (b - c)$. Since $b \geq c$, this difference is non-negative, hence the desired inequality holds in this case too.

This completes the proof of Proposition 2.

3°. Let us now prove Theorem 4.

Since every three real numbers can be sorted in the order $a \geq b \geq c$, to prove Theorem 4, it is sufficient to consider all possible differences between the terms $a \,\&\, (b \,\&\, c)$, $b \,\&\, (a \,\&\, c)$, and $c \,\&\, (a \,\&\, b)$ that correspond to $a \geq b \geq c$.

Due to Proposition 2, the largest possible difference $d$ between these three terms is the difference between the expressions $a \,\&\, (b \,\&\, c)$ and $c \,\&\, (a \,\&\, b)$. The first expression is described by the formula (6), the second by the formula (11), thus, the difference between these expressions is equal to the difference between these formulas, i.e., to:

$$d = \alpha \cdot (1-\alpha) \cdot a - \alpha \cdot (1-\alpha) \cdot c = \alpha \cdot (1-\alpha) \cdot (a-c).$$

Since $a \geq c$, the difference $a - c$ can take values between 0 and 1, the largest value 1 attained when $a = 1$ and $c = 0$. Thus, $d \leq \alpha \cdot (1-\alpha)$, and $d = \alpha \cdot (1-\alpha)$ when $a = 1$ and $c = 0$. Hence, the desired maximum of the difference $d$ is indeed equal to $\alpha \cdot (1-\alpha)$. The theorem is proven.

# 4 Third Approach

In the above text, we only talked about "and" and "or" operations. What about more complex logical operations? If we fix "and", "or", and "not" operations, then we can, in principle, knowing the degree of belief in the basic statements, determine the degree of belief in their logical combination $Q$. To do that, we represent the given formula $Q$ as a combination of &, $\vee$, and $\neg$, and then consequently use our chosen operations with degrees of belief instead of these logical symbols.

There is a problem with this approach: Every expression can be described in several different ways in terms of the basic logical operations &, $\vee$, and $\neg$. For example, $A \to B$ can be represented as $B \vee \neg A$, $(A \,\&\, B) \vee \neg A$ etc. These expressions are equivalent in normal Boolean (2-valued) logic, but if we use these expression to compute degrees of belief, we sometimes end up with different results. E.g., in the above case, if $d(A) = 0.6$ and $d(B) = 0.7$, and we use min, max, and $x \to 1 - x$ for &, $\vee$, and $\neg$, then the first expression leads to $\max(d(B), 1 - d(A)) = 0.7$, while the second leads to

$$\max(\min(d(A), d(B)), 1 - d(A)) =$$

$$\max(0.6, 0.4) = 0.6.$$

So, for as given expression $F$, instead of a single value of $d(F)$, we end up with different possible values $p_F(a, \ldots, b)$ of $d(F)$. It is therefore desirable to describe the *interval* formed by the smallest and the largest possible values of $d(F)$ for all $F$ that correspond to a given formula. This idea was first described by Türkşen in [14]. It turns out that if we use min and max, then the smallest and the largest values can be explicitly described.

By a propositional formula in a *DNF* (*disjunctive normal form*), we mean a formula of the type $C_1 \vee \ldots \vee C_m$, where each $C_j$ is of the type $x_1 \,\&\, \ldots \,\&\, x_p$, and $x_i$ are either the basic statements or their negations. We say that we have a *complete DNF* if each $C_j$ contains all variables from the formula.

By a propositional formula in a *CNF* (*conjunctive normal form*), we mean a formula of the type $D_1 \,\&\, \ldots \,\&\, D_m$, where each $D_j$ is of the type $x_1 \vee \ldots \vee x_p$, and $x_i$ are either the basic statements or their negations. We say that we have a *complete CNF* if each $D_j$ contains all variables from the formula.

Every propositional formula can be transformed into a unique complete CNF or into a uniquely defined complete DNF form. These unique formulas will be denoted by CNF($F$) and DNF($F$). For example, $A \to B$ can be transformed into a complete CNF form
$\neg A \vee B$, or into a complete DNF form $(A \,\&\, B) \vee (\neg A \,\&\, B) \vee (\neg A \,\&\, \neg B)$.

**Proposition 3** [19]**.** *Let* $\& = \min$, $\vee = \max$, *and* $\neg(x) = 1 - x$. *Then, for every propositional formula* $F(A, \ldots, B)$, *and for all values* $a, \ldots, b$,

$$p_{\mathrm{DNF}(F)}(a, \ldots, b) \le p_F(a, \ldots, b) \le p_{\mathrm{CNF}(F)}(a, \ldots, b).$$

So, for every formula $F$, we can take

$$[p_{\mathrm{DNF}(F)}(a, \ldots, b), p_{\mathrm{CNF}(F)}(a, \ldots, b)]$$

as the desired interval. In particular, for the $F = A \,\&\, B$, we get the interval $[p^-, p^+]$, where: $p^- = \min(a, b)$, and $p^+$ is equal to

$$\min(\max(1 - a, b), \max(a, 1 - b), \max(a, b)),$$

and for $F = A \vee B$, we get the interval $[p^-, p^+]$, where $p^-$ is equal to

$$\max(\min(1 - a, b), \min(a, 1 - b), \min(a, b)),$$

and $p^+ = \max(a, b)$. For these intervals, Hurwicz criterion leads to the following operations:

$$a \,\&\, b \stackrel{\text{def}}{=} \alpha \cdot \min(a, b) + (1 - \alpha) \cdot (a \otimes b), \tag{12}$$

where

$$a \otimes b \stackrel{\text{def}}{=} \min(\max(1 - a, b), \max(a, 1 - b), \max(a, b)),$$

and

$$a \vee b \stackrel{\text{def}}{=} \alpha \cdot (a \oplus b) + (a - \alpha) \cdot \max(a, b), \tag{13}$$

where

$$a \oplus b \stackrel{\text{def}}{=} \max(\min(1 - a, b), \min(a, 1 - b), \min(a, b)).$$

**Theorem 5.**

$$\max_{a,b,c} |(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)| = \frac{\alpha \cdot (1 - \alpha)}{2};$$

$$\max_{a,b,c} |(a \vee b) \vee c - a \vee (b \vee c)| = \frac{\alpha \cdot (1 - \alpha)}{2}.$$

*Comment.* This theorem was first announced in [15].

**Proof of Theorem 5.**

$1^\circ$. One can easily check that our operations $\&$ and $\vee$ are "dual" in the sense that

$$a \vee b = 1 - (1-a) \,\&\, (1-b).$$

In other words, if $c = a \,\&\, b$, then $c' = a' \vee b'$, where we denoted $a' \stackrel{\text{def}}{=} 1 - a$, $b' \stackrel{\text{def}}{=} 1 - b$, and $c' \stackrel{\text{def}}{=} 1 - c$.

We can therefore conclude that the difference $|(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)|$ corresponding to $a$, $b$, and $c$ is equal to the difference $|(a' \vee b') \vee c' - a' \vee (b' \vee c')|$ corresponding to the values $a'$, $b'$, and $c'$. Thus, any possible value of non-associativity for $\&$ is also a possible value of non-associativity for $\vee$.

Vice versa, any difference $|(a \vee b) \vee c - a \vee (b \vee c)|$ is equal to the difference $|(a' \,\&\, b') \,\&\, c' - a' \,\&\, (b' \,\&\, c')|$ corresponding to the values $a'$, $b'$, and $c'$. Thus, any possible value of non-associativity for $\vee$ is also a possible value of non-associativity for $\&$.

So, the set of possible values of non-associativity is the same for both operations $\&$ and $\vee$. We want to prove, for each of these sets, that the largest possible value of the difference is equal to $\alpha \cdot (1-\alpha)/2$. Since the two sets are equal, it is sufficient to prove this result for only one of these sets. In other words, it is sufficient to consider only one of the two operations $\&$ and $\vee$. In the following proof, we will provide the proof for $\&$.

$2^\circ$. We have already given an example that shows that the difference between $(a \,\&\, b) \,\&\, c$ and $a \,\&\, (b \,\&\, c)$ can be equal to $\alpha \cdot (1-\alpha)/2$. Thus, to prove our theorem, it is sufficient to prove that for all other possible values of $a$, $b$, and $c$, the difference cannot exceed $\alpha \cdot (1-\alpha)/2$.

$3^\circ$. Let us give a general idea of how we will prove our result.

In general, the values $a$, $b$, and $c$ must be from the interval $[0,1]$:

$$0 \le a \le 1; \quad 0 \le b \le 1; \quad 0 \le c \le 1. \tag{14}$$

Formulas for $\&$ contain the operations min and max applied to linear functions. Thus, we can consider different cases depending on which of the corresponding linear functions is larger and which is smaller. Each case is therefore described by a system of inequalities between linear functions, i.e., by a system of linear inequalities. In each case, both expressions $(a \,\&\, b) \,\&\, c$ and $a \,\&\, (b \,\&\, c)$ are linear, hence the difference between these expressions is also linear. To prove that the absolute value of this difference cannot exceed $\alpha \cdot (1-\alpha)/2$, we must prove two conclusions:

- that this difference cannot be larger than $\alpha \cdot (1-\alpha)/2$, and
- that this difference cannot be smaller than $\alpha \cdot (1-\alpha)/2$.

For each of these conclusions, we must prove that the system of linear inequalities formed by inequalities describing the case and the inequality describing the difference $(a \,\&\, b) \,\&\, c - a \,\&\, (b \,\&\, c)$ is inconsistent.

For this proof, we will use the Fourier–Motzkin elimination method (see, e.g., [11]). In this method, we eliminate the variables one by one. Specifically, we pick one variable $x$, and then describe each inequality containing this variable in an equivalent form $x \le \ldots$ or $\ldots \le x$. The value $x$ satisfying all these inequalities exists if and only if each lower bound for $x$ does not exceed (or is smaller, depending on whether the bound is strict or not) each upper bound for $x$. These inequalities between the bounds + the original inequalities that did not contain $x$ form a new system of linear inequalities. This new system is consistent if and only if the old system was consistent – but which contains one fewer variable.

After eliminating the variables one by one, we get the desired contradiction.

4°. Let us illustrate this general idea on a single case – the case that contains the above values $a = 0$ and $b = c = 0.5$.

4.1°. Let us start with the expression (12) for $a \,\&\, b$. The first term in this expression is $\min(a, b)$. Therefore, in accordance with our general idea, we must consider two possible cases: $a \le b$ and $a > b$. We will consider only one case: when

$$a \le b. \tag{15}$$

In this case, $\min(a, b) = a$.

The next term in $\min(a, 1 - b)$. We therefore have to consider two subcases: when $a \le 1 - b$ (i.e., $a + b \le 1$), and when $a + b > 1$. We will consider the subcase

$$a + b \le 1. \tag{16}$$

For this subcase, $\max(a, 1 - b) = 1 - b$. For this subcase, also $1 - a \ge b$, hence $\max(1 - a, b) = 1 - a$, and $\max(a, b) = b$ (since we are considering case (15)). Thus, the expression

$$\min(\max(a, 1 - b), \max(1 - a, b), \max(a, b))$$

takes the form $\min(1 - b, 1 - a, b)$. Due to (15), we have $1 - b \le 1 - a$, hence this expression takes the form $\min(1 - b, b)$. The value of this expression depends on whether $b \le 1 - b$, i.e., equivalently, whether $b \le 0.5$. We will have to consider both subsubcases. To illustrate our approach, we consider the subsubcase when

$$b \le 0.5. \tag{17}$$

In this case, $a \,\&\, b = \alpha \cdot a + (1 - \alpha) \cdot b$.

When $b \le 0.5$ and $a \le b$, the automatically $a \le 0.5$ and hence $a + b \le 1$. Hence, to describe this subcase, it is sufficient to consider only the inequalities (15) and (17).

4.2°. Similarly, when describing $b \,\&\, c$, we consider the case when

$$b \le c, \tag{18}$$

and

$$c \le 0.5. \tag{19}$$

In this case, $b \,\&\, c = \alpha \cdot b + (a - \alpha) \cdot c$.

4.3°. Let us now find the expression for $(a \,\&\, b) \,\&\, c$.

4.3.1°. The first term in this expression is proportional to $\min(a \,\&\, b, c)$. We know that in our case, $a \,\&\, b = \alpha \cdot a + (1 - \alpha) \cdot b$, and that $b \le c$ and (since $a \le b$ and $b \le c$) also $a \le c$. Multiplying the inequality $a \le c$ by $\alpha$ and the inequality $b \le c$ by $1 - \alpha$ and adding the resulting inequalities, we conclude that $\alpha \cdot a + (1 - \alpha) \cdot b \le c$, hence the minimum is equal to $a \,\&\, b = \alpha \cdot a + (1 - \alpha) \cdot b$.

4.3.2°. The second term in the desired expression is proportional to the minimum of $\max(a \,\&\, b, 1 - c)$, $\max(1 - (a \,\&\, b), c)$, and $\max(a \,\&\, b, c)$.

For the first of these max terms, from $a \le b$ and $b \le 0.5$, we conclude that $a \le 0.5$ and therefore, that $a \,\&\, b = \alpha \cdot a + (1 - \alpha) \cdot b \le 0.5$. Since $c \le 0.5$, we have $1 - c \ge 0.5$ and therefore, $a \,\&\, b \le 1 - c$, so $\max(a \,\&\, b, 1 - c) = 1 - c \ge 0.5$.

Similarly, $\max(1 - (a \,\&\, b), c) = 1 - (a \,\&\, b) \ge 0.5$. We already know that in our case, $a \,\&\, b \le c$, so $\max(a \,\&\, b, c) = c \le 0.5$.

Of the three max terms, one ($c$) is $\le 0.5$, and the other two are $\ge 0.5$. Therefore, the smallest of these three terms is $c$.

4.3.3°. Now, we can get the final expression for $(a \,\&\, b) \,\&\, c$: it is

$$(a \,\&\, b) \,\&\, c = \alpha \cdot (a \,\&\, b) + (1 - \alpha) \cdot c =$$

$$\alpha^2 \cdot a + \alpha \cdot (1 - \alpha) \cdot b + (1 - \alpha) \cdot c.$$

4.4°. Let us now find the expression for $a \,\&\, (b \,\&\, c)$.

4.4.1°. The first term in this expression is proportional to $\min(a, b \,\&\, c)$, where $b \,\&\, c = \alpha \cdot b + (1 - \alpha) \cdot c$. Since $a \le b$ and $a \le c$, we conclude (similarly to Part 4.3.1 of this proof) that $a \le \alpha \cdot b + (1 - \alpha) \cdot c$ and therefore, the desired minimum is equal to $a$.

4.4.2°. The second term in the desired expression is proportional to the minimum of $\max(a, 1 - (b \,\&\, c))$, $\max(1 - a, b \,\&\, c)$, and $\max(a, b \,\&\, c)$. Similarly to Part 4.3.2 of this proof, by comparing values with 0.5, we conclude that this minimum is equal to

$$\min(1 - (b \,\&\, c), 1 - a, b \,\&\, c) = b \,\&\, c.$$

4.4.3°. Now, we can get the final expression for $a \,\&\, (b \,\&\, c)$: it is

$$a \,\&\, (b \,\&\, c) = \alpha \cdot a + (1 - \alpha) \cdot (b \,\&\, c) =$$

$$\alpha \cdot a + \alpha \cdot (1 - \alpha) \cdot b + (1 - \alpha)^2 \cdot c.$$

4.5°. Subtracting the above expressions for $(a \,\&\, b) \,\&\, c$ and $a \,\&\, (b \,\&\, c)$, we conclude that the difference is equal to $\alpha \cdot (1 - \alpha) \cdot (c - a)$. To illustrate our approach, let us show that the system consisting of linear inequalities (14), (15), (17), (18), (19), and

$$\alpha \cdot (1-\alpha) \cdot (c-a) > \alpha \cdot (1-\alpha)/2, \tag{20}$$

is inconsistent.

First, let us simplify this system. We do not need all the inequalities (14): since $a \le b$ and $a \le c$, it is sufficient to require that $a \ge 0$, then automatically $b \ge 0$ and $c \ge 0$. Similarly, since $b \le 0.5$, $c \le 0.5$, and $a \le b$, we automatically get $a \le 1$, $b \le 1$, and $c \le 1$. Thus, the only inequality left from (14) is:

$$a \ge 0. \tag{14a}$$

Re (20): if $\alpha = 0$ or $\alpha = 1$, we get known associative operators, so we are only interested in the values $\alpha \in (0,1)$. For these values, the product $\alpha \cdot (1-\alpha)$ is positive. Dividing both sides of (20) by this product, we get an equivalent inequality

$$c - a > 0.5. \tag{20a}$$

Let us now eliminate variables – starting with $c$ – from the resulting system (14a), (15), (17), (19), and (20a). There are three inequalities containing $c$: $b \le c$ (18), $c \le 0.5$ (19), and to which $c > a + 0.5$ (20a). So, we have two lower bounds for $c$: $b$ and $a+0.5$, and one upper bound – 0.5. According to the general algorithm, we require that every lower bound must be smaller than every upper bound. This leads to two new inequalities: $b \le 0.5$ (which is already covered by the inequality (17)) and

$$a + 0.5 < 0.5, \tag{21}$$

or, equivalently, $a < 0$.

There is a clear contradiction (inconsistency) with (14a).

*Comment.* In this particular case, we could get this inconsistency easier, but we wanted to show how the general variable elimination approach works.

5°. Due to size limitations, we cannot present here the proofs for all cases, but we hope that the reader gets a good understanding of how this proof was done.

We have analyzed all possible cases, and in all the cases, Fourier–Motzkin elimination method does prove the desired inequalities. Thus, the theorem is proven.

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## References

[1] B. Bouchon-Meunier, V. Kreinovich, and H. T. Nguyen, "Non-Associative Operations", *Proceedings of the Second International Conference on Intelligent Technologies InTech'2001*, Bangkok, Thailand, November 27–29, 2001, pp. 39–46.

[2] L. Godo, R. Lopez de Mantaras, C. Sierra, and A. Verdaguer, "MILORD: The Architecture and management of Linguistically expressed Uncertainty", *International Journal of Intelligent Systems*, 1989, Vol. 4, pp. 471–501.

[3] I. R. Goodman, R. A. Trejo, V. Kreinovich, J. Martinez, and R. Gonzalez, "An even more realistic (non-associative) interval logic and its relation to psychology of human reasoning", *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, 2001, pp. 1586–1591.

[4] L. Hurwicz, *A criterion for decision-making under uncertainty*, Technical Report 355, Cowles Commission, 1952.

[5] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[6] J. Martinez, L. Macias, A. Esper, J. Chaparro, V. Alvarado, S. A. Starks, and V. Kreinovich, "Towards more realistic (e.g., non-associative) and- and or-operations in fuzzy logic", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7-10, 2001, pp. 2187–2192.

[7] G. A. Miller, "The magical number seven plus or minus two: some limits on our capacity for processing information", *Psychological Review*, 1956, Vol. 63, pp. 81–97.

[8] P. M. Milner, *Physiological psychology*, Holt, NY, 1970.

[9] H. T. Nguyen and V. Kreinovich, "Nested Intervals and Sets: Concepts, Relations to Fuzzy Sets, and Applications", In: R. B. Kearfott and

V. Kreinovich (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht, 1996, pp. 245–290.

[10] H. T. Nguyen and E. A. Walker, *First Course in Fuzzy Logic*, CRC Press, Boca Raton, FL, 1999.

[11] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

[12] P. Suppes, D. M. Krantz, R. D. Luce, and A. Tversky, *Foundations of measurement*, Vol. I–III, Academic Press, San Diego, CA, 1989.

[13] R. Trejo, V. Kreinovich, I. R. Goodman, J. Martinez, and R. Gonzalez, "A Realistic (Non-Associative) Logic And a Possible Explanations of 7±2 Law", *International Journal of Approximate Reasoning*, 2002, Vol. 29, pp. 235–266.

[14] I. B. Türkşen, "Interval valued fuzzy sets based on normal forms", *Fuzzy Sets and Systems*, 1986, Vol. 20, pp. 191–210.

[15] I I. B. Türkşen, A. Esper, K. Patel, S. A. Starks, and V. Kreinovich, "Selecting a Fuzzy Logic Operation from the DNF-CNF Interval: How Practical Are the Resulting Operations?", *Proceedings of the 21st International Conference of the North American Fuzzy Information Processing Society NAFIPS'2002*, New Orleans, Louisiana, June 27–29, 2002, pp. 28–33.

[16] T. Whalen, "Interval probabilities induced by decision problems", In: R. R. Yager, J. Kacprzyk, and M. Pedrizzi (Eds.), *Advances in the Dempster-Shafer Theory of Evidence*, Wiley, N.Y., 1994, pp. 353–374.

[17] T. Whalen and C. Broenn, "Hurwicz and regret criteria extended to decisions with ordinal probabilities", *Proc. NAFIPS'90*, 1990, pp. 219–222.

[18] H. H. Zimmermann and P. Zysno, "Latent connectives in human decision making", *Fuzzy Sets and Systems*, 1980, Vol. 4, pp. 37–51.

[19] Q. Zuo, I. B. Türkşen, H. T. Nguyen, and V. Kreinovich, "In expert systems, even if we fix AND/OR operations, a natural answer to a composite query is the interval of possible degrees of belief", *Reliable Computing*, 1995, Supplement (Extended Abstracts of APIC'95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995), pp. 236–240.