Interval Mathematics for Analysis of Multi-Level Granularity

Vladik Kreinovich[1] and Richard Aló[2]

[1]Computer Science, University of Texas, El Paso, TX 79968, USA, vladik@cs.utep.edu
[2]CS and Math, U. of Houston-Downtown, Houston, TX 77002, USA, ralo@uh.edu

Abstract— **The more complex the problem, the more complex the system necessary for solving this problem. For very complex problems, it is no longer possible to design the corresponding system on a single resolution level, it becomes necessary to have multi-level, multiresolutional systems, with multi-level granulation. When analyzing such systems – e.g., when estimating their performance and/or their intelligence – it is reasonable to use the multi-level character of these systems: first, we analyze the system on the low-resolution level, and then we sharpen the results of the low-resolution analysis by considering higher-resolution representations of the analyzed system. The analysis of the low-resolution level provides us with an approximate value of the desired performance characteristic. In order to make a definite conclusion, we need to know the accuracy of this approximation. In this paper, we describe interval mathematics – a methodology for estimating such accuracy. The resulting interval approach is also extremely important for tessellating the space of search when searching for optimal control. We overview the corresponding theoretical results, and present several case studies.**

I. Multi-Level (Multiresolutional) Granulation Methods are Necessary: A Brief Reminder

The more complex the problem, the more complex the system necessary for solving this problem. For very complex problems, it is no longer possible to design the corresponding system on a single resolution level, it becomes necessary to have multi-level systems, with different granulations on each level.

Granulation methods can be traced to a pioneer paper by L. Zadeh [82]. Multi-level granulation methods, namely, the methodology of multiresolutional search for the optimum solution of a control problem was first presented by A. Meystel in [42], [43]. These papers contributed to the broad interest in and dissemination of the multiresolutional approach to solving problems of the areas of intelligent control and intelligent systems.

Many algorithms based on this methodology were developed since then. The successful practical applications of these algorithms shows that multiresolutional approach is indeed necessary.

This empirical conclusion has been supported by many mathematical results; let us name a few recent ones:

- It has been proven that for general complex (NP-hard) problems, i.e., problems, for which no general feasible algorithm is possible, there always exists an appropriate granulation after which the problem becomes easy to solve. The fact that the problem is NP-hard means that there is no general algorithm for automatically finding such a granulation, this granulation requires an expert familiar with the particular problem that we are trying to solve [12].
- For noisy images $I(x)$ in which we do not know the exact statistical characteristics of the noise, only the upper bound on the noise, the optimal image processing requires representing this image as a linear combination of so-called *Haar wavelets* $e_i(x)$, i.e., functions which only take values 1 or 0. Such a wavelet representation is a known particular case of a multi-level multiresolutional granular representation [6], [7].
- In particular, when detecting a known pattern in a given image, it is provably better to use lower-resolution type techniques that look for the whole pattern as opposed to higher-resolution techniques which look for pieces of this pattern and then try to match found pieces together [66].
- Similarly to noisy images, for signal multiplexing under noise, the use of Walsh functions (similar to Haar wavelets) can be proven to be the optimal choice [2].
- In general, in function interpolation, clustering techniques – in which we combine the values into clusters before extrapolation – turn out to be optimal [36]. Such an interpolation is very useful in intelligent control, when we train a system by providing it with examples of control values used by expert human controllers in different situations.
- In general, in intelligent control, hierarchical fuzzy control is better in the sense that it requires fewer rules to describe the same quality control [37], [38], [80].
- Finally, it can be shown that for many systems, the optimal control is of "bang-bang" type, when there are finitely many preferred control values (or preferred fixed control trajectories), and the optimal control consists of optimally switching between these values (trajectories). This general result explains different empirical phenomena ranging from the empirical fact of discrete speed levels in traffic control to the phenomenon of sleep when it seems to be biologically optimal to always switch between several fixed levels of activity [31].

II. Interval Mathematics: A Methodology for Validated Analysis of Multi-Level Systems

A. Validated Analysis of Multi-Level Systems Naturally Leads to Interval Computations

When analyzing multi-level systems – e.g., when estimating their performance and/or their intelligence – it is reasonable to use the multi-level character of these systems: first, we analyze the system on the low-resolution level, and then we sharpen the results of the low-resolution analysis by considering higher-resolution representations of the analyzed system.

For example, instead of the original image with its numerous pixel-by-pixel brightness values, we consider a low-resolution image in which there is a small finite number of zones, and each zone is characterized by a single brightness value. After analyzing this image, we increase resolution, thus adding more details (more zones), etc.

The analysis of the low-resolution level provides us with an approximate value of the desired performance characteristic. In order to make a definite conclusion, we need to know the accuracy of this approximation. How can we estimate this accuracy?

In order to solve this problem, let us reformulate it in general mathematical terms. Instead of considering the *exact* system, we consider its *approximation*, analyze this approximation, and then we want to make a conclusion about the original system based on this analysis. The original system is characterized by the values of different parameters $x_1, \ldots, x_n$; e.g., for the image, these parameters are the brightness values at different pixels. We want to estimate some characteristic $q = f(x_1, \ldots, x_n)$ of the original system.

A low-resolution approximation can be usually described by fewer parameters $y_1, \ldots, y_m$, $m \ll n$; e.g., for the image, these parameters are the brightnesses of different zones. Each parameter x_i is approximated by one of the new parameters y_j; let us denote the corresponding parameter by $y_{j(i)}$. When each x_i is exactly equal to the corresponding value y_j, we get a simplified expression for q which only depends on $m \ll n$ values: $\widetilde{q} = \widetilde{f}(y_1, \ldots, y_n)$. In reality, the values x_i are somewhat different from y_j, and as a result, the estimate $\widetilde{q}$ is different from the actual value q of the desired characteristic. How can we estimate the corresponding approximation error $\widetilde{q} - q$?

In addition to the approximate model itself, we usually know, for each j, the upper bound on the error with which the value y_j approximates the corresponding values x_i. In other words, we know that the actual value of x_i belongs to the interval $\mathbf{y}_j = [y_j - \Delta_j, y_j + \Delta_j]$. Since each value x_i belongs to the interval $\mathbf{y}_{j(i)}$, the actual value of the desired characteristic belongs to the *range*

$$\mathbf{q} = f(\mathbf{y}_{i(1)}, \ldots, \mathbf{y}_{i(n)}) \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) \,|\, x_i \in \mathbf{y}_{j(i)}\}$$

of the function f on these intervals. Thus, in order to estimate the accuracy of the lower-resolution estimate $\widetilde{q}$, we can estimate the above range.

The problem of estimating the range of the function $f(x_1, \ldots, x_n)$ when we know the intervals $\mathbf{x}_i$ of possible values of x_i is a known problem in areas where the inputs are not known precisely, be it numerical methods or data processing. This problem is called the problem of *interval computations*, and methods for solving this problem are called *interval mathematics* [1], [17], [18], [20], [21], [46], [78].

B. Interval Computations are Difficult

In general, the interval computation problem is NP-hard even for quadratic functions $f(x_1, \ldots, x_n)$; see, e.g., [28]. In plain English, this means that it is highly unprovable that we will be able to find a general feasible algorithm that computes the exact range for all functions f and all intervals $\mathbf{x}_i$ in reasonable time. Since we cannot compute the exact range, what can we do instead?

We wanted to compute the exact range $\mathbf{q}$ because we wanted to get an interval that is guaranteed to contain the desired value q, and the range definitely contains this value. If we cannot compute the *exact* range in reasonable time, we can compute the *approximate* interval $\mathbf{Q}$ for the range. The only way to guarantee that the new interval still contains q is to make sure that this new intervals contains the entire range $\mathbf{q} \subseteq \mathbf{Q}$, i.e., that this interval is an *enclosure* for the desired range.

In these terms, interval mathematics is an art of computing good narrow enclosures for the range of a given function $f(x_1, \ldots, x_n)$ on given intervals $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

C. Methods of Interval Mathematics: A Very Brief Introduction

Interval mathematics started, in the 1950s, with the observation that for simple arithmetic operations $f(x_1, x_2) = x_1 + x_2$, $x_1 - x_2$, etc., the range can be computed explicitly; e.g.:

$$[x_1^-, x_1^+] + [x_2^-, x_2^+] = [x_1^- + x_2^-, x_1^+ + x_2^+];$$
$$[x_1^-, x_1^+] - [x_2^-, x_2^+] = [x_1^- - x_2^+, x_1^+ - x_2^-];$$
$$[x_1^-, x_1^+] \cdot [x_2^-, x_2^+] = [\min(x_1^- \cdot x_2^-, x_1^- \cdot x_2^+, x_1^+ \cdot x_2^-, x_1^+ \cdot x_2^+),$$
$$\max(x_1^- \cdot x_2^-, x_1^- \cdot x_2^+, x_1^+ \cdot x_2^-, x_1^+ \cdot x_2^+)].$$

The corresponding expressions are called formulas of *interval arithmetic*.

It turns out that we can use these expressions to get reasonable enclosures for arbitrary functions f. Indeed, when the computer computes the function f, it *parses* the function, i.e., it represents the computation as a sequence of elementary arithmetic operations. It can proven, by induction, that if we start with intervals and replace each arithmetic operation with the corresponding operation of interval arithmetic, at the end, we get an enclosure for f. For example, if $f(x) = x \cdot (1 - x)$, represent f as a sequence of two elementary operations:

- $r := 1 - x$ (r denotes the 1st intermediate result);
- $y := x \cdot r$.

In the interval version, perform the following computations:

- $\mathbf{r} := 1 - \mathbf{x}$;
- $\mathbf{y} := \mathbf{x} \cdot \mathbf{r}$.

In particular, when $\mathbf{x} = [0, 1]$, compute the intervals $\mathbf{r} := [1, 1] - [0, 1] = [0, 1]$, and

$$\mathbf{y} := [0, 1] \cdot [0, 1] = [\min(0 \cdot 0, 0 \cdot 1, 1 \cdot 0, 1 \cdot 1),$$
$$\max(0 \cdot 0, 0 \cdot 1, 1 \cdot 0, 1 \cdot 1)] = [0, 1].$$

The interval $[0, 1]$ is indeed an enclosure of the actual range $[0, 0.25]$.

D. Modern Methods of Interval Mathematics and Their Potential Use in Tessellating the Search Space

D.1 Methods Based on Mean Value Theorem

The enclosure obtained by using the above simple idea is often too wide. One of the main objectives of interval computations is to make this enclosure narrower. One way to do that is to use the *mean value* theorem, according to which $f(x) = f(x_0) + f'(\xi) \cdot (x - x_0)$ for some value ξ between x_0 and x. Thus, if we take, as x_0, the midpoint of the interval $\mathbf{x}$ of width w, we will have $|x - x_0| \le w/2$, $f'(\xi) \in f'(\mathbf{x})$, and thus, $f(\mathbf{x}) \subseteq f(x_0) + f'(\mathbf{x}) \cdot [-w/2, w/2]$. If we do not know the exact range $f'(\mathbf{x})$, we can use the enclosure for this range. Similar formulas can be easily written for the case of several variables.

D.2 Methods Based on Division into Subboxes and Their Relation with Multi-Level Granular Approach

In many cases, the above idea leads to a reasonable enclosure. If the enclosure is still too wide, we can divide the original box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$ into sub-boxes, compute the enclosure for each of these subboxes, and then take the union of the resulting enclosures.

It is worth mentioning that this idea is completely in line with the general multi-level granular approach: instead of considering the individual values of the function $f(x_1, \ldots, x_n)$ for all possible inputs $x_1, \ldots, x_n$, we divide the range of this function into a small number of zones, and consider the enclosure for each zone. In multi-level granular terms, we are thus considering a low-resolution approximation to the original function. If we want better results, we have to consider smaller zones, i.e., we have to consider higher-resolution approximations.

In other words, not only the *formulation* of the main problem of interval mathematics naturally comes from multi-level granular approach, but also the *methods* of interval mathematics are completely in line with this approach.

D.3 Interval Mathematics as a Method for Tessellating Search Space

The resulting interval approach is also extremely important for tessellating the space of search when searching for optimal control [20], [21]. The simplest way of using interval computations in to locate a maximum of the objective function $f(x)$ is as follows:

First, we compute the values of $f(x)$ in several points $x^{(1)}, \ldots, x^{(k)}$; we then now that $\max f(x) \ge M \stackrel{\text{def}}{=} \max(f(x^{(i)}))$. Then, we divide the original range into several zones Z_i, use interval computations to get an enclosure $\mathbf{F}_i = [F_i^-, F_i^+]$ of the range of $f(x)$ on each zone Z_i, and dismiss the zones for which $F_i^+ < M$ – because they cannot contain the global maxima.

Then, we subdivide the remaining zones into sub-zones, and repeat this procedure again – until we locate the global maxima. This idea leads to a reasonably efficient algorithms for global optimization, with can be further enhanced by using interval versions of gradient-based optimization methods.

Numerous similar methods exist for computing enclosures and optimization. Most of these methods are implemented in easily available software packages; see, e.g., [20], [21], [78].

D.4 Conclusion: Interval Mathematics Is Very Useful for Multi-Level Granular Approach

Based on the above, we can conclude that *interval mathematics is a good candidate for being "the" mathematics of multi-level granular system analysis.*

D.5 We Will Present Examples of Applying Interval Computations

In the following sections, we will describe two applications of interval mathematics in some detail. Before we go into the descriptions, we should mention that the above is the description of a "vanilla" situation. In many real-life cases, the situation is even more complex, because, in addition to a *quantitative* conclusion (about the value of some quantity q), we need to make a *qualitative* conclusion: e.g., in the following example, a conclusion on whether a plate has a hidden fault or not. Other non-trivial examples can be found, e.g., in [3] and references therein.

E. Case Study: Non-Destructive Testing

This case study is described, in detail, in [67], [75], [76], [77].

In many areas, e.g., in aerospace industry, in medicine, it is desirable to detect mechanical faults without damaging or reassembling the original system. For testing, we send a signal and measure the resulting signal. The input signal can be described by its intensity $r_1, \ldots, r_n$ at different moments of time. The intensities $s_1, \ldots, s_m$ of the resulting signal depend on r_i: $s_j = f_j(r_1, \ldots, r_n)$, where the functions f_j depend on the tested structure.

Usually, we do not know the exact analytical expression for the dependency f_j, so we can use the fact that an arbitrary continuous function can be approximated by a polynomial (of a sufficiently large order). Thus, we can take a structure, try a general linear dependency first, then, if necessary, general quadratic, etc., until we find the dependency that fits the desired data.

If a structure has no faults, then the surface is usually smooth. As a result, the dependency f_j is also smooth; we can expand it in Taylor series. Since we are sending relatively weak signals r_i (strong signals can damage the plane), we can neglect quadratic terms and only consider linear terms in these series; thus, the dependency will be *linear*.

A fault is, usually, a violation of smoothness (e.g., a crack). Thus, if there is a fault, the structure stops being smooth; hence, the function f_j stops being smooth, and therefore, linear terms are no longer sufficient. Thus, *in the absence of fault, the dependence is linear, but with the faults, the dependence is non-linear.* So, we can detect the fault by checking whether the dependency between s_j

and r_i is linear. So, we send several different inputs, measure the values $r_i^{(k)}$ and $s_j^{(k)}$ corresponding to these inputs, and check whether the dependence is linear. In this case, the values $r_i^{(k)}$ and $s_j^{(k)}$ are the inputs $x_1, \ldots, x_n$, but the desired q is a qualitative (yes-no) variable: we simply want to know whether there is a fault or not. If there is a fault, then we would also like to make a quantitative conclusion of its size, location, etc., but the most important part of the analysis is to check whether there is any fault at all.

If the measurements were ideal, all we had to do was to check whether there are values a_{ji} for which, for all j and for all measurements k, we have:

$$a_{j0} + a_{j1} \cdot r_1^{(k)} + \ldots + a_{jn} \cdot r_n^{(k)} = s_j^{(k)}.$$

Solvability of a system of linear equations is easy to check.

In reality, the situation is more complicated. Measurement are usually imprecise: the result $\widetilde{x}$ of measuring the actual value x is somewhat different from the actual value x. In many real-life situations, we do not know the probabilities of different values of measurement error $\Delta x = \widetilde{x} - x$, we only know the upper bound Δ of the corresponding measurement error. As a result, the only information that we have about the actual value x of the measured quantity is that it belongs to the interval $\mathbf{x} = [x - \Delta, x + \Delta]$. So, in practice, instead of the exact values of $r_i^{(k)}$ and $s_j^{(k)}$, we have *intervals* $\mathbf{r}_i^{(k)}$ and $\mathbf{s}_j^{(k)}$ of possible values of these quantities. The question becomes: are these intervals consistent with the linearity, i.e., are there values $r_i^{(k)} \in \mathbf{r}_i^{(k)}$ and $s_j^{(k)} \in \mathbf{s}_j^{(k)}$ for which, for some values a_{ji}, the above linearity formulas hold.

In general, the solvability of the corresponding system of interval linear equations is an NP-hard problem [28], but for some cases, efficient algorithms have been developed. For example, when we have only one (non-negative) input and only one output, with non-intersecting intervals $\mathbf{r}^{(1)} < \mathbf{r}^{(2)} < \ldots$, the solvability of the corresponding system of linear equations can be proven to be equivalent to the following inequality:

$$\max_{k<l} \frac{s^{(l)-} - s^{(k)+}}{r^{(l)+} - r^{(k)-}} \le \min_{k<l} \frac{s^{(l)+} - s^{(k)-}}{r^{(l)-} - r^{(k)+}}.$$

We tested this method on the dependence of the energy E of the ultrasound response on the voltage V that causes the original ultrasound signal. The results show that non-linearity is indeed an indication of a fault:

- For faultless plates, the above inequality is indeed true, meaning that the measurement results are consistent with linearity.
- For plates with faults, this inequality is not satisfied, meaning that the dependence is non-linear.

F. Case Study: Reliable Sub-Division of Geological Areas

This case study is described, in detail, in [8], [9].

In geophysics, appropriate subdivision of an area into segments is extremely important, because it enables us to extrapolate the results obtained in some locations within the segment (where extensive research was done) to other locations within the same segment, and thus, get a good understanding of the locations which weren't that thoroughly analyzed. The subdivision of a geological zone into segments is often a controversial issue, with different evidence and different experts' intuition supporting different subdivisions.

For example, in our area – Rio Grande rift zone – there is some geochemical evidence that this zone is divided into three segments [41]:

- the southern segment which is located, approximately, between the latitudes $y = 29°$ and $y = 34°$;
- the central segment – from $y = 34.5°$ to $y = 38°$; and
- the northern segment – from $y = 38°$ to $y = 41°$.

However, in the viewpoint of many researchers, this evidence is not yet sufficiently convincing.

It is therefore desirable to develop new techniques for zone sub-division, techniques which would be in the least possible way dependent on the (subjective) expert opinion and would, thus, be maximally reliable. To make this conclusion more reliable, we use, instead of the more rare *geological* samples, a more abundant *topographical* information (this information, e.g., comes from satellite photos). We can characterize each part of the divided zone by its topography.

In topographical analysis, we face a new problem: of too much data, most of which is geophysically irrelevant. To eliminate some of this irrelevant data, we can use the Fourier transform; indeed, it is known that while (at least some) absolute values of the map (forming a so-called spectrum) are geophysically meaningful, the phases usually are random and can be therefore ignored. So, we should only use the spectrum.

Since we are interested only in the large-scale classification, it makes sense to only use the spectrum values corresponding to relatively large spatial wavelengths, i.e., wavelengths L for which $L \ge L_0$ for some appropriate value L_0. In particular, for the sub-division of the Rio Grande rift, it makes sense to use only wavelengths of $L_0 = 1000$ km or larger.

Also, for the Rio Grande Rift, we are interested in the classification of horizontal zones, so it makes sense to divide the Rio Grande Rift into 1° zones $[y^-, y^+]$ (with y from $y^- = 30$ to $y^+ = 31$, from $y^- = 31$ to $y^+ = 32$, ..., from $y^- = 40$ to $y^+ = 41$). For each of these zones, we take the topographic data, i.e., the height $h(x, y)$ described as a function of longitude x and latitude y, compute the Fourier transform $H(\omega, y)$ with respect to x, combine all the spectral values which correspond to large wavelength (i.e., for which $\omega \le 1/L_0$), and compute the resulting spectral value

$$S(y^-) = \int_{y=y^-}^{y^+} \int_{\omega=0}^{1/L_0} |H(\omega, y)|^2 \, d\omega \, dy.$$

Since we are interested in comparing the spectral values $S(y)$ corresponding to different latitudes y, so we are not interested in the *absolute* values of $S(y)$, only in *relative* values. Thus, to simplify the data, we can *normalize* them

TABLE I

y_i	29	30	31	32	33	34
s_i	0.28	0.24	0.21	0.16	0.20	0.29
35	36	37	38	39	40	41
0.31	0.35	0.46	1.00	0.80	0.96	0.74

by, e.g., dividing each value $S(y^-)$ by the largest $S_{\max}$ of these values. In particular, for the Rio Grande rift, the resulting values of $y^- = y_1, y_2, \ldots$ and $s_i = S(y_i)/S_{\max}$ are as follows:

Based only on these spectral values s_i, we will try to classify locations into several clusters ("segments").

From the geophysical viewpoint, the desired zones correspond to "monotonicity regions": in the first zone, the values s_i are (approximately) decreasing, in the next zone, they are (approximately) increasing, etc. So, we must look for the monotonicity regions of the (unknown) function $s(y)$.

The problem is that the values s_i are only approximately known, so we cannot simply compare the values to determine whether a function increases or decreases. The heights are measured pretty accurately, so the only errors in the values s_i come from discretization. In other words, we would like to know the values of the function $s(y) = S(y)/S_{\max}$ for all y, but we only know the values $s_1 = s(y_1)$, …, $s_n = s(y_n)$ of this function for the points $y_1, \ldots, y_n$. For each y which is different from y_i, it is reasonable to estimate $s(y)$ as the value $s_i = s(y_i)$ at the point y_i which is the closest to y (and, ideally, which belongs to the same segment as y_i). For each point y_i, what is the largest possible error Δ_i of the corresponding approximation?

When $y > y_i$, the point y_i is still the closest until we reach the midpoint $y_{\rm mid} = (y_i + y_{i+1})/2$ between y_i and y_{i+1}. It is reasonable to assume that the largest possible approximation error $|s(y) - s_i|$ for such points is attained when the distance between y and y_i is the largest, i.e., when y is this midpoint; in this case, the approximation error is equal to $|s(y_{\rm mid}) - s_i|$.

If the points y_i and y_{i+1} belong to *the same* segment, then the dependence of $s(y)$ on y should be reasonably smooth for $y \in [y_i, y_{i+1}]$. Therefore, on a narrow interval $[y_i, y_{i+1}]$, we can, with reasonable accuracy, ignore quadratic and higher terms in the expansion of $s(y_i + \Delta y)$ and thus, approximate $s(y)$ by a linear function. For a linear function $s(y)$, the difference $s(y_{\rm mid}) - s(y_i)$ is equal to the half of the difference $s(y_{i+1}) - s(y_i) = s_{i+1} - s_i$; thus, for $y > y_i$, the approximation error is bounded by $0.5 \cdot |s_{i+1} - s_i|$.

If the points y_i and y_{i+1} belong to *different* segments, then the dependence $s(y)$ should exhibit some non-smoothness, and it is reasonable to expect that the difference $|s_{i+1} - s_i|$ is much higher than the approximation error.

In both cases, the approximation error is bounded by

$$0.5 \cdot |s_{i+1} - s_i|.$$

Similarly, for $y < y_i$, the approximation error is bounded by $0.5 \cdot |s_i - s_{i-1}|$ if the points y_i and y_{i-1} belong to the same segment, and is much smaller if they don't. In both cases, the approximation error is bounded by

$$0.5 \cdot |s_i - s_{i-1}|.$$

We have two bounds on the approximation error and we can therefore conclude that the approximation error cannot exceed the smallest Δ_i of these two bounds, i.e., the value

$$\Delta_i = 0.5 \cdot \min(|s_i - s_{i_1}|, |s_{i+1} - s_i|).$$

As a result, instead of the *exact* values s_i, for each i, we get the *interval* $\mathbf{s}_i = [s_i^-, s_i^+]$ of possible values of $s(y)$, where $s_i^- = s_i - \Delta_i$ and $s_i^+ = s_i + \Delta_i$. In particular, for the Rio Grande rift, we get:

$$\mathbf{s}_1 = [0.26, 0.30], \mathbf{s}_2 = [0.225, 0.255], \mathbf{s}_3 = [0.195, 0.225],$$

$$\mathbf{s}_4 = [0.14, 0.18], \mathbf{s}_5 = [0.18, 0.22], \mathbf{s}_6 = [0.28, 0.30],$$

$$\mathbf{s}_7 = [0.30, 0.32], \mathbf{s}_8 = [0.33, 0.37], \mathbf{s}_9 = [0.405, 0.515],$$

$$\mathbf{s}_{10} = [0.80, 1.10], \mathbf{s}_{11} = [0.72, 0.88], \mathbf{s}_{12} = [0.88, 1.04],$$

$$\mathbf{s}_{13} = [0.63, 0.85].$$

We want to find regions of uncertainty of a function $s(y)$, but we do not know the exact form of this function; all we know is that for every i, $s(y_i) \in \mathbf{s}_i$ for known intervals $\mathbf{s}_i$. How can we find the monotonicity regions in the situation with such interval uncertainty? Of course, since we only know the values of the function $s(y)$ in finitely many points y_i, this function can have as many monotonicity regions between y_i and y_{i+1} as possible. What we are interested in is funding the subdivision into monotonicity regions which can be deduced from the data. The first natural question is: can we explain the data by assuming that the dependence $s(y)$ is monotonic? If not, then we can ask for the possibility of having a function $s(y)$ with exactly two monotonicity regions:

- if such a function is possible, then we are interested in possible locations of such regions;
- if such a function is not possible, then we will try to find a function $s(y)$ which is consistent with our interval data and which has three monotonicity regions, etc.

This problem was first formalized and solved in [71], [72], where we developed a linear-time algorithm for solving this problem. By applying this algorithm, we find three monotonicity regions: $[29, 34]$, $[31, 41]$, and $[37, 41]$ – in good accordance with the geochemical data from [41].

G. Other Applications: A Brief Overview

Other successful applications of interval techniques include:

- telemanipulation [10], [27], [67];
- robot navigation [67];
- analysis of multi-spectral satellite images [65], [67].

Since a fuzzy set can be naturally represented as a nested family of intervals (corresponding to different levels of certainty), methods of fuzzy data processing actively use interval computations and be considered as natural applications of interval techniques [23], [52], [56], [67].

III. Multi-D Generalizations of Interval Mathematics and Symmetry Approach

A. General Idea

In addition to the upper bound on the approximation error for each quantity x_i, we often have an additional information. For example, in some cases, in addition to the upper bounds Δ_i for the differences $\widetilde{x}_i - x_i$, we also know the upper bound on their distance between the vectors $\widetilde{x}$ and x, i.e., the upper bound on $\sqrt{(\widetilde{x}_1 - x_1)^2 + \ldots + (\widetilde{x}_n - x_n)^2}$. In this case, we know that the actual values of $x_1, \ldots, x_n$ belongs to the intersection of a box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$ and a ball. We may have more complex shapes. Processing complex shapes is computationally difficult (see, e.g., [34]), so we must find good approximations for such shapes. Ideally, we should find approximations which are *optimal* in some reasonable sense.

A similar problem of finding the optimal shapes arises in the selection of "clusters" (zones) corresponding to the low-resolution approximation. Here also, it is desirable to find the optimal zones.

Let us show, on the example of selecting zones on the plane, how this problem can be solved (a more general case is described in [49]).

Of course, the more parameters we allow, the better the approximation. So, the question can be reformulated as follows: for a given number of parameters (i.e., for a given dimension of approximating family), which is the best family?

For simplicity, we will restrict ourselves to families of sets have analytical (or piece-wise analytical) boundaries, i.e., boundaries that can be described by an equation $F(x,y) = 0$ for some analytical function $F(x,y) = a + bx + cy + dx^2 + exy + fy^2 + \ldots$ Since we are interested in finite-dimensional families of sets, it is natural to consider finite-dimensional families of functions, i.e., families of the type $\{C_1 \cdot F_1(x,y) + \ldots + C_d \cdot F_d(x,y)\}$, where $F_i(z)$ are given analytical functions, and $C_1, \ldots, C_d$ are arbitrary (real) constants. So, the question is: which of such families is the best?

When we say "the best", we mean that on the set of all such families, there must be a relation $\geq$ describing which family is better or equal in quality. This relation must be transitive (if A is better than B, and B is better than C, then A is better than C). This relation is not necessarily asymmetric, because we can have two approximating families of the same quality. However, we would like to require that this relation be *final* in the sense that it should define a unique *best* family $A_{\rm opt}$ (i.e., the unique family for which $\forall B\,(A_{\rm opt} \geq B)$. Indeed:

- If none of the families is the best, then this criterion is of no use, so there should be *at least one* optimal family.
- If *several* different families are equally best, then we can use this ambiguity to optimize something else: e.g., if we have two families with the same approximating quality, then we choose the one which is easier to compute. As a result, the original criterion was not final: we get a new criterion ($A \geq_{\rm new} B$ if either A gives a better approximation, or if $A \sim_{\rm old} B$ and A is easier to compute), for which the class of optimal families is narrower. We can repeat this procedure until we get a final criterion for which there is only one optimal family.

It is reasonable to require that the relation $A \geq B$ should be invariance relative to natural geometric symmetries, i.e., shift-, rotation- and scale-invariant.

Now, we are ready for the formal definitions.

Definition 1. *Let $d > 0$ be an integer. By a d-dimensional family, we mean a family A of all functions of the type $\{C_1 \cdot F_1(x,y) + \ldots + C_d \cdot F_d(x,y)\}$, where $F_i(z)$ are given analytical functions, and $C_1, \ldots, C_d$ are arbitrary (real) constants. We say that a set is defined by this family A if its border consists of pieces described by equations $F(x,y) = 0$, with $F \in A$.*

Definition 2. *By an optimality criterion, we mean a transitive relation $\geq$ on the set of all d-dimensional families. We say that a criterion is final if there exists one and only one optimal family, i.e., a family $A_{\rm opt}$ for which $\forall B\,(A_{\rm opt} \geq B)$. We say that a criterion $\geq$ is shift- (corr., rotation- and scale-invariant) if for every two families A and B, $A \geq B$ implies $TA \geq TB$, where TA is a shift (rotation, scaling) of the family A.*

Theorem [35], [74]. ($d \leq 4$) *Let $\geq$ be a final optimality criterion which is shift-, rotation-, and scale-invariant, and let $A_{\rm opt}$ be the corresponding optimal family. Then, the border of every set defined by this family $A_{\rm opt}$ consists of straight line intervals and circular arcs.*

For $d = 5$ and $d = 6$, we also get hyperbolas, parabolas, and ellipses [57].

A similar symmetry-based optimization technique can be used to find the optimal technique for subdividing boxes in interval range estimation and interval optimization; see, e.g., [22].

B. Case Studies: Brief Overview

B.1 Analyzing Cotton Images

The above approach has been very helpful in the automatic analysis of cotton images [57], [63]. Specifically, the above symmetry-based approach helps in classifying trash (bark, leaves, etc.) in ginned cotton and in classifying insects by their shapes. The symmetry approach enables us not only to find the optimal shapes, but also to find the optimal geometric characteristics for distinguishing between different shapes and different sizes of the same size. The same symmetry approach leads to the conclusion that the optimal approximations to sizes form a geometric progression; this conclusion is in good accordance with the actual insect sizes.

B.2 Half-Orders of Magnitude

A similar geometric progression result explains why, when people make crude estimates, they feel comfortable choosing between alternatives which differ by a half-order of magnitude (e.g., were there 100, 300, or 1,000 people in the crowd), and less comfortable making a choice on a more detailed scale, with finer granules, or on a coarser scale (like 100 or 1,000) [19]. This empirical fact is difficult to explain within standard uncertainty formalisms like fuzzy logic; see, e.g., [33].

B.3 Analyzing Geospatial Data II

Computer processing can drastically improve the quality of an image and the reliability and accuracy of a spatial database. A large image (database) does not easily fit into the computer memory, so we process it by downloading pieces of the image. Each downloading takes a lot of time, so, to speed up the entire processing, we must use as few pieces as possible.

Many algorithms for processing images and spatial databases consist of comparing the value at a certain spatial location with values at nearby locations. For such algorithms, we must select (possibly overlapping) sub-images in such a way that for each point, its neighborhood (of given radius) belongs to a single sub-image. In [4], we formulate the corresponding optimization problem in precise terms, and show (in good accordance with the above optimization result) that the optimal sub-images should be bounded by straight lines or circular arcs.

B.4 Analyzing Geospatial Data III

Geospatial databases often contain erroneous measurements. For some such databases such as gravity databases, the known methods of detecting erroneous measurements – based on regression analysis – do not work well. As a result, to clean such databases, experts use manual methods which are very time-consuming. In [73], we propose a (natural) multi-level granular (localized) version of regression analysis as a technique for automatic cleaning. Specifically, we subdivide the original image into zones, and apply regression analysis separately within each zone (on the high-resolution level) and between different zones (on a low-resolution level).

In this physical problem, natural requirements lead to the following optimality criterion for selecting zones: minimizing the zone's diameter (that describes the variance within the zone) under given area (that describes the number of measurements within the zones). The efficiency of the resulting optimal zones is shown on the example of the gravity database, where our algorithm not only detected all erroneous measurements found manually by the experts;, but it also uncovered several suspicious points that the experts overlooked.

B.5 Non-Destructive Testing II

A standard way of detecting faults is to measure a certain quantity x at different points on the analyzed plate, and to classify the point as faulty is when the value x of the measured quantity at this point differs from the average a of measurement results by more than two or three σ.

Based on the results of measuring a single quantity (e.g., ultrasonic signal), we often miss some faults. To improve the quality of fault detection, it is necessary to measure several different quantities, and combine the results of these measurements. A natural idea is to classify the point as faulty is one of the measurement detects a fault. However, one of the measurements may be erroneous, we would rather consider a point a fault location if at least one other measured quantity at this or nearby point indicates a fault.

In other words, to improve the quality of fault detection, we replace the original point-by-point analysis by a new method which involves high-resolution clustering. When the corresponding neighborhoods are selected in an optimal way, this replacement indeed improves the quality of fault detection [60], [61].

A further improvement in fault detections comes when we treat the physically different points near the plate's edge as a different zone, and classify a point as faulty only if the corresponding value x differs from the average a_z within this zone by more than two or three standard deviations σ_z measured within this zone z. In other words, a further improvement in fault detection comes when we supplement the above high-resolution technique by additional low-resolution subdivision into zones.

B.6 Why Two Sigma

In the above example, and in statistics in general, a two-sigma criterion is used. The normal justification for this criterion is that for $k \approx 2$, the dependence of the probability to be outside the $k \cdot \sigma$ interval $[a - k \cdot \sigma, a + k \cdot \sigma]$ on the (unknown) probability distribution is the smallest. In [54], [55], we provide a theoretical explanation for this empirical fact, and thus, for the "2σ" criterion.

For that, we take into consideration the fact that an arbitrary probability distribution can be represented as $f(\eta)$, where η is normally distributed, so the choice of a distribution is equivalent to the choice of a function $f(x)$. An symmetry-based approach similar to the one presented above leads to the family $f(x) = x^\alpha$, and for this family, in the vicinity of normal distribution (when $\alpha \approx 1$), the smallest dependence on α is indeed attained for $k \approx 2$.

B.7 Acupuncture Points

The above approach to describing optimal shapes can be successfully applied to finding a good approximation for the location of the acupuncture points, i.e., points in which acupuncture treatment is the most efficient [48].

B.8 Towards Optimal Image Compression

In the above image processing problems, we process the image as it appears. In many situations, we must store the image for future use, and there is not enough storage space to store all the images, so we need to compress the image. In other situations, there is not enough bandwidth to send the entire image, so again, compression is needed.

It is proven that finding the optimal compression of a given image, be it an optimal lossless compression or an optimal lossy compression with a given bound on allowable loss of information, is a computationally difficult problem [68]. Since we cannot find the optimal compression, a natural idea is to consider several compression techniques and find the best one. The problem is to quantify what "the best" means, especially in the situations when we may have several possible applications of the compressed image, and since we do not know where exactly this image will be used, it is difficult to quantify the quality of the compression. In [24], [51], we consider the optimal choice of quality metric most appropriate for a given problem. First, we use a similar-based optimization approach to find the optimal family of possible quality metrics (which turns out to be L^p-metrics), and then, we find p based on a specific problem.

B.9 Pattern Matching

In many real-life situations, we are interested in finding the known pattern in a given image. For example, in the analysis of geospatial data, we may be looking for certain geophysical patterns indicative of, say, presence of water. In [11], [13], [14], [15], [64], [81], a similar symmetry-based optimality approach is used to develop optimal FFT-based techniques for such matching.

B.10 Guaranteed Quality Estimation for Approximately Given Systems

Our final example bring us back to the original problem – of quality estimation for an approximately given system. Symmetry-based approach can help in designing optimal methods for such quality estimation for the situations when the system is treated as a "black box", a low-resolution approximation to the original system in which we are not allowed to use the high-resolution details [25], [69]. In particular, in [25], [69], we describe modified Monte-Carlo techniques which provide us with validated results even when we do not know the exact values of the statistical characteristics of the system – only intervals of possible values of such characteristics.

IV. Multi-Level Granular Approach to Reasoning and Logic: A Brief Overview

A. Reasoning and Logic: Successes and Problems

Multi-level granular approach can be applied not only to the systems themselves, but also to the way we reason about these systems, i.e., to the *logic* of human reasoning. Specifically, in many areas (medicine, geophysics, military decision-making, etc.), top quality experts make good decisions, but they cannot handle all situations. It is therefore desirable to incorporate their knowledge into a decision-making computer system.

Experts describe their knowledge by statements $S_1, \ldots, S_n$ (e.g., by if-then rules). Experts are often not 100% sure about these statements S_i; this uncertainty is described by the *subjective probabilities* p_i (degrees of belief, etc.) which experts assign to their statements. The conclusion C of an expert system normally depends on several statements S_i. For example, if we can deduce C either from S_2 and S_3, or from S_4, then the validity of C is equivalent to the validity of a Boolean combination $(S_2 \,\&\, S_3) \vee S_4$. So, to estimate the reliability $p(C)$ of the conclusion, we must estimate the probability of Boolean combinations. In this paper, we consider the simplest possible Boolean combinations are $S_1 \,\&\, S_2$ and $S_1 \vee S_2$.

In general, the probability $p(S_1 \,\&\, S_2)$ of a Boolean combination can take different values depending on whether S_1 and S_2 are independent or correlated. So, to get the precise estimates of probabilities of all possible conclusions, we must know not only the probabilities $p(S_i)$ of individual statements, but also the probabilities of all possible Boolean combinations. To get all such probabilities, it is sufficient to describe 2^n probabilities of the combinations $S_1^{\varepsilon_1} \,\&\, \ldots \,\&\, S_n^{\varepsilon_n}$, where $\varepsilon_i \in \{+,-\}$, S^+ means S, and S^- means $\neg S$. The only condition on these probabilities is that their sum should add up to 1, so we need to describe $2^n - 1$ different values. A typical knowledge base may contain hundreds of statements; in this case, the value $2^n - 1$ is astronomically large. We cannot ask experts about all 2^n such combinations, so in many cases, we must estimate $p(S_1 \,\&\, S_2)$ or $p(S_1 \vee S_2)$ based only on the values $p_1 = p(S_1)$ and $p_2 = p(S_2)$. There exist many possible "and"-operations $f_\& : [0,1] \times [0,1] \to [0,1]$ which transform the degrees p_1 and p_2 into an estimate $f_\&(p_1, p_2)$ for $p(S_1 \,\&\, S_2)$. Similarly, there exist many "or"-operations which transform degrees the p_1 and p_2 into an estimate $f_\vee(p_1, p_2)$ for $p(S_1 \vee S_2)$.

Many such operations have been successfully used in fuzzy logic and intelligent control; see, e.g., [23], [58]. In spite of the successes, there are still major problems with these operations:

- First, these operations are not perfect. Indeed, some of these operations, although very natural and useful at first glance, seem to violate natural commonsense requirements; we will give an example later).
- Second, there are so many different possible "and"- and "or"-operations that it is difficult to meaningfully select one of them. Any guidance for decreasing the class of possible operations is very welcome.

B. Reasoning and Logic: Multi-Level Granular Approach

In our viewpoint, the above problems of the existing logical methodologies come, to a large extent, from the fact that researchers often combine different degrees of certainty together. In reality, the degrees have a clear multi-level granular character, and if we fully take this character into consideration, we can make a large progress in solving the above problems.

Let us explain why expert degrees of uncertainty are of multi-level granular nature. An expert rarely provides us with numbers describing his or her degrees of uncertainty. A more natural way for an expert to describe his/her degree of belief in a certain statement is to use a word from natural language such as "most probably" or "possibly", and then we translate this word into a number. There are only few such words, and these words form the lowest-resolution level of the uncertainty description. On this level, several different statements with slightly different degrees of uncertainty may be described by the same word and thus, lumped into a single cluster. To avoid this lumping, we may ask an expert to provide us with a more detailed description of the expert's degree, e.g., by using hedged combinations of words like "slightly less certain but still reasonably certain". The more details we ask, the more higher-resolution description we get.

Another possibility to describe the expert's degrees in numerical terms is to ask the expert to describe his/her degrees on a scale from, say, 0 to 10. We can start with a low-resolution scale, e.g., with a scale consisting of only two values "yes" and "no" that corresponds to the use of the classical (two-valued) logic. As we increase the number of elements on the scale, we get a higher- and higher-resolution description. Eventually, we get real numbers describing uncertainty.

In both cases, we get numbers as a result, but these numbers appear as a result of a multi-level granular procedure. It is therefore natural, when resolving the above problems – of seeming inconsistency with common sense and of too many options – to consider not only the resulting assignments of numbers, but also the multi-level granular approximations to these assignments. This consideration indeed helps in solving the above problems.

C. Multi-Level Granular Character of Uncertainty Reasoning Resolves the Inconsistency Between Uncertainty Operations and Common Sense

Let us give one example of such inconsistency and show how the multi-level granular character of human reasoning can help with this particular example. It is known that for given $p_1 = p(S_1)$ and $p_2 = p(S_2)$, possible values of $p(S_1 \,\&\, S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1+p_2-1, 0)$ and $p^+ = \min(p_1, p_2)$; and possible values of $p(S_1 \vee S_2)$ form an interval $\mathbf{p} = [p^-, p^+]$, where $p^- = \max(p_1, p_2)$ and $p^+ = \min(p_1+p_2, 1)$ (see, e.g., a survey [50] and references therein). So, in principle, we can use such interval estimates and get an interval $\mathbf{p}(C)$ of possible values of $p(C)$. Sometimes, this idea leads to meaningful estimates, but often, it leads to a useless $\mathbf{p}(C) = [0, 1]$ [49], [59]. In such situations, it is reasonable, instead of using the entire interval $\mathbf{p}$, to select a point within this interval as a reasonable estimate for $p(S_1 \,\&\, S_2)$ (or, correspondingly, for $p(S_1 \vee S_2)$).

Since the only information we have, say, about the unknown probability $p(S_1 \,\&\, S_2)$ is that it belongs to the interval $[p^-, p^+]$, it is natural to select a *midpoint* of this interval as the desired estimate:

$$f_\&(p_1, p_2) \stackrel{\rm def}{=} \frac{1}{2} \cdot \max(p_1 + p_2 - 1, 0) + \frac{1}{2} \cdot \min(p_1, p_2);$$

$$f_\vee(p_1, p_2) \stackrel{\rm def}{=} \frac{1}{2} \cdot \max(p_1, p_2) + \frac{1}{2} \cdot \min(p_1 + p_2, 1).$$

This midpoint selection is not only natural from a common sense viewpoint; it also has a deeper justification. Namely, in accordance of our above discussion, for $n = 2$ statements S_1 and S_2, to describe the probabilities of all possible Boolean combinations, we need to describe $2^2 = 4$ probabilities $x_1 = p(S_1 \,\&\, S_2)$, $x_2 = p(S_1 \,\&\, \neg S_2)$, $x_3 = p(\neg S_1 \,\&\, S_2)$, and $x_4 = p(\neg S_1 \,\&\, \neg S_2)$; these probabilities should add up to 1: $x_1 + x_2 + x_3 + x_4 = 1$. Thus, each probability distribution can be represented as a point $(x_1, \ldots, x_4)$ in a 3-D simplex $\mathcal{S} = \{(x_1, x_2, x_3, x_4) \mid x_i \geq 0 \,\&\, x_1 + \ldots + x_4 = 1\}$. We know the values of $p_1 = p(S_1) = x_1 + x_2$ and $p_2 = p(S_2) = x_1 + x_3$, and we are interested in the values of $p(S_1 \,\&\, S_2) = x_1$ and $p(S_1 \vee S_2) = x_1 + x_2 + x_3$. It is natural to assume that *a priori*, all probability distributions (i.e., all points in a simplex $\mathcal{S}$) are "equally possible", i.e., that there is a uniform distribution ("second-order probability") on this set of probability distributions. Then, as a natural estimate for the probability $p(S_1 \,\&\, S_2)$ of $S_1 \,\&\, S_2$, we can take the conditional mathematical expectation of this probability under the condition that the values $p(S_1) = p_1$ and $p(S_2) = p_2$:

$$E(p(S_1 \,\&\, S_2) \mid p(S_1) = p_1 \,\&\, p(S_2) = p_2) =$$

$$P(x_1 \mid x_1 + x_2 = p_1 \,\&\, x_1 + x_3 = p_2).$$

The problem is that these operations are non-associative. Why is this a problem? If we are interested in estimating the degree of belief in a conjunction of three statements $S_1 \,\&\, S_2 \,\&\, S_3$, then we can either apply the "and" operation to p_1 and p_2 and get an estimate $f_\&(p_1, p_2)$ for the probability of $S_1 \,\&\, S_2$ and then, we apply the "and" operation to this estimate and p_3, and get an estimate $f_\&(f_\&(p_1, p_2), p_3)$ for the probability of $(S_1 \,\&\, S_2) \,\&\, S_3$. Alternatively, we can start by combining S_2 and S_3, and get an estimate $f_\&(p_1, f_\&(p_2, p_3))$. Intuitively, we would expect these two estimates to coincide, but, e.g., $(0.4 \,\&\, 0.6) \,\&\, 0.8 = 0.2 \,\&\, 0.8 = 0.1$, while $0.4 \,\&\, (0.6 \,\&\, 0.8) = 0.4 \,\&\, 0.5 = 0.2 \neq 0.1$.

How can we solve this problem? Since we know that the numerical values are only an approximation, we can analyze how non-associative the above operations can be. If the difference is below the natural resolution level, then, from the practical point of view, the above operations are as good as associative ones. The following is true:

Theorem [16], [40], [70]**.**

$$\max_{a,b,c} |f_\&(f_\&(a, b), c) - f_\&(a, f_\&(b, c))| = \frac{1}{9};$$

$$\max_{a,b,c} |f_\vee(f_\vee(a, b), c) - f_\vee(a, f_\vee(b, c))| = \frac{1}{9}.$$

Each word describing a degree of belief is a "granule" covering the entire sub-interval of values. Thus, non-associativity is negligible if the corresponding realistic "granular" degree of belief have granules of width $\geq 1/9$. One can fit no more than 9 granules of such width in the interval $[0, 1]$. This may explain why humans are most comfortable with ≤ 9 items to choose from – the famous "7 plus minus 2" law; see, e.g., [44], [45].

D. *Multi-Level Granular Character of Uncertainty Reasoning Helps to Drastically Narrow Down the Class of Possible Logics*

These results cover both the logics in which the set of different degrees is an interval $[0, 1]$, and more complex logics.

D.1 $[0, 1]$-Based Logics

For *numerical* operations, if we interpret the degree of belief in a statement S as (proportional to) the number of arguments in favor of S, then we arrive at a natural choice of "and"- and "or" operations: $f_{\&}(a, b) = a \cdot b$, $f_{\vee}(a, b) = a + b$, and $f_{\vee}(a, b) = b^a$. As one of the unexpected consequences, we get a surprising relation with the entropy techniques, well known in probabilistic approach to uncertainty [62].

A similar conclusion can be made if we require that the operations be consistent with their multi-level granular structure: namely, for a discrete low-resolution level, we define "derivatives" of these operations as finite differences, and then require that the corresponding continuous limit operations have exactly the same expressions for the derivatives [5].

The multi-level granular character of human reasoning also explains why in logic, only unary and binary operations are normally used: because although in principle, there exist ternary operations on $[0, 1]$ (in the limit case) which cannot be represented as compositions of natural unary and binary ones, but on each resolution level, when we have only finitely many degrees, every operation can be naturally represented as such a composition [53].

D.2 More General Logics

The need for more general logics comes from the fact that just like experts are not sure about the statement S, they are also not sure about their own degrees of belief $d(S)$. Thus, instead of a single number $d(S)$, we can consider several possible numbers d, with degrees $d_2(d)$ describing to what extent these numbers are adequate descriptions of the original expert's uncertainty. This "second-order" approach has several successful applications. In principle, it is possible to go further and consider the fact that the degrees $d_2(d)$ are also not given precisely, so we seem to need the third-, fourth-order etc, approaches. However, in practice, such theoretically possible approaches turned out to be not useful. This fact can be explained if we take the multi-level granular character of reasoning into consideration:

- On the one hand, every "first-order" and "second-order" logic, in which the set of degree of belief is an ordered set, can be naturally described as a limit of an interval-related multi-level granular procedure [29], [30], [47], [79].
- On the other hand, if degrees come from words, then the third order is no longer necessary [32].

It is natural to select a continuous approach which best reflects the multi-level granular character of human reasoning, i.e., in which there is a qualitative difference between different pairs of degrees. A natural way to describe this difference in continuous case is to use the approach of non-standard analysis, with the actual infinitesimal elements (= lexicographic ordering). The optimal selection of such logics is described in [39], [56].

Conclusion

A natural way to analyze complex systems is to apply granulation. Granulation simplifies the analysis of the system and thus makes this analysis feasible, but it also makes the results of this analysis less accurate. To combine feasibility and accuracy, we must supplement the original low-level granulation, with few granules, with more high-resolution granulations. In other words, to analyze complex systems, we need to use multi-level granulation techniques. These techniques enable us first, to analyze the system on the low-resolution level, and then, to sharpen the results of the low-resolution analysis by considering higher-resolution representations of the analyzed system.

The analysis of each resolution level provides us with an approximate value of a desired characteristic of a system. In order to make a definite conclusion, we need to know the accuracy of this approximation. In this paper, we overview interval mathematics – a methodology for estimating such accuracy. This interval approach is also extremely important for tessellating the space of search when searching for optimal control. We survey the corresponding theoretical results, and present several case studies.

Acknowledgments

This work was supported in part by NASA under cooperative agreement NCC5-209 and grant NCC 2-1232, by the Future Aerospace Science and Technology Program (FAST) Center for Structural Integrity of Aerospace Systems, effort sponsored by the Air Force Office of Scientific Research, Air Force Materiel Command, USAF, under grant number F49620-00-1-0365, by NSF grants CDA-9522207 and 9710940 Mexico/Conacyt, and by the Grant No. W-00016 from the U.S.-Czech Science and Technology Joint Fund. A preliminary version of this paper was presented at the NIST-sponsored conference [26].

The authors are very grateful to Andrzej Bargiela, Witold Pedrycz, and Alex Meystel for their support, encouragement, and helpful discussions.

References

[1] G. Alefeld and J. Herzberger, *Einführung in die Intervallrechnung*, Bibliographisches Institut, Mannheim, Wien, Zürich, 1974.

[2] R. Al-Jamal, S. Manoli, A. E. Brito, and O. Kosheleva, "Interval + Communications = Walsh: For Signal Multiplexing Under In-

terval Uncertainty, Walsh Functions are Optimal", *Proc. the Joint 9th World Congress of the Int'l Fuzzy Systems Association and 20th Int'l Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1677–1681, 2001.

[3] A. Bargiela, "Interval and ellipsoidal uncertainty models", In: W. Pedrycz (ed.), *Granular Computing: An Emerging Paradigm*, Springer-Verlag, Heidelberg, 2001, pp. 23–57.

[4] J. Beck, V. Kreinovich, and B. Penn, "Geombinatoric aspects of processing large images and large spatial databases", *Geombinatorics*, Vol. 11, No. 4, pp. 6–12, 2001.

[5] B. Bouchon-Meunier, H. T. Nguyen, and V. Kreinovich, "Discrete (set) derivatives and 'algebraic' fuzzy logic operations", *Proc. FUZZ-IEEE'2001*, Melbourne, Australia, 2001.

[6] A. E. Brito and O. Kosheleva, "Interval + Image = Wavelet: For Image Processing under Interval Uncertainty, Wavelets are Optimal", *Reliable Computing*, Vol. 4, No. 3, pp. 291–301, 1998.

[7] A. E. Brito, O. M. Kosheleva, and S. D. Cabrera, "Multi-Resolution Data Processing is Optimal: Case Study of Detecting Surface Mounted Devices", *Proceedings of the International Conference on Intelligent Systems and Semiotics (ISAS'97)*, National Institute of Standards and Technology Publ., Gaithersburg, MD, pp. 157–161, 1997.

[8] D. D. Coblentz, V. Kreinovich, B. S. Penn, and S. A. Starks, "Towards Reliable Sub-Division of Geological Areas: Interval Approach" *Proc. 19th Int'l Conference of the North American Fuzzy Information Society NAFIPS'2000*, Atlanta, Georgia, July 13–15, pp. 368–372, 2000.

[9] D. D. Coblentz, V. Kreinovich, B. S. Penn, and S. A. Starks, "Towards Reliable Sub-Division of Geological Areas: Interval Approach", In: L. Reznik and V. Kreinovich (eds.), *Soft Computing in Measurements and Information Acquisition*, Springer-Verlag, 2002 (to appear).

[10] L. O. Fuentes and V. Kreinovich, "Towards Intelligent Virtual Environment for Teaching Telemanipulation Operators: Virtual Tool Approach and its Interval-Based Justification", *Proceedings of the 2nd International Workshop on Intelligent Virtual Environments*, Xalapa, Veracruz, Mexico, September 11–12, pp. 8–11, 1998.

[11] F. Garcia, R. Araiza, and B. Rzycki, "Towards Optimal Mosaicking of Multi-Spectral Images", *Proceedings of the Second National NASA Student Conference on Earth Sciences, Aero-Space Technology, Human Exploration and Development of Space, Space Science, and Crosscutting Technologies*, Nashville, Tennessee, April 7–10, pp. 261–266, 2000.

[12] O. N. Garcia, V. Kreinovich, L. Longpré, and H. T. Nguyen, "Complex problems: granularity is necessary, granularity helps", In: Nguyen Hoang Phuong and A. Ohsato (eds.), *Proceedings of the Vietnam-Japan Bilateral Symposium on Fuzzy Systems and Applications VJFUZZY'98*, HaLong Bay, Vietnam, 30th September–2nd October, pp. 449–455, 1998.

[13] S. Gibson, *An optimal FFT-based algorithm for mosaicking images*, Master Thesis, Department of Computer Science, University of Texas at El Paso, 1999.

[14] S. Gibson, O. Kosheleva, L. Longpre, B. Penn, and S. A. Starks, "An Optimal FFT-Based Algorithm for Mosaicking Images, With Applications to Satellite Imaging and Web Search" *Proceedings of the Joint Conferences in Information Sciences JCIS'2000*, Atlantic City, NJ, February 27–March 3, Vol. I, pp. 248–251, 2000.

[15] S. Gibson, V. Kreinovich, L. Longpré, B. Penn, and S. A. Starks, "Intelligent Mining in Image Databases, With Applications to Satellite Imaging and to Web Search", In: A. Kandel, H. Bunke, and M. Last (eds.), *Data Mining and Computational Intelligence*, Springer-Verlag, Berlin, pp. 309–336, 2001.

[16] I. R. Goodman, R. A. Trejo, V. Kreinovich, J. Martinez, and R. Gonzalez, "An even more realistic (non-associative) interval logic and its relation to psychology of human reasoning", *Proc. Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1586–1591, 2001.

[17] R. Hammer, M. Hocks, U. Kulisch, D. Ratz, *Numerical toolbox for verified computing. I. Basic numerical problems*, Springer Verlag, Heidelberg, N.Y., 1993.

[18] E. R. Hansen, *Global optimization using interval analysis*, Marcel Dekker, N.Y., 1992.

[19] J. R. Hobbs and V. Kreinovich, "Optimal Choice of Granularity In Commonsense Estimation: Why Half-Orders of Magnitude", *Proc. Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1343–1348, 2001.

[20] R. B. Kearfott, *Rigorous Global Search: Continuous Problems*, Kluwer, Dordrecht, 1996.

[21] R. B. Kearfott and V. Kreinovich (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht, 1996.

[22] R. B. Kearfott and V. Kreinovich, "Where to Bisect a Box? A Theoretical Explanation of the Experimental Results", In: Goetz Alefeld and Raul A. Trejo (eds.), *Interval Computations and its Applications to Reasoning Under Uncertainty, Knowledge Representation, and Control Theory. Proceedings of MEXICON'98, Workshop on Interval Computations, 4th World Congress on Expert Systems*, México City, México, 1998.

[23] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[24] O. Kosheleva, V. Kreinovich, and Y. Yam, "On the optimal choice of quality metric in image compression", *Proceedings of the International Symposium on Smart Structures and Microsystems (IS3M)*, Hong Kong, October 19–21, 2000, Paper C3-3.

[25] V. Kreinovich, "A simplified version of the tomography problem can help to estimate the errors of indirect measurements", In: A. Mohamad-Djafari (ed.), *Bayesian Inference for Inverse Problems*, Proceedings of the SPIE/International Society for Optical Engineering, Vol. 3459, San Diego, CA, pp. 106–115, 1998.

[26] V. Kreinovich and R. Aló, "Interval Mathematics for Analysis of Multiresolutional Systems", *Proceedings of the International Workshop on Measuring Performance and Intelligence of Intelligent Systems PERMIS'01*, Mexico City, September 4, 2001.

[27] V. Kreinovich and L. O. Fuentes, "Telemanipulation: The Virtual Tool Approach and Its Interval-Based Justification", In: G. Alefeld and R. A. Trejo (eds.), *Interval Computations and its Applications to Reasoning Under Uncertainty, Knowledge Representation, and Control Theory. Proceedings of MEXICON'98, Workshop on Interval Computations, 4th World Congress on Expert Systems*, México City, México, 1998.

[28] V. Kreinovich, A. Lakeyev, J. Rohn, and P. Kahl, *Computational complexity and feasibility of data processing and interval computations*, Kluwer, Dordrecht, 1998.

[29] V. Kreinovich, M. Mukaidono, "Intervals (Pairs of Fuzzy Values), Triples, etc.: Can We Thus Get an Arbitrary Ordering?", *Proceedings of the 9th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE'2000)*, San Antonio, Texas, May 7–10, Vol. 1, pp. 234–238, 2000.

[30] V. Kreinovich, M. Mukaidono, and K. Atanassov, "From Fuzzy Values To Intuitionistic Fuzzy Values To Intuitionistic Fuzzy Intervals etc.: Can We Get an Arbitrary Ordering?", *Notes on Intuitionistic Fuzzy Sets (NIFS)*, Vol. 5, No. 3, pp. 11–18, 1999.

[31] V. Kreinovich and H. T. Nguyen, "Granularity as an Optimal Approach to Uncertainty – A General Mathematical Idea With Applications to Sleep, Consumption, Traffic Control, Learning, etc.", *Proc. 19th Int'l Conference of the North American Fuzzy Information Society NAFIPS'2000*, Atlanta, Georgia, July 13–15, pp. 316–320, 2000.

[32] V. Kreinovich and H. T. Nguyen, "1st Order, 2nd Order, What Next? Do We Really Need Third-Order Descriptions: A View From A Realistic (Granular) Viewpoint", *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1908–1913, 2001.

[33] V. Kreinovich and H. T. Nguyen, and W. Pedrycz, "How to Make Sure That "≈ 100" + 1 Is ≈ 100 in Fuzzy Arithmetic: Solution and Its (Inevitable) Drawbacks", *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1653–1658, 2001.

[34] V. Kreinovich and J. Wolff von Gudenberg, "Arithmetic of complex sets: Nickel's classical paper revisited from a geometric viewpoint", *Geombinatorics*, Vol. 9, No. 1, pp. 21–26, 1999.

[35] V. Kreinovich and J. Wolff von Gudenberg, "An optimality criterion for arithmetic of complex sets", *Geombinatorics*, Vol. 10, No. 1, pp. 31–37, 2000.

[36] V. Kreinovich and Y. Yam, "Why Clustering in Function Approximation? Theoretical Explanation", *International Journal of Intelligent Systems*, 2002 (to appear).

[37] M. Krishna, *Using symmetries to reduce the size of a fuzzy knowledge base, with an application to fault detection*, Master Thesis, Department of Computer Science, University of Texas at El Paso, 1999.

[38] M. Krishna, V. Kreinovich, and R. Osegueda, "Fuzzy Logic in Non-Destructive Testing of Aerospace Structures", In: Jaime Ramírez-Angulo (ed.), *Proceedings of the 1999 IEEE Midwest Symposium on Circuits and Systems*, Las Cruces, New Mexico, August 8–11, 1999, Vol. 1, pp. 431–434.

[39] C. Langrand, V. Kreinovich, and H. T. Nguyen, "Two-dimensional fuzzy logic for expert systems", *Sixth International Fuzzy Systems Association World Congress*, San Paulo, Brazil, July 22–28, Vol. 1, pp. 221–224, 1995.

[40] J. Martinez, L. Macias, A. Esper, J. Chaparro, V. Alvarado, S. A. Starks, and V. Kreinovich, "Towards more realistic (e.g., non-associative) and- and or-operations in fuzzy logic", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2187–2192.

[41] N. J. McMillan, "Temporal and spatial magmatic evolution of the Rio Grande rift", *New Mexico Geological Survey Guidebook, 49th Field Conference*, Las Cruces County, 107–116, 1998.

[42] A. Meystel, "Planning in a hierarchical nested controller for autonomous robots," *Proc. IEEE 25th Conf. on Decision and Control*, Athens, Greece, pp. 1237–1249, 1986.

[43] A. Meystel, "Theoretical Foundations of Planning and Navigation for Autonomous Mobile Systems", *International Journal of Intelligent Systems*, Vol. 2, No. 2, pp. 73–128, 1987.

[44] G. A. Miller, "The magical number seven plus or minus two: some limits on our capacity for processing information", *Psychological Review*, Vol. 63, pp. 81–97, 1956.

[45] P. M. Milner, *Physiological psychology*, Holt, NY, 1970.

[46] R. Moore, *Methods and Applications of Interval Analysis*, SIAM, Philadelphia, 1979.

[47] M. Mukaidono, Y. Yam, and V. Kreinovich, "Intervals is All We Need: An Argument", *Proceedings of The Eighth International Fuzzy Systems Association World Congress IFSA'99*, Taipei, Taiwan, August 17–20, 1999, pp. 147–150.

[48] Hoang Phuong Nguyen, S. Starks, and V. Kreinovich, "Towards foundations for traditional oriental medicine", In: Nguyen Hoang Phuong and A. Ohsato (eds.), *Proceedings of the Vietnam-Japan Bilateral Symposium on Fuzzy Systems and Applications VJFUZZY'98*, HaLong Bay, Vietnam, 30th September–2nd October, pp. 704–708, 1998.

[49] H. T. Nguyen and V. Kreinovich, *Applications of continuous mathematics to computer science*, Kluwer, Dordrecht, 1997.

[50] H. T. Nguyen, V. Kreinovich, and Q. Zuo, "Interval-valued degrees of belief: applications of interval computations to expert systems and intelligent control", *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems (IJUFKS)*, 1997, Vol. 5, No. 3, pp. 317–358.

[51] H. T. Nguyen, O. Kosheleva, V. Kreinovich, and L. Ding, "On the Optimal Choice of Quality Metric In Image Compression: A Soft Computing Approach", *Proceedings of the Sixth International Conference on Control, Automation, Robotics and Vision ICARCV'2000*, Singapore, December 5–8, 2000.

[52] H. T. Nguyen and V. Kreinovich, "Nested Intervals and Sets: Concepts, Relations to Fuzzy Sets, and Applications", In: R. B. Kearfott et al (eds.), *Applications of Interval Computations*, Kluwer, Dordrecht, pp. 245–290, 1996.

[53] H. T. Nguyen, V. Kreinovich, and I. R. Goodman, "Why Unary and Binary Operations in Logic: General Result Motivated by Interval-Valued Logics", *Proceedings of the Joint 9th World Congress of the International Fuzzy Systems Association and 20th International Conference of the North American Fuzzy Information Processing Society IFSA/NAFIPS 2001*, Vancouver, Canada, July 25–28, pp. 1991–1996, 2001.

[54] H. T. Nguyen, V. Kreinovich, G. N. Solopchenko, and C.-W. Tao, "Why Two Sigma? A Theoretical Justification", In: L. Reznik and V. Kreinovich (eds.), *Soft Computing in Measurements and Information Acquisition*, Springer-Verlag, 2002 (to appear).

[55] H. T. Nguyen, V. Kreinovich, and C.-W. Tao, "Why 95% and Two Sigma? A Theoretical Justification for an Empirical Measurement Practice", *Proc. International Workshop on Intelligent Systems Resolutions: The 8th Bellman Continuum*, Taipei, Taiwan, December 11–12, pp. 358–362, 2000.

[56] H. T. Nguyen, V. Kreinovich, and B. Wu, "Fuzzy/probability ∼ fractal/smooth", *International Journal of Uncertainty, Fuzziness, and Knowledge-Based Systems (IJUFKS)*, Vol. 7, No. 4, pp. 363–370, 1999.

[57] H. T. Nguyen, N. R. Prasad, V. Kreinovich, and H. Gassoumi, "Some Practical Applications of Soft Computing and Data Mining", In: A. Kandel, H. Bunke, and M. Last (eds.), *Data Mining and Computational Intelligence*, Springer-Verlag, Berlin, pp. 273–307, 2001.

[58] H. T. Nguyen and E. A. Walker, *First Course in Fuzzy Logic*, CRC Press, Boca Raton, FL, 1999.

[59] N. J. Nilsson, "Probabilistic logic", *Artificial Intelligence*, Vol. 28, pp. 71–87, 1986.

[60] R. A. Osegueda, C. Ferregut, V. Kreinovich, S. Seetharami, and H. Schulte, "Fuzzy (Granular) Levels of Quality, With Applications to Data Mining and to Structural Integrity of Aerospace Structures", *Proceedings of the 19th International Conference of the North American Fuzzy Information Society NAFIPS'2000*, Atlanta, Georgia, July 13–15, pp. 348–352, 2000.

[61] R. A. Osegueda, S. R. Seelam, A. C. Holguin, V. Kreinovich, and C.-W. Tao, "Statistical and Dempster-Shafer Techniques in Testing Structural Integrity of Aerospace Structures", *Proc. of International Conference on Intelligent Technologies*, Bangkok, Thailand, December 13–15, 2000, pp. 383–389.

[62] P. Santiprabhob, H. T. Nguyen, W. Pedrycz, and V. Kreinovich, "Logic-motivated choice of fuzzy logic operators", *Proc. FUZZ-IEEE'2001*, Melbourne, Australia, 2001.

[63] M. Siddaiah, M. A. Lieberman, N. R. Prasad, and V. Kreinovich, "A geometric approach to classification of trash in ginned cotton", *Geombinatorics*, Vol. 10, No. 2, pp. 77–82, 2000.

[64] S. Srikrishnan, R. Araiza, H. Xie, S. A. Starks, and V. Kreinovich, "Automatic referencing of satellite and radar images", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2176–2181.

[65] S. A. Starks and V. Kreinovich, "Multi-spectral inverse problems in satellite image processing", In: A. Mohamad-Djafari (ed.), *Bayesian Inference for Inverse Problems*, Proceedings of the SPIE/International Society for Optical Engineering, Vol. 3459, San Diego, CA, pp. 138–146, 1998.

[66] S. A. Starks and V. Kreinovich, "Locating the whole pattern is better than locating its pieces: a geometric explanation of an empirical phenomenon," *Geombinatorics*, Vol. 8, No. 4, pp. 116–121, 1999.

[67] S. A. Starks and V. Kreinovich, "Aerospace applications of soft computing and interval computations (with an emphasis on multi-spectral satellite imaging)", In: M. Jamshidi, M. Fathi, and T. Furunashi (eds.), *Soft Computing, Multimedia, and Image Processing. Proceedings of the 2000 World Automation Congress WAC'2000, Maui, Hawaii, June 11–16, 2000*, TSI Press, Albuquerque, pp. 644–651, 2000.

[68] S. Subbaramu, A. Q. Gates, and V. Kreinovich, "Application of Kolmogorov Complexity to Image Processing: It Is Possible to Have a Better Compression, but It Is Not Possible to Have the Best One", *Bulletin of the European Association for Theoretical Computer Science (EATCS)*, Vol. 69, pp. 145–150, 1999.

[69] R. Trejo and Vladik Kreinovich, "Error Estimations for Indirect Measurements: Randomized vs. Deterministic Algorithms For 'Black-Box' Programs", S. Rajasekaran, P. Pardalos, J. Reif, and J. Rolim (eds.), *Handbook on Randomized Computing*, Kluwer, 2001.

[70] R. Trejo, V. Kreinovich, I. R. Goodman, J. Martinez, and R. Gonzalez, "A Realistic (Non-Associative) Logic And a Possible Explanations of 7+-2 Law", *International Journal of Approximate Reasoning*, 2002, Vol. 29, pp. 235–266.

[71] K. Villaverde and V. Kreinovich, "A linear-time algorithm that locates local extrema of a function of one variable from interval measurement results," *Interval Computations*, No. 4, pp. 176–194, 1993.

[72] K. Villaverde and V. Kreinovich, "Parallel algorithm that locates local extrema of a function of one variable from interval measurement results", *Reliable Computing*, Supplement (Extended Abstracts of APIC'95: International Workshop on Applications of Interval Computations, El Paso, TX, Febr. 23–25, 1995), pp. 212–219, 1995.

[73] Q. Wen, A. Q. Gates, J. Beck, V. Kreinovich, and G. R. Keller, "Towards automatic detection of erroneous measurement results in a gravity database", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2170–2175.

[74] J. Wolff von Gudenberg and V. Kreinovich, "Candidate Sets for Complex Interval Arithmetic", In: Hrushikesha Mohanty and Chitta Baral (eds.), *Trends in Information Technology, Proceeedings of the International Conference on Information Technology ICIT'99, Bhubaneswar, India, December 20–22, 1999*, Tata McGraw-Hill, New Delhi, pp. 230–233, 2000.

[75] K. Worden, R. Osegueda, C. Ferregut, S. Nazarian, D. L. George, M. J. George, V. Kreinovich, O. Kosheleva, and S. Cabrera, "Interval Methods in Non-Destructive Testing of Aerospace Structures and in Mammography", *International Conference on Interval Methods and their Application in Global Optimization (INTERVAL'98), April 20–23, Nanjing, China, Abstracts*, pp. 152–154, 1998.

[76] K. Worden, R. Osegueda, C. Ferregut, S. Nazarian, E. Rodriguez, D. L. George, M. J. George, V. Kreinovich, O. Kosheleva, and S. Cabrera, "Interval Approach to Non-Destructive Testing of Aerospace Structures and to Mammography", In: Goetz Alefeld and Raul A. Trejo (eds.), *Interval Computations and its Applications to Reasoning Under Uncertainty, Knowledge Representation, and Control Theory. Proceedings of MEXICON'98, Workshop on Interval Computations, 4th World Congress on Expert Systems*, México City, México, 1998.

[77] K. Worden, R. Osegueda, C. Ferregut, S. Nazarian, D. L. George, M. J. George, V. Kreinovich, O. Kosheleva, and S. Cabrera, "Interval Methods in Non-Destructive Testing of Material Structures", *Reliable Computing*, Vol. 7, No. 4, pp. 341–352, 2001.

[78] website on interval computations http://www.cs.utep.edu/interval-comp

[79] Y. Yam, M. Mukaidono, and V. Kreinovich, "Beyond [0,1] to Intervals and Further: Do We Need All New Fuzzy Values?", *Proceedings of The Eighth International Fuzzy Systems Association World Congress IFSA'99*, Taipei, Taiwan, August 17–20, pp. 143–146, 1999.

[80] Y. Yam, H. T. Nguyen, and V. Kreinovich, "Multi-Resolution Techniques in the Rules-Based Intelligent Control Systems: A Universal Approximation Result", *Proceedings of the 14th IEEE International Symposium on Intelligent Control/Intelligent Systems and Semiotics ISIC/ISAS'99*, Cambridge, Massachusetts, September 15–17, pp. 213–218, 1999.

[81] H. Xie, N. Hicks, G. R. Keller, H. Huang, and V. Kreinovich, "Automatic image registration based on a FFT algorithm and IDL/ENVI", *Proceedings of the ICORG-2000 International Conference on Remote Sensing and GIS/GPS*, Hyderabad, India, December 1–4, 2000.

[82] L. A. Zadeh, "Fuzzy sets and information granularity", In: M. Gupta, R. Ragade, and R. Yager (eds.), *Advances in Fuzzy Set Theory and Application*, North Holland, Amsterdam, 1979, pp. 3–18.