Exact Upper Bound on the Mean of the Multiple Product

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Abstract

In practice, in addition to the intervals $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ of possible values of inputs $x_1, \ldots, x_n$, we sometimes also know their means E_i. For such cases, we provide an explicit exact (= best possible) upper bound for the mean of the product $x_1 \cdot \ldots \cdot x_n$ of positive values x_i.

1 Formulation of the Problem

One of the main applications of interval computations [3, 4] is application to *indirect measurement*, when we are interested in the value of some quantity y that is difficult (or even impossible) to measure directly). To estimate y, we therefore measure the values of several easier-to-measure quantities $x_1, \ldots, x_n$, and then use the known relation $y = f(x_1, \ldots, x_n)$ between x_i and y to reconstruct the value y as $\widetilde{y} = f(\widetilde{x}_1, \ldots, \widetilde{x}_n)$, where $\widetilde{x}_i$ is the result of measuring x_i.

In many real-life situations, the only information that we have about the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$ is that this error cannot exceed a known bound Δ_i, i.e., that $|\Delta x_i| \le \Delta_i$. In such situations, after measuring x_i, the only information that we get about the actual (unknown) value of x_i is that this value belongs to the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i] = [\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. In this case, we are interested in the interval $\mathbf{y}$ of possible value of y, i.e., in the range of the function $f(x_1, \ldots, x_n)$ over the corresponding box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$.

Interval computations provide the exact range for the case when $f(x_1, \ldots, x_n)$ is a simple arithmetic operation, and provide an enclosure for the general case.

In some practical situations, in addition to the upper bound on the measurement error Δx_i, we have partial information about the probabilities of different values within this interval. A very typical case is when we know the mean value of this error. Thus, in addition to knowing the interval of possible values $\mathbf{x}_i$ for x_i, we know the mathematical expectation E_i for x_i [5]. In such situations, in addition to the interval of possible values of $y = f(x_1, \ldots, x_n)$, we want to know the range of possible values of the mathematical expectation E of y.

In [2], we have shown how to compute the exact range of E for the case when $f(x_1, \ldots, x_n)$ is a simple arithmetic operation – i.e., $x_1 + x_2$, $x_1 - x_2$, $x_1 \cdot x_2$, etc. – and provide an enclosure for the general case.

In many ecological applications (see, e.g., [1] and references therein), the corresponding function $f(x_1, \ldots, x_n)$ is a multiple product $x_1 \cdot \ldots \cdot x_n$ of positive values x_i. For example, pollutant often comes from the industrial source to, say, a lake, via a chain of transitions, so the resulting concentration can be estimated as $x_1 \cdot x_2 \cdot \ldots \cdot x_n$, where x_1 is the original pollutant amount and the parameters x_i ($i \geq 2$) describe what portion of the pollutant goes from one link to the next one. For example, x_2 may describe the portion of the pollutant that seeps into the soil, x_3 the portion of the soil pollutant that goes from the soil into the creeks, and x_4 describes the portion of the creek's pollutant that stays in the lake. For each of these parameters, we know the interval $\mathbf{x}_i = [\underline{x}_i, \overline{x}_i]$ of possible values, and we often know the mean value E_i. Our goal is to find the interval of possible values of the product y, and the bounds on the mean of the product. In ecological problems, we are mainly interested in the worst-case estimates, so we mainly interested in the upper bound $\overline{y}$ for the interval $\mathbf{y}$ and in the upper bound $\overline{E}$ for the mean E.

Computing $\overline{y}$ is easy: since all the values x_i are positive, we have $\overline{y} = \overline{x}_1 \cdot \ldots \cdot \overline{x}_n$. When x_i are independent, computing $\overline{E}$ is also easy: in this case, $\overline{E} = E = E_1 \cdot \ldots \cdot E_n$.

The situation becomes less trivial in the general case when we cannot assume independence, and we therefore have to consider all possible distributions on the box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$. For this case, in principle, we can use algorithms presented in [1] for a product of two variables, and, by applying this algorithm $n-1$ times, get estimates for $x_1 \cdot x_2$, $(x_1 \cdot x_2) \cdot x_3$, ..., and finally, for $y = x_1 \cdot \ldots \cdot x_n$. However, the resulting algorithmic estimate cannot be easily described in an explicit form and therefore, it is difficult to analyze – and the analysis of possible changes is one of the main objectives of ecological research. It is therefore desirable to produce an explicit easy-to-analyze expression for $\overline{E}$. Such an expression is provided in this paper.

2 Main Result

In formal terms, in this paper, we solve the following problem:

GIVEN: positive values $\underline{x}_1$, $\overline{x}_1$, ..., $\underline{x}_n$, $\overline{x}_n$, E_1, ..., E_n,

FIND: the value

$$\overline{E} \stackrel{\text{def}}{=} \max\{E(x_1 \cdot \ldots \cdot x_n) \mid \text{ all distributions of } (x_1, \ldots, x_n) \text{ for which}$$
$$x_1 \in [\underline{x}_1, \overline{x}_1], \ldots, x_n \in [\underline{x}_n, \overline{x}_n], E[x_1] = E_1, \ldots, E[x_n] = E_n\}.$$

To describe the value $\overline{E}$, we first compute the values $p_i \stackrel{\text{def}}{=} (E_i - \underline{x}_i)/(\overline{x}_i - \underline{x}_i)$ and then order the variables in the decreasing order of p_i. Without losing generality, we can assume that the variables $x_1, \ldots, x_n$ are already ordered in this way, i.e., that $p_1 \geq p_2 \geq \ldots \geq p_n$. Then:

$$\overline{E} = (1 - p_1) \cdot \underline{x}_1 \cdot \underline{x}_2 \cdot \ldots \cdot \underline{x}_n +$$
$$(p_1 - p_2) \cdot \overline{x}_1 \cdot \underline{x}_2 \cdot \ldots \cdot \underline{x}_n +$$
$$\ldots +$$
$$(p_i - p_{i+1}) \cdot \overline{x}_1 \cdot \ldots \cdot \overline{x}_i \cdot \underline{x}_{i+1} \cdot \ldots \cdot \underline{x}_n +$$
$$\ldots +$$
$$p_n \cdot \overline{x}_1 \cdot \ldots \cdot \overline{x}_n.$$

The proof of this result is given in the Proofs section. Before we present this proof, let us describe the intuitive meaning of the above formula.

3 Intuitive Meaning of the Above Formula

The probability p_i can be interpreted as follows: if we only allow values $\underline{x}_i$ and $\overline{x}_i$, then there is only one probability distribution on x_i for which the average is exactly E_i. In this probability distribution, the probability $p[\overline{x}_i]$ of $\overline{x}_i$ is equal to p_i, and the probability $p[\underline{x}_i]$ of $\underline{x}_i$ is equal to $1 - p_i$.

In general, when we have two events A and B with known probabilities $p(A)$ and $p(B)$, then the probability of $A \,\&\, B$ can take any value from the interval $[p(A) \underline{\&} p(B), p(A) \overline{\&} p(B)]$, where $a \underline{\&} b \text{def} = \max(a + b - 1, 0)$ and $a \overline{\&} b \stackrel{\text{def}}{=} \min(a, b)$ (see, e.g., [6]). Indeed:

- the largest possible intersection is the smallest of the two sets, and
- the smallest possible intersection is when they are as far apart as possible:
  - if $p(A) + p(B) \leq 1$, they can be completely disjoint hence $\underline{p}(A \,\&\, B) = 0$,

– else we spread them as much as possible, so that $p(A \vee B) = 1$ hence $p(A \,\&\, B) = p(A) + p(B) - p(A \,\&\, B) = p(A) + p(B) - 1$.

Thus, we can introduce a natural notation $\neg p \stackrel{\text{def}}{=} 1 - p$ and rewrite the above formula as follows:

$$\overline{E} = \sum_{I \subseteq N} E_I,$$

where, for $I = \{i_1, \ldots, i_k\}$ and $N - I = \{j_1, \ldots, j_l\}$, we denoted:

$$E_I \stackrel{\text{def}}{=} (p_{i_1} \,\overline{\&}\, \ldots \,\overline{\&}\, p_{i_k}) \,\underline{\&}\, (\neg p_{j_1} \,\overline{\&}\, \ldots \,\overline{\&}\, \neg p_{j_l}) \cdot \overline{x}_{i_1} \cdot \ldots \cdot \overline{x}_{i_k} \cdot \underline{x}_{j_1} \cdot \ldots \cdot \underline{x}_{j_l}.$$

Indeed, we have

$$p_{i_1} \,\overline{\&}\, \ldots \,\overline{\&}\, p_{i_k} = \min(p_{i_1}, \ldots, p_{i_k}),$$

$$\neg p_{j_1} \,\overline{\&}\, \ldots \,\overline{\&}\, \neg p_{j_l} = \min(1 - p_{j_1}, \ldots, 1 - p_{j_l}) = 1 - \max(p_{j_1}, \ldots, p_{j_l}),$$

and therefore, a p_i-dependent factor in E_I can be rewritten as

$$\max(\min(p_{i_1}, \ldots, p_{i_k}) - \max(p_{j_1}, \ldots, p_{j_l}), 0).$$

The only possibility for the corresponding difference to be ≥ 0 is when each value p_{i_m} is larger than each value p_{j_q} – in other words, when all the values $p_{i_1}, \ldots, p_{i_k}$ precede all the values $p_{j_1}, \ldots, p_{j_l}$ in the decreasing order of p_i.

4 Proof of the Main Result

1°. To get the desired bound for $\overline{E}$, we must consider the values $E[x_1 \cdot \ldots \cdot x_n]$ for all possible probability distributions on the box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$ for which $E[x_1] = E_1$, …, $E[x_n] = E_n$. To describe a general probability distribution, we must use infinitely many parameters, and hence, this problem is difficult to solve directly.

To make the problem simpler, we will show that a general distribution with $E[x_i] = E_i$ can be simplified without changing the values $E[x_i]$ and $E[x_1 \cdot \ldots \cdot x_n]$. Thus, to describe possible values of $E[x_1 \cdot \ldots \cdot x_n]$, we do not need to consider all possible distributions, it is sufficient to consider only the simplified ones.

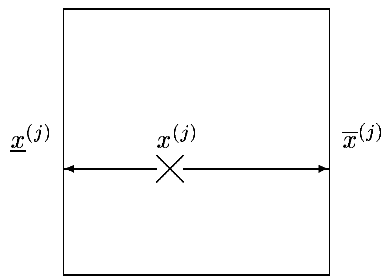
We will describe the simplification for discrete distributions that concentrate on finitely many points $x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)})$, $1 \leq j \leq N$. An arbitrary probability distribution can be approximated by such distributions, so we do not lose anything by this restriction.

So, we have a probability distribution in which the point $x^{(1)}$ appears with the probability $p^{(1)}$, the point $x^{(2)}$ appears with the probability $p^{(2)}$, etc. Let us modify this distribution as follows: pick a point $x^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$ that occurs with probability $p^{(j)}$, and replace it with two points: $\overline{x}^{(j)} = (\overline{x}_1, x_2^{(j)}, \ldots)$

with probability $p^{(j)} \cdot \overline{p}^{(j)}$ and $\underline{x}^{(j)} = (\underline{x}_1, x_2^{(j)}, \ldots)$ with probability $p^{(j)} \cdot \underline{p}^{(j)}$, where

$$\overline{p}^{(j)} \stackrel{\text{def}}{=} \frac{x_1^{(j)} - \underline{x}_1}{\overline{x}_1 - \underline{x}_1}$$

and $\underline{p}^{(j)} \stackrel{\text{def}}{=} 1 - \overline{p}^{(j)}$:



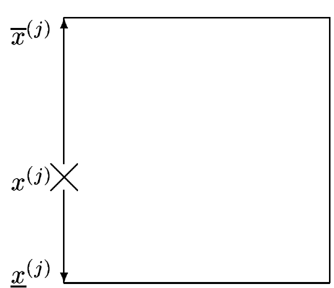
Here, the values $\overline{p}^{(j)}$ and $\underline{p}^{(j)} = 1 - \overline{p}^{(j)}$ are chosen in such a way that $\overline{p}^{(j)} \cdot \overline{x}_1 + \underline{p}^{(j)} \cdot \underline{x}_1 = x_1^{(j)}$. Due to this choice,

$$p^{(j)} \cdot \overline{p}^{(j)} \cdot \overline{x}_1 + p^{(j)} \cdot \underline{p}^{(j)} \cdot \underline{x}_1 = p^{(j)} \cdot x_1^{(j)},$$

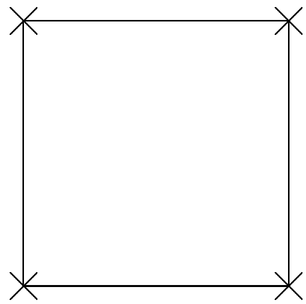
hence for the new distribution, the mathematical expectation $E[x_1]$ is the same as for the old one. Similarly, we can prove that the values $E[x_2]$, ..., $E[x_n]$, and $E[x_1 \cdot \ldots \cdot x_n]$ do not change.

We started with a general discrete distribution with N points for each of which $x_1^{(j)}$ could be inside the interval $\mathbf{x}_1$, and we have a new distribution for which $\leq N-1$ points have the value x_1 inside this interval. We can perform a similar replacement for all N points and get a distribution with the same values of $E[x_1]$, ..., $E[x_n]$, and $E[x_1 \cdot \ldots \cdot x_n]$ as the original one but for which, for every point, x_1 is equal either to $\underline{x}_1$, or to $\overline{x}_1$.

For the new distribution, we can perform a similar transformation relative to x_1 and end up – without changing the values x_1 – with the distribution for which always either $x_2 = \underline{x}_1$ or $x_2 = \overline{x}_2$:



Similarly, we can perform such a transformation for x_3, etc. Thus, instead of considering all possible distributions, it is sufficient to consider only distributions for which $x_1 \in \{\underline{x}_1, \overline{x}_1\}$, ..., $x_n \in \{\underline{x}_n, \overline{x}_n\}$. In other words, it is sufficient to consider only distributions which are located in 2^n corner points of the box $\mathbf{x}_1 \times \ldots \times \mathbf{x}_n$:



2°. Let us now show that, if we are looking for the maximum $\overline{E}$ of E, it is sufficient to consider only distributions with the following property: for every two points $x^{(i)}$ and $x^{(j)}$ with non-zero probability, if $x_k^{(i)} < x_k^{(j)}$ for some coordinate k, then $x_l^{(i)} \le x_l^{(j)}$ for all other coordinates l.

We will prove this statement as follows. Let us assume that the above property is not satisfied. This means that for some k and l, we have $x_k^{(i)} < x_k^{(j)}$ and $x_l^{(i)} > x_l^{(j)}$.

Let $p^{(i)} > 0$ and $p^{(j)} > 0$ be the probabilities of these two points. We will show that, if with probability $p \stackrel{\text{def}}{=} \min(p^{(i)}, p^{(j)})$, we "swap" the coordinates of the points $x^{(i)}$ and $x^{(j)}$, we thus increase (or keep unchanged) the value E. Therefore, when we are looking for the maximum of E, it is sufficient to consider only distributions for which the above property holds.

Specifically, let $k_1, \ldots, k_q$ be coordinates for which $x_{k_m}^{(i)} \le x_{k_m}^{(j)}$, and let $l_1, \ldots, l_s$ be coordinates for which $x_{l_t}^{(i)} > x_{l_t}^{(j)}$. With probability p, we replace the points $x^{(i)}$ and $x^{(j)}$ with two new points $x_{\text{new}}^{(i)}$ and $x_{\text{new}}^{(j)}$ for which coordinates k_m remain the same while the coordinates l_t are swapped: $x_{\text{new},k_m}^{(i)} = x_{k_m}^{(i)}$, $x_{\text{new},k_m}^{(j)} = x_{k_m}^{(j)}$, $x_{\text{new},l_t}^{(i)} = x_{l_t}^{(j)}$, and $x_{\text{new},l_t}^{(j)} = x_{l_t}^{(i)}$. It is easy to see that this swap does not change the averages $E[x_i]$. How does it affect the mathematical expectation of the product $E[x_1 \cdot \ldots \cdot x_n]$? The only two terms that changed are terms corresponding to $x^{(i)}$ and $x^{(j)}$ with probability p:

- For the original points, the sum of these two terms is equal to

$$p \cdot \left(\prod_{z=1}^{n} x_z^{(i)} + \prod_{z=1}^{n} x_z^{(j)} \right) = p \cdot (\Pi_k^{(i)} \cdot \Pi_l^{(i)} + \Pi_k^{(j)} \cdot \Pi_l^{(j)}),$$

where we denoted:

$$\Pi_k^{(i)} \stackrel{\rm def}{=} \prod_{m=1}^{q} x_{k_m}^{(i)}, \quad \Pi_l^{(i)} \stackrel{\rm def}{=} \prod_{t=1}^{s} x_{l_t}^{(i)},$$

$$\Pi_k^{(j)} \stackrel{\rm def}{=} \prod_{m=1}^{q} x_{k_m}^{(j)}, \quad \Pi_l^{(j)} \stackrel{\rm def}{=} \prod_{t=1}^{s} x_{l_t}^{(j)}.$$

- For the new points, the corresponding sum is equal to

$$p \cdot (\Pi_k^{(i)} \cdot \Pi_l^{(j)} + \Pi_k^{(j)} \cdot \Pi_l^{(i)}).$$

- Therefore, the difference between the new and the old values of $E[x_1 \cdot \ldots \cdot x_n]$ is equal to:

$$p \cdot (\Pi_k^{(i)} \cdot \Pi_l^{(j)} + \Pi_k^{(j)} \cdot \Pi_l^{(i)} - \Pi_k^{(i)} \cdot \Pi_l^{(i)} - \Pi_k^{(j)} \cdot \Pi_l^{(j)}).$$

One can easily see that this difference is equal to

$$p \cdot (\Pi_k^{(i)} - \Pi_k^{(j)}) \cdot (\Pi_l^{(j)} - \Pi_l^{(i)}).$$

By definition of k_m, we have $x_{k_m}^{(i)} \le x_{k_m}^{(j)}$; multiplying these inequalities between positive numbers, we conclude that $\Pi_k^{(i)} \le \Pi_k^{(j)}$. Similarly, from $x_{l_t}^{(i)} > x_{l_t}^{(j)}$, we conclude that $\Pi_l^{(i)} > \Pi_l^{(j)}$. Thus, the difference between the new and the old values is indeed non-negative.

The statement is proven.

3°. Due to Part 2° of the proof, for every two different points $x^{(i)} \ne x^{(j)}$:

- either $x_k^{(i)} \le x_k^{(j)}$ for all k and $x_k^{(i)} < x_k^{(j)}$ some all k; we will denote this by $x^{(i)} \prec x^{(j)}$;
- or $x_k^{(j)} \le x_k^{(i)}$ for all k and $x_k^{(j)} < x_k^{(i)}$ some all k – i.e., $x^{(j)} \prec x^{(i)}$.

So, the relation $\prec$ defines a linear (total) order on the set of all the points $x^{(i)}$. Without losing generality, let us assume that the points $x^{(i)}$ are ordered according to this order, i.e., that

$$x^{(1)} \prec x^{(2)} \prec \ldots \prec x^{(N)}.$$

By definition of $\prec$, we can conclude that for each coordinate k, we have:

$$x_k^{(1)} \le x_k^{(2)} \le \ldots \le x_k^{(N)}.$$

In Part 1° of the proof, we have already shown that for every point $x^{(i)}$, each coordinate $x_k^{(i)}$ is equal either to smallest possible value $\underline{x}_k$ or to the largest possible value $\overline{x}_k$. Due to the above inequality, once $x_k^{(i)}$ is equal to its largest possible value, i.e., once $x_k^{(i)} = \overline{x}_k$, all the following values of x_k must also be equal to the same largest possible value, i.e., $x_k^{(i+1)} = \ldots = x_k^{(N)} = \overline{x}_k$.

Therefore, when we move from $x^{(i)}$ to $x^{(i+1)}$, the overall number of coordinates equal to $\overline{x}_k$ cannot decrease; it cannot also stay the same because otherwise, we would have $x^{(i)} = x^{(i+1)}$. Thus, this number can only increase. This overall number can take values from 0 to n, and this overall number increases once we go from $x^{(i)}$ to $x^{(i+1)}$; thus, we cannot have more than n such increases, and so, we can have no more than $n+1$ different points $x^{(i)}$.

Based on the order between the points $x^{(i)}$, we can defined the order between the coordinates x_k: namely, we say that x_k precedes x_l if in the sequence $x^{(i)}$, the first appearance of $\overline{x}_k$ precedes the first appearance of $\overline{x}_l$. One can easily see that this relation is an order. This is, in general, partial order; let us arbitrarily extend it to a linear order on the set of n coordinates $x_1, \ldots, x_n$.

For simplicity, let us assume that the variables $x_1, \ldots, x_n$ are already ordered according to this order, i.e., that $\overline{x}_1$ first appears in the sequence $x^{(i)}$ before (or at the same time as) $\overline{x}_2$, etc. Due to this order, if for some point $x^{(i)}$, we have a "small" value of some coordinate $x_k^{(i)} = \underline{x}_k$, then all the following coordinates are also "small": $x_{k+1}^{(i)} = \underline{x}_{k+1}, \ldots, x_n^{(i)} = \underline{x}_n$. In other words, each vector $x^{(i)}$ can take one of the following values:

$$(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n), (\overline{x}_1, \underline{x}_2, \ldots, \underline{x}_n), \ldots, (\overline{x}_1, \ldots, \overline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_n), \ldots, (\overline{x}_1, \ldots, \overline{x}_n).$$

These are exactly the vectors corresponding to the expression for $\overline{E}$ that we are proving. To complete the proof, we must therefore show that these expressions occur with probabilities, correspondingly, $1 - p_1$, $p_1 - p_2$, etc.

Indeed:

- let $p^{(1)}$ be the probability of $(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$;
- let $p^{(2)}$ be the probability of $(\overline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$;
- ...
- let $p^{(i+1)}$ be the probability of $(\overline{x}_1, \ldots, \overline{x}_i, \underline{x}_{i+1}, \ldots, \underline{x}_n)$;
- ...
- let $p^{(n+1)}$ be the probability of $(\overline{x}_1, \ldots, \overline{x}_n)$.

The sum of all these probabilities should be equal to 1:

$$p^{(1)} + p^{(2)} + \ldots + p^{(n+1)} = 1.$$

For each i, the mean value of x_i (that should be equal to E_i) is equal to

$$\underline{x}_i \cdot (p^{(1)} + \ldots + p^{(i)}) + \overline{x}_i \cdot (p^{(i+1)} + \ldots + p^{(n+1)}).$$

By definition, p_i is the probability with which we must take $\overline{x}_i$ so that if we take $\underline{x}_i$ with probability $1 - p_i$, we get the desired mean p_i. Thus, for every i, we have:

$$p_i = p^{(i+1)} + \ldots + p^{(n+1)}.$$

In particular, for $i < n$, we have

$$p_{i+1} = p^{(i+2)} + \ldots + p^{(n+1)};$$

thus,

$$p_i - p_{i+1} = (p^{(i+1)} + p^{(i+2)} + \ldots + p^{(n+1)}) - (p^{(i+2)} + \ldots + p^{(n+1)}) = p^{(i+1)}.$$

For $i = n$, we have $p_n = p^{(n+1)}$. Finally, the probability $p^{(1)}$ can be determined as

$$p^{(1)} = 1 - (p^{(2)} + \ldots + p^{(n+1)}) =$$

$$1 - ((p_1 - p_2) + (p_2 - p_3) + \ldots + (p_{n-1} - p_n) + p_n) = 1 - p_1.$$

For the values $x^{(i)}$ with these probabilities, the mathematical expectation of the product $x_1 \cdot \ldots \cdot x_n$ is exactly equal to the expression from our Main Result. The theorem is proven.

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