Robust Methodology for Characterizing System Response to Damage: Approach Based on Partial Order

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Abstract. To describe the response of engineering complex systems to various damage mechanics, engineers have traditionally use probability-based reliability approach. For complex components and/or systems, the failure or damage probabilities cannot be directly statistically determined, they are not probabilities in the frequentist sense but rather "subjective" probabilities—scalar quantities that characterize our knowledge and which are treated like frequentist ("physical") probabilities. In this paper, we describe a new approach based on partial order.

1 Introduction

To describe the response of engineering complex systems to various damage mechanics, engineers have traditionally use probability-based reliability approach. For complex components and/or systems, the failure or damage probabilities cannot be directly statistically determined, they are not probabilities in the frequentist sense but rather "subjective" probabilities—scalar quantities that characterize our knowledge and which are treated like frequentist ("physical") probabilities.

There are several different techniques for describing our uncertainty by a numerical value, and it is a known fact that different techniques lead to somewhat different numerical values. The traditional statistical approach is not "robust": by using slightly different values of subjective probability (which describe the experts' uncertainty as well as the original values), we may end up with radically different conclusions.

It is therefore desirable to develop robust methodologies for characterizing response of systems to various damage mechanisms. The probabilistic approach to decision making is based on a solid foundations: there are axioms, principles that—if true—uniquely lead to probabilities and probability-based techniques for decision making. Most of these principles are pretty reasonable, with the exception of one: that the corresponding ordering of alternatives is "total" ("linear"). Traditional decision theory (see, e.g., [1, 2]) is based on the assumption that a person whose preferences we want to describe can always (linearly) order his preferences, i.e., that for every two alternatives a and a', he can decide:

- whether a is better than a' (we will denote it by $a' \prec a$);
- or whether a' is better than a ($a \prec a'$);
- or whether a and a' are (for this person) of the same quality (we will denote it by $a \sim a'$).

A similar assumption (often implicit) underlies the traditional description of degrees of belief ("subjective probabilities") by numbers from the interval [0,1].

In real life, an expert may not be able to always compare two different alternatives. In this paper, we provide an exact description of decision making under partial ordering of alternatives. In turns out that in general, the uncertainty of each situation is characterized not by a scalar linearly ordered quantity (probability), but by a matrix-type partially ordered quantity (ordered operator).

Important particular cases are interval-valued probabilities and more general algebraic structures described by S. Markov and his group; see, e.g., [7], [8].

Our results were partially published in [5] and [6].

2 Traditional Utility Theory: A Brief Reminder

In this section, we will mainly follow standard definitions (see, e.g., [1, 2]), but we will not always follow them exactly: in some cases, we will slightly rephrase these definitions (without changing their mathematical contents) so as to make the following transition to partially ordered preferences as clear as possible.

Definition 1. *Let $\mathcal{A}$ be a set; this set will be called the set of* alternatives *(or the set of* pure alternatives*). By a* lottery *on $\mathcal{A}$ we understand a a probability measure on $\mathcal{A}$ with finite support.*

In other words, a lottery is a pair $\langle A, p \rangle$, where $A = \{a_1, \ldots, a_n\} \subseteq \mathcal{A}$ is a finite subset of $\mathcal{A}$, and p is a mapping $p : A \to [0, 1]$ for which $p(a_i) \geq 0$ and $\sum p(a_i) = 1$. A lottery will also denoted as $p(a_1) \cdot a_1 + \ldots + p(a_n) \cdot a_n$. We do not consider lotteries with infinite numbers of alternatives, because every real-life randomizing device, be it a dice or a computer-based random number generator, produces only finitely many possibilities.

The set of lotteries will be denoted by L. On this set L, we can naturally define an operation of *probability combination* as a convex combination of the corresponding probability measures: namely, if we have m values $q_1, \ldots, q_m \in$

$[0,1]$ with $\sum q_j = 1$, and m lotteries $\ell_j = \langle A_j, p_j \rangle$, then we can define the probability combination $\ell = q_1 \cdot \ell_1 + \ldots + q_m \cdot \ell_m$ as a lottery $\ell = \langle A, p \rangle$ with $A = \cup A_j$ and $p(a) = \sum q_j \cdot p_j(a)$, where the sum is taken over all j for which $a \in A_j$.

Definition 2. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$. By a* preference relation*, we mean a pair $\langle \prec, \sim \rangle$, where $\prec$ is a (strict) order on L, $\sim$ is an equivalence relation on L, and for every $\ell, \ell', \ell'' \in L$ and every $p \in (0,1)$, the following conditions hold:*

1. *if $\ell \sim \ell'$ and $\ell' \prec \ell''$, then $\ell \prec \ell''$;*
2. *if $\ell \prec \ell'$ and $\ell' \sim \ell''$, then $\ell \prec \ell''$;*
3. *if $\ell \prec \ell'$, then $p \cdot \ell + (1-p) \cdot \ell'' \prec p \cdot \ell' + (1-p) \cdot \ell''$;*
4. *if $p \cdot \ell + (1-p) \cdot \ell'' \prec p \cdot \ell' + (1-p) \cdot \ell''$, then $\ell \prec \ell'$;*
5. *if $\ell \sim \ell'$, then $p \cdot \ell + (1-p) \cdot \ell'' \sim p \cdot \ell' + (1-p) \cdot \ell''$;*
6. *if $p \cdot \ell + (1-p) \cdot \ell'' \sim p \cdot \ell' + (1-p) \cdot \ell''$, then $\ell \sim \ell'$.*

Definition 3. *A preference relation is called* linearly ordered *(or* linear*, for short) if for every $\ell, \ell' \in L$, either $\ell \preceq \ell'$, or $\ell' \preceq \ell$ (where $\ell \preceq \ell'$ means that either $\ell \prec \ell'$ or $\ell \sim \ell'$).*

It is known that linearly ordered preference relations can be characterized in terms of special functions called *utility functions*:

Definition 4. *A function u from the set L of all lotteries to an ordered set V is called a* utility function*. For each $\ell \in L$, $u(\ell)$ will be called a* value *of the utility function. We say that a utility function u* describes *the preference relation if for every $\ell, \ell' \in L$, the following two conditions hold:*

- *$\ell \prec \ell'$ if and only if $u(\ell) < u(\ell')$;*
- *$\ell \sim \ell'$ if and only if $u(\ell) = u(\ell')$.*

Definition 5. *A utility function $u : L \to V$ is called* convexity-preserving *if on the set V, convex combination $p_1 \cdot v_1 + \ldots + p_n \cdot v_n$ is defined for all $p_i \geq 0$, $\sum p_i = 1$, and if for every p_i and ℓ_i, we have $u(p_1 \cdot \ell_1 + \ldots + p_m \cdot \ell_m) = p_1 \cdot u(\ell_1) + \ldots + p_m \cdot u(\ell_m)$.*

To describe linearly ordered preference relations, we use *scalar* utility functions, i.e., convexity-preserving utility functions for which $V = R$. It is known that for every convexity-preserving function $u : L \to R$, the relations $u(\ell) < u(\ell')$ and $u(\ell) = u(\ell')$ define a linearly ordered preference relation. It is also known that this utility function is determined uniquely modulo a linear transformation, i.e.:

- If two different scalar utility functions $u : L \to R$ and $u' : L \to R$ describe the same preference relation, then there exists a linear function $T(z) = k \cdot z + m$, with $k > 0$, such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.

– Vice versa, if a scalar utility function $u : L \to R$ describes a preference relation, and $k > 0$ and m are real numbers, then the function $u'(\ell) = T(u(\ell))$ (where $T(z) = k \cdot z + m$) is also a scalar utility function which describes the same preference relation.

One can also show that every *Archimedean* (in some reasonable sense) linearly ordered preference relation $\langle \prec, \sim \rangle$ can be described by an appropriate scalar utility function.

In other words, each (Archimedean) *linearly* ordered preference relation can be described by a utility function, and this utility function is determined uniquely modulo a linear transformation. This is not necessarily true for *non-Archimedean* preference relations, e.g., for a lexicographic ordering $(x_1, x_2) > (y_1, y_2)$ iff either $x_1 > y_1$ or ($x_1 = y_1$ and $x_2 > y_2$). It turns out that non-Archimedean linearly ordered preferences can be described by utilities with values in linearly ordered affine spaces (for a general introduction into ordered algebraic structures, see, e.g., [3]):

3 Utilities with Values in Linearly Ordered Affine Spaces: Brief Reminder

An *affine space* (see, e.g., [4] and references therein) is "almost" a vector space, the main difference between them is that in the linear space, there is a fixed starting point (0), while in the affine space, there is no fixed point. More formally:

– A *linear space* is defined as a set V with two operations: addition $v + v'$ and multiplication $\lambda \cdot v$ of elements from V by real numbers $\lambda \in R$ (operations which must satisfy some natural properties). With this two basic operations, we can define an arbitrary linear combination $\lambda_1 \cdot v_1 + \ldots + \lambda_n \cdot v_n$ of elements $v_1, \ldots, v_n \in V$.
– In the *affine space*, we can only define those linear combination which are shift-invariant, i.e., linear combinations with $\sum \lambda_i = 1$.

The relation between a linear space and an affine space is rather straightforward:

– if we have an affine space V, then we can pick an arbitrary point $v_0 \in V$, are define a linear space in which this point is 0. Namely, we can define $v + v'$ as $1 \cdot v + 1 \cdot v' - 1 \cdot v_0$: since we took v_0 as 0, this linear combination will be exactly $v + v'$.
– Vice versa, if we have a hyperplane H in a linear space, then (unless this hyperplane goes through 0) this hyperplane is *not* a linear space, but it is *always* an affine space.

Definition 6. *A vector space V with a strict order $<$ is called an* ordered vector space *if for every $v, v', v'' \in V$, and for every real number $\lambda > 0$ the following two properties are true:*

– *if $v < v'$, then $v + v'' < v' + v''$;*

– *if* $v < v'$, *then* $\lambda \cdot v < \lambda \cdot v'$.

Since this ordering does not change under shift, it, in effect, defines an ordering on the affine space.

Definition 7. *By a* vector utility function, *we mean a convexity-preserving utility function with values in an ordered affine space* V.

To analyze uniqueness of vector utility functions, we must consider isomorphisms. A mapping T between two affine spaces is called *affine* if it preserves the affine structure, i.e., if $T(\sum \lambda_i \cdot v_i) = \sum \lambda_i \cdot T(v_i)$ whenever $\sum \lambda_i = 1$. For finite-dimensional affine spaces, affine mappings are just linear transformations $(x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$, i.e., transformations in which each resulting coordinate y_i is determined by a linear function $y_i = a_i + \sum b_{ij} \cdot x_j$.

Definition 8. *A one-to-one affine transformation* $T : V \to V'$ *of two ordered affine spaces is called an* isomorphism *if for every* $v_1, v_2 \in V$, $v < v'$ *if and only if* $T(v) < T(v')$.

Recall that for every subset $S \subseteq V$ of an affine space, its *affine hull* $A(S)$ can be defined as the smallest affine subspace containing S, i.e., equivalently, as the set of all affine combinations $\sum \lambda_i \cdot s_i$ $(\sum \lambda_i = 1)$ of elements from S.

Theorem 1. *Let* $\mathcal{A}$ *be a set, and let* L *be the set of all lotteries over* $\mathcal{A}$.

– (consistency) *For every convexity-preserving function* $u : L \to V$ *from* L *to a linearly ordered affine space* V, *the relations* $u(\ell) < u(\ell')$ *and* $u(\ell) = u(\ell')$ *define a linearly ordered preference relation.*
– (existence) *For every linearly ordered preference relation* $\langle \prec, \sim \rangle$, *there exists a vector utility function (with values in a linearly ordered affine space) which describes this preference.*
– (uniqueness) *The utility function is determined uniquely modulo an isomorphism:*
  - *If two different vector utility functions* $u : L \to V$ *and* $u' : L \to V'$ *describe the same linearly ordered preference relation, then there exists an isomorphism* $T : A(u(L)) \to A(u'(L))$ *between the affine hulls of the images of the functions, such that for every lottery* ℓ, $u'(\ell) = T(u(\ell))$.
  - *Vice versa, if a vector utility function* $u : L \to V$ *describes a preference relation, and* $T : A(u(L)) \to V'$ *is an isomorphism of ordered affine spaces, then the function* $u'(\ell) = T(u(\ell))$ *is also a vector utility function, and it describes the same preference relation.*

4 New Approach: Utility Theory for Partially Ordered Preferences

It turns out that a similar result holds for *partially* ordered references as well:

Theorem 2. *Let $\mathcal{A}$ be a set, and let L be the set of all lotteries over $\mathcal{A}$.*

- (consistency) *For every convexity-preserving function $u : L \to V$ from L to an ordered affine space, the relations $u(\ell) < u(\ell')$ and $u(\ell) = u(\ell')$ define a preference relation.*
- (existence) *For every preference relation $\langle \prec, \sim \rangle$, there exists a vector utility function which describes this preference.*
- (uniqueness) *The utility function is determined uniquely modulo an isomorphism:*
  - *If two different vector utility functions $u : L \to V$ and $u' : L \to V'$ describe the same preference relation, then there exists an isomorphism $T : A(u(L)) \to A(u'(L))$ between the affine hulls of the images of the functions, such that for every lottery ℓ, $u'(\ell) = T(u(\ell))$.*
  - *Vice versa, if a vector utility function $u : L \to V$ describes a preference relation, and $T : A(u(L)) \to V'$ is an isomorphism of ordered affine spaces, then the function $u'(\ell) = T(u(\ell))$ is also a vector utility function, and it describes the same preference relation.*

Example. Let us consider a simple case in which the quality of each alternative a is described by the value of a single quantity $q(a)$ (e.g., profit), and the partialness of the preference relation is caused by the fact that we do not know the exact values of this quantity; instead, for each alternative a, we know the *interval* $[q^-(a), q^+(a)]$ of possible values of this quantity. In such a case, it is natural to define preference as follows:

- $a \preceq a'$ if and only if $q^-(a) \le q^-(a')$ and $q^+(a) \le q^+(a')$;
- $a \sim a'$ if and only if $q^-(a) = q^-(a')$ and $q^+(a) = q^+(a')$.

5 How to Describe Degrees of Belief ("Subjective Probabilities") for Partially Ordered Preferences?

In traditional (scalar) utility theory, it is possible to describe our degree of belief $ps(E)$ in each statement E, e.g., as follows: We pick two alternatives a_0 and a_1 with utilities 0 and 1, and as the degree of belief in E, we take the utility of a conditional alternative "if E then a_1 else a_0" (or $(E|a_1|a_0)$, for short). This utility is also called *subjective probability* because if E is a truly random event which occurs with probability p, then this definition leads to $ps(E) = p$: Indeed, according to the convexity-preserving property of a utility function, we have

$$ps(E) = u(E|a_1|a_0) = p \cdot u(a_1) + (1 - p) \cdot u(a_0) = p \cdot 1 + (1 - p) \cdot 0 = p. \quad (1)$$

How can a similar description look like for *partially* ordered preferences? Before we formulate our result, let us first explain our reasoning that led to this result. The linear-ordered case definition of subjective probability $ps(E)$ can be rewritten as follows: for every two lotteries $\ell, \ell' \in L$, we have

$$u(E|\ell|\ell') = ps(E) \cdot u(\ell) + (1 - ps(E)) \cdot u(\ell'), \quad (2)$$

or, equivalently,

$$u(E|\ell|\ell') = ps(E) \cdot (u(\ell) - u(\ell')) + u(\ell'). \tag{3}$$

In other words, we can interpret $ps(E)$ as a *linear operator* which transforms the utility difference $u(\ell) - u(\ell')$ into an expression

$$u(E|\ell|\ell') - u(\ell') = ps(E) \cdot (u(\ell) - u(\ell')). \tag{4}$$

It is, therefore, reasonable to expect that for *partially* ordered preferences, when we have multi-dimensional (vector) utilities with values in a vector space V, $ps(E)$ would also be a linear operator, but this time from V to V (and not from R to R). We will now show that this expectation is indeed true.

Definition 9. *Let $\mathcal{A}$ be a set, let L be the set of all lotteries over $\mathcal{A}$, and let E be a formula (called* event*). By a* conditional lottery*, we mean an expression of the type $\sum p_i \cdot \ell_i + \sum q_k \cdot (E|\ell'_k|\ell''_k)$, where $\sum p_i + \sum q_k = 1$, and ℓ_i, ℓ'_k, and ℓ''_k are lotteries. We will denote the set of all conditional lotteries by $L(E)$.*

The meaning of a conditional lottery is straightforward: with probability p_i, we run a lottery ℓ_i, and with probability q_k, we run a conditional event "if E then ℓ'_k else ℓ''_k".

Definition 10. *Let $\mathcal{A}$ be a set, and let $L(E)$ be the set of all conditional lotteries over $\mathcal{A}$. By a* preference relation*, we mean a pair $\langle \prec, \sim \rangle$, where $\prec$ is a (strict) order on $L(E)$, $\sim$ is an equivalence relation on $L(E)$, which satisfies conditions 1)–6) from Definition 2 plus the following additional conditions:*

1. *if $\ell \sim \ell'$, then $(E|\ell|\ell'') \sim (E|\ell'|\ell'')$;*
2. *if $\ell' \sim \ell'''$, then $(E|\ell|\ell') \sim (E|\ell|\ell'')$;*
3. *$(E|\ell|\ell) \sim \ell$;*
4. *$(E|p \cdot \ell + (1-p) \cdot \ell'|\ell'') \sim p \cdot (E|\ell|\ell'') + (1-p) \cdot (E|\ell'|\ell'')$;*
5. *$(E|\ell|p \cdot \ell' + (1-p) \cdot \ell'') \sim p \cdot (E|\ell|\ell') + (1-p) \cdot (E|\ell|\ell'')$;*
6. *$(E|p \cdot \ell + (1-p) \cdot \ell''|p \cdot \ell' + (1-p) \cdot \ell'') \sim p \cdot (E|\ell|\ell') + (1-p) \cdot \ell''$;*
7. *if $\ell \preceq \ell'$, then $\ell \preceq (E|\ell|\ell') \preceq \ell'$.*

The meaning of all these conditions is straightforward; e.g., 7) means that $(E|\ell|\ell')$ is better (or of the same quality) than ℓ because in the conditional alternative, both possibilities ℓ and ℓ' are at least as good as A.

In accordance with our Theorem 2, the utility of such events can be described by a vector utility function.

Definition 11. *Let V be an ordered vector space.*

- *A linear operator $T : V \to V$ is called* non-negative *(denoted $T \geq \mathbf{0}$) if $x > 0$ implies $Vx \geq 0$.*
- *A linear operator T is called a* probability operator *if both T and $\mathbf{1} - T$ are non-negative (where $\mathbf{1}$ is a unit transformation $v \to v$).*

Theorem 3.

- *Let* $u : L \to V$ *be a vector utility function and let* $T : V \to V$ *be a strict probability operator. Then, a function* $u^* : L(E) \to V$ *defined as*

$$u^*\left(\sum_i p_i \cdot \ell_i + \sum_k q_k \cdot (E|\ell'_k|\ell''_k)\right) = \sum_i p_i \cdot u(\ell_i) + \sum_k q_k \cdot u^*(E|\ell'_k|\ell''_k), \quad (5)$$

with $u^*(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell')$, *is a vector utility function which describes a preference relation on* $L(E)$.
- *Let* $\langle \prec, \sim \rangle$ *be a preference relation on* $L(E)$, *and let* $u : L(E) \to V$ *be a vector utility function which describes this preference. Then, there exists a probability operator* $T : A(u(L)) \to V$ *for which*

$$u(E|\ell|\ell') = Tu(\ell) + (\mathbf{1} - T)u(\ell') \quad (6)$$

for all ℓ *and* ℓ'.

Thus, we get a generalization of subjective probabilities, from scalar values $p \in [0, 1]$ (which, in our description, correspond to scalar matrices) to general linear *probability operators*. It can be shown that for a fixed order on an n-dimensional space, the set of all such matrices is at most n-dimensional.

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