Second-Order Uncertainty as a Bridge Between Probabilistic and Fuzzy Approaches

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Abstract

On the example of physics, we show that the traditional one-level description is not completely adequate. For a more adequate structure, a hierarchical description of uncertainty is necessary, which supplements the more traditional first-order uncertainty with second-order, third-order and more sophisticated models. In particular, the second-order approach seems to provide a bridge between probabilistic and fuzzy approaches to uncertainty.

Keywords: Second Order, Probabilistic, Fuzzy, Algorithmic Information Theory, Kolmogorov Complexity.

1 It Is Important to Define the Notion of a Random Sequence

Modern physics is based on the quantum paradigm. Quantum physics does not predict the exact values of measured quantities. It predicts the wave function $\psi(x)$ and hence the probabilities of different measurement results. For example, the probability density is $\rho(x) = |\psi(x)|^2$.

The only information about the actual sequence of measurement results is that this sequence must be random relative to the corresponding probability measure m.

Traditional probability theory does not describe what it means for a sequence to be random. So, to adequately apply quantum physics, we must supplement the traditional probability theory with a definition of randomness.

2 Random Sequence: Definition from Algorithmic Information Theory

A definition of a random sequence is provided by the so-called Algorithmic Information Theory; see, e.g., [2, 4]. The main idea behind the corresponding definition of randomness comes from practice.

In practice, if we prove, e.g., that for almost all sequences (i.e., with probability 1), the frequency tends to 1/2, then for the actual random sequence, the frequency will also tend to 1/2. In other words, whenever we have a constructively checkable property of a sequence which occurs with probability 1, we expect this property to hold for the actual physically random sequence.

Kolmogorov and Martin-Löf made this property the definition of a random sequence. Namely, they defined a sequence to be *random* if it satisfies every computable property P whose probability is 1.

The notion of a property is normal in logic, but it is somewhat alien to the traditional foundations of probability theory, which are usually based on the set-theoretic language. In set-theoretic terms:

- satisfying a property P is the same as belonging to the set $\{x \mid P(x)\}$ of all the objects x which satisfy this property, and
- the probability of a property P is equal to the probability measure of this set.

So, in set-theoretic terms, a sequence is called random if it belongs to every computable set of measure 1.

This definition can be further reformulated. Belonging to a set is the same as not belonging to its complement. Complements to sets of probability measure

1 are exactly sets of measure 0. Thus, we can define a random sequence as a sequence which does not belong to any computable set of measure 0.

3 The Above Definition of a Random Sequence Is Consistent

Let us show that this definition is consistent, i.e., that there are sequences which are random according to this definition.

Indeed, according to the above definition, a sequence is random if it does not belong to the union U of all computable sets of measure 0. How big is this union?

Every computable set is uniquely determine by the corresponding computational program. Each program is a finite sequence of symbols. There are only countably many such combinations, so there are only countably many computable sets. Thus, U is a union of countably many sets of measure 0. Hence, the set U is itself of measure 0.

A set of measure 0 does not exhaust all the probability space – which is of measure 1. Thus, there are sequences which do not belong to this set U. Be definition, such sequences are random. Thus, we have proved that there exist random sequences – i.e., that the above definition of a random sequence is consistent.

4 Not Only There Exist Random Sequence, but Almost All Sequences Are Random

A similar argument shows not only that there exist random sequences, but also that there exist many of them.

Indeed, according to the above definition, a sequence is random if and only if it does not belong to a set U of measure 0. The probability of belonging to the set U is 0, so the probability of *not* belonging to this set is equal to 1.

Thus, not only we proved that there are random sequences (i.e., that the notion of a random sequence is consistent), but we have actually proved that almost all sequences are random.

5 Uncertainty in Physics: Second-Order Uncertainty is Needed

The above brief description does not capture all the nuances of quantum physics. Namely, this description corresponds to the *first* (particle) quantization, when the wave function is described by the Schroedinger equation

$$\frac{\partial\psi(x,t)}{\partial t} = H(\psi(x,t)),$$

where H is a known operator; e.g., $H = \Delta^2 + V(x)$, where $V(x)$ is a potential energy function.

For a known (and computable) operator H, and for a computable $\psi(x,0)$, the entire function $\psi(x,t)$ is computable and hence, the probability measure m is also computable.

A more adequate description of quantum phenomena requires *second* quantization, in which, instead of assuming that we know exactly how the potential energy $V(x)$ depends on the particle locations, we assume that fields like $V(x)$ can have quantum fluctuations as well. In other words, we assume that $V(x)$ is not a deterministically computable function, but a function which is random relative to some (computable) probability measure m_1.

6 Possibly, We Also Need Third-Order and Higher-Order Uncertainty

Theoretical physicists discuss the possibility of the *third* quantization, in which m_1 is also not computable, but random relative to some other measure m_2, etc.

7 Second-Order and Higher-Order Uncertainty in Physics: Towards Fundamental Justification

We will show that the semi-heuristic hierarchy of 1-st order, 2-nd order, higher-order uncertainties can be theoretically justified within the above Kolmogorov-Martin-Löf definition of randomness.

Indeed, as we have mentioned, if we are given a probability measure m on the set of all binary sequences, then this definition describes a *random* sequence as a sequence which does not belong to any constructive set of m-measure 0.

We start with computable measures, i.e., measures for which the function $x_1 \ldots x_n \to m(x_1 \ldots x_n)$ describing the probability of a random sequence to start with $x_1 \ldots x_n$ is computable; sequences which are random relative to such measures will be assigned to level 1.

Now, we can take measures for which the sequence $m(x_1 \ldots x_n)$ is a level 1 sequence, and assign sequences which are random relative to such measures to level 2, etc.

8 Uncertainty in Physics: Our Main Result

Our main result is that this hierarchy is provably inhabitable, i.e., that on every level, there is a sequence which does not belong to the previous levels.

9 Proof: Main Idea

The proof of this result is reasonably straightforward. It is based on the following two known facts about random sequences:

- The first fact that we use in this proof is the fact that if a sequence if random relative to some probability measure m, then this measure can be uniquely determined from this sequence: indeed, the corresponding probabilities can be determined as limits of frequencies corresponding to this sequence.

  From this fact, we can conclude that a sequence cannot be random relative to two different probability measures.

- The second fact is that, as we have shown, for every probability measure m, almost all sequence are random relative to m.

So:

- If we take a computable measure m, then almost all sequence are random relative to m. It is known that random sequences are not computable, so we get examples of random sequences of level 1.

- If we take a measure which is defined by a (1st order) random measure m_1, then almost all sequences are random relative to this measure. Since a sequence cannot be random relative to two different measures, a sequence which is random relative to m_1 cannot be random relative to any computable measure m and thus, it cannot be random of level 1. So, we have examples of sequences of level 2 which do not belong to levels 0 and 1.

- Similarly, starting with a measure m_k which is random of level k, we get sequences which are random of level $k+1$ which do not belong to any previous level.

Thus, on every level, there is a sequence which does not belong to the previous levels. The theorem is proven.

10 Objective Uncertainty Outside Quantum Physics: Image Processing

As another example of the need for second order uncertainty, let us consider image compression. In image compression, the standard statistical approach is based on:

- finding the actual probability distributions, and
- finding the compression algorithm which is the best for this particular distribution.

The problem is that for real images, this traditional statistical description is not very adequate: as shown in [5],

- while the empirical averages do converge,
- the empirical standard deviations don't.

So:

- instead of the first-order probabilistic approach, where we have a single probability distribution,
- we must consider the "second-order" probabilistic approach, in which we have a *class* of probability distributions.

11 Subjective Uncertainty: Second-Order Uncertainty as a Bridge Between Probabilistic and Fuzzy

A similar need for second-order approaches appears not only in physics (where uncertainty is objective), but also in the description of *subjective* uncertainty. For example:

- First-order fuzzy approach can be interpreted in probabilistic terms. Specifically, a fuzzy set with a membership function $\mu(x)$ can be interpreted as a class of probability distributions [1, 3, 7, 8, 9], namely, as the class of all random sets (probability distributions on the class of all sets S) for which, for every point x, we have

$$P(S \mid x \in S) = \mu(x).$$

- However, not all classes of probability distributions are thus covered. The second-order fuzzy approach can be also interpreted in this way, and covers more classes – thus approximating each class more adequately.

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