# Separating Components in Interval-Valued Images

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**Abstract**

In many applications of imaging, we would like to know whether we have an image of a single-component object or an image of an object that consists of several components. Many algorithms have been designed to solve this problem; however, these algorithms are all heuristic. Often, according to some reasonable methods, we have a single component, while according to some other equally reasonable methods, the same image have multiple components. It is desirable to produce reliable methods, so that if a method claims that there are multiple components, then it should mean that the observed data is incompatible with the assumption that there is only one component. At present, there exist reliable methods for separating components in a (interval-valued) 1D source. In this paper, we develop an efficient algorithm for separating components in a general (interval-valued) 2D source.

# 1 Introduction

This Section – as well as the following Section 2 – describes the basic relevant problems, definitions, and ideas from (low level) image processing and digital and combinatorial geometry. Our main objective in providing these two sections is to enable readers – who may not be familiar with these basic notions – to fully understand both the problem that we are solving and our solution. Readers who are reasonably familiar with the main ideas of image processing are welcome to browse through the first two sections and to go directly to Section 3 that contains original results.

**The problem of separating components is important.** In many applications of imaging, we would like to know whether we have an image of a single-component object or of an object that consists of several components. For example, in astronomical images, it is desirable to know whether we have a single-component source or if the observed source consists of several components; see, e.g., [23].

Similar problems occur when we analyze satellite images and/or geophysical maps to check whether, e.g., a given geological zone is indeed a single zone or if it is more appropriate to divide it into several "components" (segments). In geophysics, appropriate subdivision of an area into segments is extremely important, because it enables us to extrapolate the results obtained in some locations within the segment (where extensive research was done) to other locations within the same segment, and thus, get a good understanding of the locations which weren't that thoroughly analyzed.

**The problem of separating components is difficult to solve.** If we had a perfect image, then, by showing this image to an expert, we could usually easily determine whether we have a single component or multiple components. In practice, however, due to measurement inaccuracy, this problem is difficult to solve. For example, in astronomy, if we observe a narrow gap between two point sources, it may be an indication that there are two components or it may be an artifact created by noise. To show how difficult this problem is, it is sufficient to say that, e.g., in radioastronomy, the ability to distinguish between a single-component source and a source consisting of two close components is one of the main metrics for gauging the quality of a radiotelescope; see, e.g., [21].

**How this problem is solved now.** Since, as we have mentioned, the problem of separating components is very important, many algorithms have been designed to solve this problem. These algorithms use different criteria to determine the boundaries between the components – if no such boundary is found, this means that we have a single-component source, otherwise the source consists of several components. An overview of the existing approaches is given, e.g., in [22] (for latest developments, see, e.g., [14]).

For example, one such approach identifies boundary points as points $x$ for which the Laplacian

$$\Delta f \stackrel{\text{def}}{=} \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} \tag{1}$$

is close to 0; the value of the Laplacian is estimated by using numerical differentiation.

**Main limitation of the existing methods: they are heuristic.** From the practical viewpoint, the main problem with these methods is that they are all heuristic methods. Each of these methods is based on reasonable ideas, but there are many of them, and often, different methods give different results. Often, according to some reasonable methods, we have a single component, and according to some other equally reasonable methods, the same image has multiple components.

**Example.** Let us explain, e.g., the problem with the Laplacian criterion with respect to geospatial data. Geospatial data describes the dependence $f(x_1, x_2)$ of different quantities such as elevation, gravity value, magnetic field, etc., on the spatial coordinates $x_1$ and $x_2$. It is well known that since the Earth's surface is a sphere, there is no way to map a reasonably large area of the Earth surface into a plane in such a way that the distances and angles and sizes are all preserved – a distortion is inevitable. In each problem, some distortions are more tolerable while others are less tolerance. For example, if the main purpose of the map is to be used in navigation, then it is best to use the projection that preserves the angles – but that may be highly distortive of the actual sizes of different geographic areas. On the other hand, if we are interested in the relative sizes of different areas, then we may prefer a projection that keeps the sizes intact – but that may distort angles and distances. There are many different types of projection (see, e.g., [6]), and it is desirable that a subdivision of a geographical area into zones and sub-zones should not depend on what exactly projection we use.

From the mathematical viewpoint, the use of a different projection means that instead of the original coordinates $x_1$ and $x_2$ we use new coordinates $x_1' = T_1(x_1, x_2)$ and $x_2' = T_2(x_1, x_2)$ for some nonlinear functions $T_1$ and $T_2$. If we know the image $f'(x_1', x_2')$ in the new coordinates, i.e., if we know how the described physical quantity depends on $x_1'$ and $x_2'$, then the image in the original coordinates has the form $f(x_1, x_2) = f'(T_1(x_1, x_2), T_2(x_1, x_2))$.

The problem is that the Laplacian is not invariant under arbitrary non-linear transformations. As a result, the border $\Delta f' = 0$ that the Laplacian method is computing in the new coordinates is, in general, different from the border $\Delta f = 0$ of the same area that this method computes when using the original coordinates.

In other words, the subdivision into zones produced by this method depends on what exactly projection we use.

**Reliable methods are desirable.** It is desirable to produce reliable methods, in the sense that if a method claims that there are multiple components, then it should mean that the observed data is incompatible with the assumption that there is only one component.

**Existing reliable methods.** At present, there exist reliable methods for separating components in a 1D source; the corresponding algorithms are described in [24].

Similar algorithms have been used to separate components in a 2D case in situations when the experts provide us with a preferred coordinate system [1, 3, 4].

**What we are planning to do.** In this paper, we develop an efficient algorithm for separating components in a general 2D source.

Before we describe algorithms for separating components in an interval-valued image (e.g., image known with interval uncertainty), let us first analyze the problem of separating components in precisely known images.

# 2 Separating Components in Precisely Known Images

**Simplest case: black and white images.** Let us start with the simplest case when we have a black and white image, i.e., an image in which every point is either black or white. In precise terms, black and white images can be described as follows: we have two values $b < w$, and at every point $x$, the intensity $f(x)$ of the image is equal either to $b$ ("black") or to $w$ ("white").

To describe such an image, it is sufficient to describe the source as the set of all its white points. For such images, it is intuitively clear how to define a component:

- if the source is a connected set, then we have a single component;
- if the source is not a connected set, then, from the topological viewpoint, this source consists of several connected components; these topological components are exactly the source components that we are looking for.

**How to define connectedness in discrete images.** The notion of connectedness is well-defined for "continuous" sets – i.e., sets that are subsets of $R^2$. However, in practice, when we get an image, we only have the values of the intensity $f(x)$ on finitely many points $x$ ("pixels"). In this case, the set $P$ of all points at which we know the value of intensity is discrete; therefore, the set $W$ of all white points – which is a subset of $P$ – is also discrete. What is the natural way to define connectedness for such sets?

Some of the pixels are immediate neighbors to each other. The set $P$ of all the pixels with the (symmetric) neighborhood relation $\sim$ forms a *graph*. In a 1D

grid, each point inside the source has 2 immediate neighbors – i.e., points for which each coordinate differs by no more than 1 grid step; each of the endpoints has only 1 immediate neighbor. In a 2D rectangular grid, each point has $\leq 8$ immediate neighbors. In general, we have a small constant $C$ such that each point has no more than $C$ immediate neighbors.

It is natural to say that two pixels $a$ and $b$ from a set $L$ are *connected* in this set $L$ if we can reach $b$ from $a$ by going from a point to its neighbor, i.e., if there exists a sequence $a \sim a_1 \sim a_2 \sim \ldots \sim a_k \sim b$ connecting $a$ and $b$ in which all the intermediate points $a_1, \ldots, a_k$ belong to $L$; see, e.g, [8, 12, 13, 15, 19] and references therein. This connectedness relation is an equivalence relation (it is actually the transitive closure of the neighborhood relation $\sim$) and therefore, it divides each set of pixels $S \subseteq P$ (in particular, the set $W$ of all white points) into several equivalence classes – components of the original black and white source.

*Comment* 1. It is worth mentioning that the whole problem of separating the source into components only makes sense when the original set of pixels $P$ is itself connected. If $P$ is itself disconnected, we clearly should analyze different connected components of $P$ separately.

*Comment* 2. The above description is good for astronomical sources, for which the background is simply vacuum. In geospatial analysis, "white" points correspond to elevations, while "black" points correspond to depressions. Both elevations and depressions are meaningful zones. Thus, in addition to subdividing the set of all elevations into connected components, we must also classify the set of all depressions into connected components.

**How to compute components of a black and white source.** There exists a known algorithm that, given a graph with $n$ pixels, computes its connected components in linear ($O(n)$) time; see, e.g., [5].

This algorithm computes the components one by one; as a component is computed, the pixels that are already known to be from this component are marked by the number of this component. In the beginning, no pixels are marked. Once a component is computed, we check whether there are any unmarked points left; if yes, we start uncovering the next component, if no, we stop.

To uncover each component, we mark its pixels one by one. We start with the first unmarked pixel. After that, we mark all the un-marked immediate neighbors of all the pixels that are already known to belong to this component; when no such unmarked immediate neighbors remain, we stop – the component has been computed.

Since we mark each pixel only once, this algorithm indeed requires linear time.

**Gray-scale images.** In general gray-scale images, the intensity values $f(x)$ can be arbitrary non-negative real numbers, not just $b$ and $w$. How do we define

components for such images?

Intuitively, we say, e.g., that the source consists of two components, if we can visually represent these two components as two disconnected sets representing these components. In order to represent a component as a set, we need to choose a threshold $\mu$ so that all the points $x$ in which the intensity is higher than this threshold (i.e., in which $f(x) \geq \mu$) belong to the set, while all the points $x$ in which the intensity is smaller than this threshold (i.e., in which $f(x) < \mu$) are outside this set.

Thus, we say that a source consists of two components if for some threshold $\mu$, the superlevel set $\{x \mid f(x) \geq \mu\}$ is disconnected and consists of two connected components.

In principle, we can have arbitrary real values $\mu$ as thresholds. However, since in practice, we only know the values $f(x)$ of the intensity function at finitely many pixels $x$, it is sufficient to consider only thresholds that coincide with one of the values $f(x)$ – for any other threshold, the superlevel set is exactly the same as for one of these.

Thus, to separate an image into components, we should first sort $n$ values $f(x)$ $(x \in P)$ into a increasing sequence $\mu_1 < \mu_2 < \ldots < \mu_m$ for some $m \leq n$. If all $n$ intensities are different, then $m = n$; otherwise, we have $m < n$. For example, for a black and white image, $m = 2$, $\mu_1 = b$ and $\mu_2 = w$.

Once the values are sorted, we consider superlevel sets $L_i \stackrel{\text{def}}{=} \{x \mid f(x) \geq \mu_i\}$ one by one. We start with the set $L_1$, check if it is connected, and if it is not, we divide it into connected components. We then do the same with the set $L_2$, etc.

Since the values $\mu_i$ are sorted in increasing order, we have $L_1 \supseteq L_2 \supseteq \ldots \supseteq L_m$. When $i < j$, we have $L_i \supseteq L_j$ and therefore, each connected component of $L_j$ is a subset of $L_i$. Therefore, each connected component of $L_j$ is a subset of one of the connected components of $L_i$.

We start with a connected set $P$; we can say that this set constitutes a single connected component. On each level $i$, we get one or several connected components $S_{ij}$. On the next level $i+1$, three things can happen to the component $S_{ij}$:

- it may simply shrink, i.e., on the level $i+1$, there will be exactly one connected component that is contained in $S_{ij}$;
- it may split, i.e., on the level $i+1$, we may have several connected components all of which are contained in $S_{ij}$;
- it may disappear, i.e., on the level $i+1$, there will be no connected component that is contained in $S_{ij}$.

When a component shrinks, we simply have a subset of the original component, the structure does not change. Thus, to describe the structure of the original source, we must only keep the sets $S_{ij}$ corresponding to components that appeared for the first time – by splitting one of the connected components on the previous level $i-1$.

These "kept" connected components correspond to different levels. Not only we have components corresponding to different levels $\mu_i$; we also have a hierarchical structure, according to which each component on the next level is obtained by splitting one of the components that first appeared on the previous levels. This hierarchical structure is exactly what we mean by separating an astronomical-type image into components.

For geospatial zones, as we have mentioned before, we must analyze not only elevations but also depressions. Thus, in addition to the components corresponding to a superlevel set classification of $f(x)$, we should also compute components corresponding to the classification of $f'(x) \stackrel{\text{def}}{=} -f(x)$ – i.e., equivalently, to the classification corresponding to sublevel sets $L'_i = \{x \mid f(x) \le \mu_i\}$. For sublevel sets, we have $L'_1 \subseteq L'_2 \subseteq \ldots \subseteq L'_m$; therefore, we start with the largest sublevel set $L'_m$, then consider the next largest sublevel set $L'_{m-1}$, etc.

**Computational complexity of the resulting algorithm.** We already know that on each level, separation into components requires linear time $O(n)$. For a grid of size $n$, there are $\le n$ levels; thus, the overall computational complexity of this algorithm is $\le n \cdot O(n) = O(n^2)$.

**Comment: relation to Morse theory.** In this paper, we describe components by finding out how the topology of superlevel sets $\{x \mid f(x) \ge \mu\}$ and sublevel sets $\{x \mid f(x) \le \mu\}$ changes with $\mu$.

The idea of describing the shape of a manifold – in our case, the shape of a graph of a function $f(x)$ – by describing how the topology of the superlevel sets $\{x \mid f(x) \ge \mu\}$ changes with the change in $\mu$ is well known in mathematics: it is one of the main ideas behind *Morse theory*; for a classical exposition of Morse theory, see [16]. Morse theory has received much attention in the last two decades as a result of the paper by Witten [25] which relates Morse theory to quantum field theory; for latest developments in Morse theory, see. e.g., [2, 7, 18].

# 3 Separating Components in Interval-Valued Images: Formulation of the Problem, Propositions, and the Resulting Algorithm

## 3.1 Formulation of the problem

**Interval-valued images.** In the above algorithms, we assumed that for every pixel $x$, we know the exact value $f(x)$ of the image's intensity at this pixel. In reality, due to measurement uncertainty, the measured intensity value $\widetilde{f}(x)$ is only approximately equal to the actual (unknown) intensity value $f(x)$. If we know the upper bound $\Delta$ on the measurement error, we can conclude that for every pixel $x$, we know the interval $[\widetilde{f}(x) - \Delta, \widetilde{f}(x) + \Delta]$ that is guaranteed to contain the actual (unknown) values of $f(x)$.

In order words, instead of a number-valued image $f(x)$, we have an interval-valued image, i.e., for every pixel $x$, we know the interval $\mathbf{f}(x) = [\underline{f}(x), \overline{f}(x)]$ that is guaranteed to contain the actual (unknown) value of $f(x)$.

**What is the problem, and why this problem is difficult.** Based on this interval-valued image, we must classify the source into components. As we have mentioned in the previous section, to separate the source into components, we must check whether for each possible value $\mu$, the superlevel set $\{x \mid f(x) \geq \mu\}$ is connected or not, and if not, how many connected components it contains. We want a reliable method, i.e., we want such a method that if it concludes that there are several components, then this conclusion must be valid for all possible functions $f(x) \in \mathbf{f}(x)$.

There are infinitely many such functions $f(x)$, so we cannot simply check this property by checking all possible $f(x) \in \mathbf{f}(x)$, we must come up with an algorithm that is better than exhaustive search.

**What we are planning to do.** The following results provide the foundations for such an algorithm. To describe these results, we will introduce the notion of an "subgraph interval", and extend the definition of connectivity to such subgraph interval.

## 3.2 Definitions and results important for the analysis of interval-valued images

A subgraph $L$ of a finite graph $(P, \sim)$ is usually defined as a subset $L \subset P$ with edges inherited from $(P, \sim)$. An interval $\mathbf{a} = [\underline{a}, \overline{a}]$, where $\underline{a} \leq \overline{a}$, is usually defined as the set of all the real numbers between $\underline{a}$ and $\overline{a}$; in a computer, an interval is usually represented as a pair of floating-point numbers $\underline{a}$ and $\overline{a}$. Similarly, we want to describe a subgraph interval $\mathbf{L} = [\underline{L}, \overline{L}]$ of a graph $(P, \sim)$, where $\underline{L} \subseteq \overline{L} \subseteq P$, as the set of all subsets between $\underline{L}$ and $\overline{L}$; in a computer, this interval will be represented as a pair of sets $\underline{L}$ and $\overline{L}$. Thus, we arrive at the following definition:

**Definition 1.** *Let $(P, \sim)$ be a finite graph. By an* subgraph interval $\mathbf{L}$*, we mean a pair $[\underline{L}, \overline{L}]$, where $\underline{L} \subseteq \overline{L} \subseteq P$. We say that a set $L$* belongs *to $\mathbf{L}$ if $\underline{L} \subseteq L \subseteq \overline{L}$.*

**Definition 2.** *Let $(P, \sim)$ be a finite graph, and let $\mathbf{L} = [\underline{L}, \overline{L}]$ be its subgraph interval. We say that $\mathbf{L}$ is* possibly connected *if for every $a, b \in \underline{L}$, there exists a sequence $a \sim a_1 \sim a_2 \sim \ldots \sim a_n \sim b$ connecting $a$ and $b$ in which all intermediate elements $a_i$ belong to $\overline{L}$.*

The term "possibly connected" is justified by the following result:

**Proposition 1.** *Let $(P, \sim)$ be a finite graph, and let $\mathbf{L} = [\underline{L}, \overline{L}]$ be a subgraph interval of this graph; then, the following two conditions are equivalent to each other:*

- $\mathbf{L}$ *is possibly connected;*
- *there exists a connected set $L \in \mathbf{L}$.*

**Proof.** Let us first assume that there exists a connected set $L \in \mathbf{L}$, i.e., a connected set $L$ for which $\underline{L} \subseteq L \subseteq \overline{L}$. By definition of connectivity, this means that for every $a, b \in L$, there exists a sequence $a \sim a_1 \sim a_2 \sim \ldots \sim a_n \sim b$ connecting $a$ and $b$ in which all intermediate elements $a_i$ belong to $L$. Since $\underline{L} \subseteq L$, this means, in particular, that for every $a, b \in \underline{L}$, there exists a sequence $a \sim a_1 \sim a_2 \sim \ldots \sim a_n \sim b$ connecting $a$ and $b$ in which all intermediate elements $a_i$ belong to $L$. Since $L \subseteq \overline{L}$, this means that all intermediate elements $a_i$ in the connecting sequence also belong to $\overline{L}$. Thus, the subgraph interval $\mathbf{L}$ is indeed possibly connected.

Vice versa, let us assume that $\mathbf{L}$ is possibly connected; let us show that there exists a connected set $L \in \mathbf{L}$. Indeed, since the subgraph interval $\mathbf{L}$ is possibly connected, for every $a, b \in \underline{L}$, there exists a sequence of elements $a_i \in \overline{L}$ that connects $a$ and $b$. As the desired $L$, we can now take the union of the set $\underline{L}$ and of all such sequences. By definition, this set contains $\underline{L}$; since $\underline{L} \subseteq \overline{L}$ and all the sequences are formed only from elements of $\overline{L}$, we conclude that $L \subseteq \overline{L}$; hence, $L \in \mathbf{L}$. To complete the proof, it is therefore sufficient to prove that the set $L$ is connected.

Indeed, let $a, b \in L$. By definition of $L$, $a$ is either an element of $\underline{L}$, or it is an element of $\overline{L}$ that is connected to some element $a' \in \underline{L}$ by a sequence; by the construction of $L$, all the elements of this sequence belong to $L$. Thus, $a$ is either an element of $\underline{L}$, or it is connected to an element $a'$ from $\underline{L}$. Similarly, $b$ is either an element of $\underline{L}$, or it is connected to an element $b'$ from $\underline{L}$. Since a subgraph interval $\mathbf{L}$ is possibly connected, the elements $a', b' \in \underline{L}$ are connected by a sequence of elements from $\overline{L}$ – and thus, by a sequence of elements from $L$. Thus, $a$ is connected to $a'$, $a'$ is connected to $b'$, and $b'$ is connected to $b$ – hence, $a$ is connected to $b$, so $L$ is connected. The proposition is proven.

*Discussion.* Proposition 1 enables us to better understand the meaning of the notion of "possible connectedness".

Informally, there is a graph $L$ about which we only have partial information:

- we know the vertices that are guaranteed to be inside $L$; these vertices form the lower endpoint $\underline{L}$;
- we also know that all the vertices from $L$ must be from the given set $\underline{L}$.

About the vertices that are in $\overline{L}$ but not in $\underline{L}$, we do not know whether they belong to the graph $L$ or not. Our goal is to find out whether it is possible that

$L$ is a connected graph (and if $L$ cannot be connected, to extract as much information as possible about the number and location of the connected components of the graph $L$).

Let us illustrate this problem on two extreme examples: an example when we have a full knowledge of the graph $L$, and an example in which we do not have any knowledge about $L$.

Complete knowledge means that for every vertex, we know exactly whether this vertex belong to $L$ or not. In other words, in this example, $\underline{L} = \overline{L}$. Not surprisingly, such a "degenerate" subgraph interval is possibly connected if and only if the corresponding graph $\underline{L} = \overline{L}$ is connected.

In the other extreme, we may have no or almost no information about the graph $L$. For example, we may have the case when $\underline{L}$ consists of a single vertex $\{a\}$. This means that only about this vertex $a$, we are confident that this vertex belongs to the (unknown) graph $L$; about all other vertices, we do not know whether they belong to $L$ or not. In this case, it is quite possible that $L$ actually consists of this same single vertex; in this case, the graph $L$ is, of course, connected. Thus, the corresponding subgraph interval is possibly connected.

**Corollary.** *Let $(P, \sim)$ be a finite graph, and let $\mathbf{L} = [\underline{L}, \overline{L}]$ be a subgraph interval of this graph; then, the following two conditions are equivalent to each other:*

- *$\mathbf{L}$ is not possibly connected (in the sense of Definition 2);*
- *every set $L \in \mathbf{L}$ is disconnected.*

In our proof of Proposition 1, we have actually proven a stronger result:

**Definition 3.** *Let $\mathbf{L} = [\underline{L}, \overline{L}]$ be a subgraph interval of a graph $(P, \sim)$. By a* connectivity representative *of a subgraph interval, we mean the set $L_0$ of all the points $a \in \overline{L}$ that are connected with some point $b \in \underline{L}$ by a sequence from $\overline{L}$, i.e., for which there exists a sequence $a \sim a_1 \sim a_2 \sim \ldots \sim a_k \sim b$ in which all the intermediate points $a_1, \ldots, a_k$ belong to $\overline{L}$ and the final point $b$ belongs to $\underline{L}$.*

**Proposition 2.** *Let $(P, \sim)$ be a finite graph, and let $\mathbf{L} = [\underline{L}, \overline{L}]$ be a subgraph interval of this graph; then, the following three conditions are equivalent to each other:*

- *$\mathbf{L}$ is possibly connected;*
- *there exists a connected set $L \in \mathbf{L}$;*
- *the connectivity representative $L_0$ of the subgraph interval $\mathbf{L}$ is connected.*

**Corollary.** *Let $(P, \sim)$ be a finite graph, and let $\mathbf{L} = [\underline{L}, \overline{L}]$ be a subgraph interval of this graph; then, the following three conditions are equivalent to each other:*

- *$\mathbf{L}$ is not possibly connected (in the sense of Definition 2);*
- *every set $L \in \mathbf{L}$ is disconnected;*
- *the connectivity representative $L_0$ of the subgraph interval $\mathbf{L}$ is disconnected.*

Now, we are ready to describe the main result. Let $\mathbb{I}$ denote the set of all intervals.

**Definition 4.** *Let $(P, \sim)$ be a finite graph, let $\mathbf{f} : P \to \mathbb{I}$, and let $\mu$ be a real number. By a* superlevel set interval *$\{x \,|\, \mathbf{f}(x) \geq \mu\}$, we mean a subgraph interval*

$$[\{x \,|\, \underline{f}(x) \geq \mu\}, \{x \,|\, \overline{f}(x) \geq \mu\}]. \tag{2}$$

**Proposition 3.** *For every finite graph $(P, \sim)$, for every interval-valued function $\mathbf{f} : P \to \mathbb{I}$, and for every real number $\mu$, the following two conditions are equivalent to each other:*

- *there exists a function $f(x) \in \mathbf{f}(x) = [\underline{f}(x), \overline{f}(x)]$ for which the superlevel set $\{x \,|\, f(x) \geq \mu\}$ is connected;*
- *the superlevel set interval $\{x \,|\, \mathbf{f}(x) \geq \mu\}$ is possibly connected (in the sense of Definition 2).*

**Proof.** This Proposition is, in effect, an immediate corollary of Proposition 1, with $\underline{L} = \{x \,|\, \underline{f}(x) \geq \mu\}$ and $\overline{L} = \{x \,|\, \overline{f}(x) \geq \mu\}$. For readers who are not very familiar with digital and computational geometry, let us explain this in more detail.

Let us first assume that there exists a function $f(x) \in \mathbf{f}(x)$ for which the superlevel set is connected. Since $f(x) \in \mathbf{f}(x)$, we have $\underline{f}(x) \leq f(x) \leq \overline{f}(x)$. Therefore,

$$\{x \,|\, \underline{f}(x) \geq \mu\} \subseteq \{x \,|\, f(x) \geq \mu\} \subseteq \{x \,|\, \overline{f}(x) \geq \mu\}, \tag{3}$$

i.e., the superlevel set of $f(x)$ belongs to the superlevel set interval $\{x \,|\, \mathbf{f}(x) \geq \mu\}$. Since the superlevel set of $f(x)$ is connected, Proposition 1 enables us to conclude that the superlevel set interval $\{x \,|\, \mathbf{f}(x) \geq \mu\}$ is possibly connected.

Vice versa, let us assume that the superlevel set interval $\{x \,|\, \mathbf{f}(x) \geq \mu\}$ is possibly connected. Due to Proposition 1, we can now conclude that there exists a connected set $L$ for which

$$\{x \,|\, \underline{f}(x) \geq \mu\} \subseteq L \subseteq \{x \,|\, \overline{f}(x) \geq \mu\}. \tag{4}$$

Let us define the function $f(x)$ as follows: $f(x) = \overline{f}(x)$ for all $x \in L$ and $f(x) = \underline{f}(x)$ for all $x \notin L$. Let us show that for this function $f(x)$, the superlevel set $\{x \mid \overline{f}(x) \geq \mu\}$ is exactly $L$ – and therefore, this set is connected.

Indeed, if $x \in L$, then, by our construction of $f(x)$, we have selected, for this value $x$, $f(x) = \overline{f}(x)$. Since $L \in \mathbf{L}$, we conclude that $L \subseteq \{x \mid \overline{f}(x) \geq \mu\}$ hence, for every $x \in L$, we have $\overline{f}(x) \geq \mu$ hence $f(x) \geq \mu$. Thus, $L \subseteq \{x \mid f(x) \geq \mu\}$.

Vice versa, let $x \in \{x \mid f(x) \geq \mu\}$ (i.e., $f(x) \geq \mu$); let us show that in this case, $x \in L$. We will prove it by reduction to a contradiction. Let $x \notin L$; in this case, $f(x) = \underline{f}(x)$. Since $f(x) \geq \mu$, we can thus conclude that $\underline{f}(x) \geq \mu$, thus, $x$ belongs to the set $\{x \mid \underline{f}(x) \geq \mu\}$. We know, however, that $L$ belongs to the superlevel set interval $\{x \mid \mathbf{f}(x) \geq \mu\}$, hence $\{x \mid \underline{f}(x) \geq \mu\} \subseteq L$ and so $x$ is an element of $L$. The contradiction proves that the case $x \notin L$ is impossible, thus, $x \in L$. The proposition is proven.

**Corollary.** *For every finite graph $(P, \sim)$, for every interval-valued function $\mathbf{f} : P \to \mathbb{I}$, and for every real number $\mu$, the following two conditions are equivalent to each other:*

- *for every function $f(x) \in \mathbf{f}(x) = [\underline{f}(x), \overline{f}(x)]$, the superlevel set $\{x \mid f(x) \geq \mu\}$ is disconnected;*
- *the superlevel set interval $\{x \mid \mathbf{f}(x) \geq \mu\}$ is not possibly connected (in the sense of Definition 2).*

## 3.3 Resulting algorithm

**Main idea.** How can we use these results to check the connectedness? According to the proof of Proposition 1, a subgraph interval $\mathbf{L}$ is possibly connected if and only if its connectivity representative is connected. One can also easily see that if this representative is not connected, then the number of components in a connectivity representative indicates the smallest possible number of components in a superlevel set $\{x \mid f(x) \geq \mu\}$.

We already know that checking connectivity – and finding connected components – requires linear time; thus, to show that we can check possible connectedness of a subgraph interval, it is sufficient to be able to compute the connectivity representative in linear time.

**Proposition 4.** *There exists a linear time algorithm that, given a subgraph interval $\mathbf{L} = [\underline{L}, \overline{L}]$, computes its connectivity representative in linear time.*

**Proof.** This algorithm is similar to the above algorithm for computing the connected components. In this algorithm, we deal with elements of $\overline{L}$. At each stage of the algorithm, some of these elements are marked. We start with marking all the elements from $\underline{L}$. After that, we mark all the un-marked immediate neighbors of all the marked pixels; when no such unmarked immediate

neighbors remain, we stop – the marked points form the desired connectivity representative.

Since we mark each pixel only once, this algorithm requires linear time.

**Choice of thresholds.** What thresholds $\mu$ should we choose? To cover all possible superlevel set intervals, it is sufficient to consider only values of $\mu$ that coincide either with $\underline{f}(x)$ or with $\overline{f}(x)$ for some $x \in P$.

Thus, to separate an interval-valued image into components, we should first sort $2n$ values $\underline{f}(x)$ and $\overline{f}(x)$ $(x \in P)$ into a increasing sequence $\mu_1 < \mu_2 < \ldots < \mu_m$ for some $m \le 2n$. If all $2n$ intensities are different, then $m = 2n$; otherwise, we have $m < 2n$.

**Final description of the algorithm.** Once the values are sorted, we consider superlevel set intervals $\mathbf{L}_i \stackrel{\text{def}}{=} \{x \,|\, \mathbf{f}(x) \ge \mu_i\}$ one by one. We start with the set interval $\mathbf{L}_1$, check if it is possibly connected, and if it is not, we divide it into connected components. We then do the same with the set $\mathbf{L}_2$, etc.

For geospatial applications, we should also perform a similar analysis for sublevel set intervals $\mathbf{L}'_i \stackrel{\text{def}}{=} \{x \,|\, \mathbf{f}(x) \le \mu_i\}$ one by one. We start with the set interval $\mathbf{L}'_m$, check if it is possibly connected, and if it is not, we divide it into connected components. We then do the same with the set $\mathbf{L}_{m-1}$, etc.

**The new algorithm is free from the above-mentioned limitations of the existing heuristic algorithms.** As we have mentioned, one of the problems with heuristic methods like the Laplacian method is that their results change if we use a different projection to describe geospatial data.

The proposed new algorithm bases its decisions (on the connectedness and on the location and number of components) only on the topological structure of the superlevel sets $\{x \,|\, f(x) \ge \mu\}$ and the sublevel sets $\{x \,|\, f(x) \le \mu\}$. In contrast to, e.g., the Laplacian method, this topological structure does not change if we simply move to a new map projections, i.e., if we replace the function $f'(x_1, x_2)$ with a new function $f(x_1, x_2) = f'(T_1(x_1, x_2), T_2(x_1, x_2))$ – as long as the transformation $x'_1 = T_1(x_1, x_2)$, $x'_2 = T_2(x_1, x_2)$ is continuous in both directions (from $x_i$ to $x'_i$ and from $x'_i$ to $x_i$). Thus, for the new method, the resulting components do not depend on what projection we use.

**Computational complexity of the resulting algorithm.** We already know that on each level, separation into components requires linear time $O(n)$. For a grid of size $n$, there are $\le 2n$ levels; thus, the overall computational complexity of this algorithm is $\le 2n \cdot O(n) = O(n^2)$.

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