USING 1-D RADAR OBSERVATIONS TO DETECT A SPACE EXPLOSION CORE AMONG THE EXPLOSION FRAGMENTS: SEQUENTIAL AND DISTRIBUTED ALGORITHMS

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ABSTRACT

A radar observes the result of a space explosion. Due to radar's low horizontal resolution, we get a 1-D signal $x(t)$ representing different 2-D slices. Based on these slices, we must distinguish between the body at the core of the explosion and the slowly out-moving fragments. We propose new algorithms for processing this 1-D data. Since these algorithms are time-consuming, we also exploit the possibility of parallelizing these algorithms.

1. FORMULATION OF THE PROBLEM

Most astronomical processes are slow; however, sometimes, space explosions happen: starts become supernovae, planetoids are torn apart by tidal and gravitational forces, etc. Even the Universe itself is currently viewed as a result of such an explosion – the Big Bang.

From the astrophysical viewpoint, these explosions are very important, because, e.g., supernovae explosions is how heavy metals spread around in the Universe.

The explosion processes are very rare and very fast, so unless they are very powerful and spectacular – like an explosion of a nearby supernovae that happened in 1054 – they are very difficult to observe. As a result, space explosion processes often go unnoticed.

What we do observe in most cases is the *result* of the space explosion, i.e., the explosion core – the remainder of the original celestial body – surrounded by the explosion fragments. The most well known example of such a result is the Crab Nebula formed after the 1054 supernovae explosion.

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In order to better understand the corresponding physical process, it is extremely important to identify the explosion core.

In space, there is not much friction, so, due to inertia, most of the fragments travel with approximately the same speed as in the beginning of the explosion. Dividing the distance between the two fragments by their relative speed, we can determine – reasonably accurately – when the explosion occurred (this is how we know that the supernovae in the Crab Nebulae exploded in the year 1054). At that explosion time, all the fragments and the core were located at the same point, so it is difficult to distinguish between the core and the fragments.

In general, we have a 2-D (and sometimes even 3-D) image of the result of the explosion. In such situations, detecting the explosion core is an image processing problem.

However, there is one important case when we only have 1-D data. In this case, we cannot use image processing techniques, we have to use techniques for processing 1-D data – i.e., DSP techniques.

This is the case of nearby space explosions, when the radar is the main source of information. A radar sends a pulse signal toward an object; this signal reflects from the object back to the station. We can measure, very accurately, the overall time that the signal traveled, which gives us the distance to the object. We can also measure the velocity, or, to be more precise, the rate with which the distance changes. It is, however, very difficult to separate the signals from different fragments located at the same distance.

As a result, what we observe is a 1-D signal $s(t)$, where each value $s(t)$ represents the intensity of the reflection from all the fragments located at distance $c \cdot t$ from the radar – i.e., from the 2-D slice corresponding to this distance. Based on these slices, we must distinguish between the body at the core of the explosion and the (slowly ex-

panding) fragments.

In this paper, we describe a new method of identifying a core based on the slice observations.

2. A NEW METHOD FOR SOLVING THE PROBLEM: MAIN IDEA

2.1. Repeated signal measurements at several different moments of time T_k

At first glance, there may seem to be no difference between the signals reflected by the fragments and the signal reflected by the core. However, in the process of an explosion, fragments usually start rotating fast, at random rotation frequencies, with random phases. As a result, the signals reflected from the fragments oscillate, while the signal from the original core practically does not change.

As a result, the reflected signals change with time. Therefore, it makes sense to measure the signal $s(t)$ not just once, but at several consequent moments of time, i.e., to consider the signals $s_1(t), \ldots, s_N(t)$ measured at moments $T_1 < \ldots < T_N$, and use the difference between the dynamic character of the fragments and the static character of the core to identify the core.

2.2. Relating measurements performed at different moments of time $T_k \neq T_l$: the corresponding t-scales are linearly related

In order to compare signals measured at different moments of time $T_k \neq T_l$, we must identify the layers measured at different moments of time.

Let T_0 be the moment of explosion, and let x_0 be the initial distance between the radar and the core (and the fragments) at that initial moment of time T_0. We assume that our coordinate system has the radar as its origin, and that the x axis is the axis in the direction of the analyzed "cloud". For each fragment i, let $v_x^{(i)}$ be the x-component of the velocity of i-th fragment (velocity relative to the radar). Hence, at moment T_k, the x-coordinate of i-th fragment in our coordinate system – i.e., its distance from the radar – is equal to $x^{(i)}(T_k) = x_0 + v_x^{(i)} \cdot (T_k - T_0)$. Therefore, the radar signal reflected from this fragment corresponds to the time

$$t_k^{(i)} = \frac{x_k^{(i)}}{c} = \frac{x_0}{c} + v_x^{(i)} \cdot \frac{T_k - T_0}{c}. \qquad (1)$$

Similarly, when we repeat the radar measurement at time $T_l \neq T_k$, the radar signal reflected from the i-th fragment corresponds to the time

$$t_l^{(i)} = \frac{x_0}{c} + v_x^{(i)} \cdot \frac{T_l - T_0}{c}. \qquad (2)$$

What is the relation between the corresponding times $t_k^{(i)}$ and $t_l^{(i)}$? From the equation (1), we conclude that

$$v_x^{(i)} = \frac{c \cdot t_k^{(i)} - x_0}{T_k - T_0}.$$

Substituting this expression into the formula (2), we conclude that

$$t_l^{(i)} = \frac{x_0}{c} + \frac{c \cdot t_k^{(i)} - x_0}{T_k - T_0} \cdot \frac{T_l - T_0}{c} = a_{kl} \cdot t_k^{(i)} + b_{kl}, \quad (3)$$

where

$$a_{kl} = \frac{T_l - T_0}{T_k - T_0} > 0$$

and

$$b_{kl} = \frac{x_0}{c} - \frac{x_0}{T_k - T_0} \cdot \frac{T_l - T_0}{c}$$

do not depend on i.

In other words, the t-scales of the signals $s_k(t)$ and $s_l(t)$ are related by a linear dependence $t_k \to t_l = a_{kl} \cdot t_k + b_{kl}$.

2.3. How can we experimentally find the coefficients of this linear relation?

At each moment of time T_k, we get the observed signal $s_k(t)$. Let $\underline{t}_k$ be the smallest time at which we get some reflection from the fragments cloud, and let $\overline{t}_k$ be the largest time at which we observe the radar reflection from this cloud. This means that there is a fragment i for which $t_k^{(i)} = \underline{t}_k$, there is a fragment j for which $t_k^{(j)} = \overline{t}_k$, and for every other fragment f, the corresponding moment of time is in between $\underline{t}_k$ and $\overline{t}_k$: $t_k^{(f)} \in [\underline{t}_k, \overline{t}_k]$.

As we have mentioned, for every other observation T_l, the relation between the corresponding times $t_k^{(i)}$ and $t_l^{(i)}$ is linear, with a positive coefficient a_{kl}. Since $a_{kl} > 0$, the corresponding linear functions $t \to a_{kl} \cdot t + b_{kl}$ is monotonically increasing. Thus, the value t_l is the smallest for the same fragment i for which t_k was the smallest. Hence, $\underline{t}_l = t_l^{(i)} = a_{kl} \cdot t_k^{(i)} + b_{kl}$, i.e.,

$$\underline{t}_l = a_{kl} \cdot \underline{t}_k + b_{kl}. \qquad (4)$$

Similarly,

$$\overline{t}_l = a_{kl} \cdot \overline{t}_k + b_{kl}. \qquad (5)$$

The values $\underline{t}_k$, $\overline{t}_k$, $\underline{t}_l$, and $\overline{t}_l$ are directly observable. Thus, by solving the system of two linear equations (4) and (5) with 2 unknowns, we get explicit expressions for a_{kl} and b_{kl} in terms of these observable values:

$$a_{kl} = \frac{\overline{t}_l - \underline{t}_l}{\overline{t}_k - \underline{t}_k}; \quad b_{kl} = \frac{\overline{t}_k \cdot \underline{t}_l - \underline{t}_k \cdot \overline{t}_l}{\overline{t}_k - \underline{t}_k}.$$

2.4. How can we transform signals $s_k(t)$ and $s_l(t)$ to the same scale?

Our main idea is that after we measure the fragments cloud at two different moments of time T_k and T_l, we should compare the values $s_k(t)$ and $s_l(t)$ corresponding to the same fragments.

We know that for each moment of time t, the value $s_k(t)$ describes the same fragment(s) as the value $s_l(t')$, where $t' = a_{kl} \cdot t + b_{kl}$. We also know how to experimentally determine the coefficients a_{kl} and b_{kl}. So, to make the desired comparison easier, it is reasonable to "re-scale" the signals to the same t-scale, so that the compared values correspond to exactly the same value t. In other words, we would like to generate a re-scaled signal

$$\widetilde{s}_l(t) \stackrel{\text{def}}{=} s_l(a_{kl} \cdot t + b_{kl}). \tag{6}$$

If the measurements were absolutely accurate, i.e., if we had the values $s_k(t)$ corresponding to each individual time t, then such a re-scaling would be easy: we could simply explicitly use the formula (6).

In real life, however, each value $s_l(t)$ corresponds not just to a single time t, but to the entire "bin" of values, from some value $\underline{t}$ to the value $\underline{t} + \Delta t$, where Δt is the accuracy with which the radar can measure the time t (in other words, $\Delta t = \Delta x/c$, where Δx is the accuracy with which the radar can measure the distance). In other words, what we actually observe is a sequence of values $\ldots$, $s((i-1)\cdot\Delta t)$, $s(i\cdot\Delta t)$, $s((i+1)\cdot\Delta)$, $\ldots$ Crudely speaking, each observed value $s(i \cdot \Delta t)$ represent the overall intensity of all the fragments for which the actual reflection time $t = x/c$ is in the interval

$$I_i \stackrel{\text{def}}{=} [(i-0.5)\cdot\Delta t, (i+0.5)\cdot\Delta t]. \tag{7}$$

Because of this discreteness, we cannot directly use the formula (6) to match the signals: Indeed, from the moment T_k to the moment T_l, the cloud slightly expands. At the moment T_k, the value $s_k(i \cdot \Delta t)$ is the overall intensity of all the fragments for which t_k belongs to the interval (6) of width Δt. At moment T_l, the times $t_l = a_{kl} \cdot t_k + b_{kl}$ corresponding to these same fragments occupy a wider interval – of width $a_{kl} \cdot \Delta t > \Delta t$. Thus, these fragments are no longer in the same bin, they may be in different bins.

How can we match the values? A natural idea is to use linear extrapolation. In other words, to estimate $\widetilde{s}(t)$ for $t = i \cdot \Delta t$, we apply the linear transformation $a_{kl} \cdot t + b_{kl}$ to the interval I_i. The resulting interval $\widetilde{I}_i$ consists of several parts from different intervals I_j. As $\widetilde{s}_l(t)$, we take a linear combination of the corresponding values $s_l(j \cdot \Delta t)$, with weights proportional to the relative length $|\widetilde{I}_i \cap I_j|/\Delta t$ of the intersection $\widetilde{I}_i \cap I_j$:

$$\widetilde{s}_l(i\cdot\Delta t) \stackrel{\text{def}}{=} \sum_j \frac{|\widetilde{I}_i \cap I_j|}{\Delta t} \cdot s_l(j\cdot\Delta t).$$

For example, if $\widetilde{I}_i$ consists of the entire interval I_j, 0.1 of I_{j-1}, and 0.05 of I_{i-1}, then $\widetilde{s}_l(i\cdot\Delta t)$ is equal to:

$$0.1 \cdot s_l((i-1)\cdot\Delta t) + s_l(i\cdot\Delta t) + 0.05 \cdot s_l((i+1)\cdot\Delta t).$$

In the following text, we will assume that the signals $s_i(t)$ have already been thus rescaled.

2.5. Algorithm: main idea

Each layer ("bin") contains several fragments. These fragments oscillate with random (uncorrelated) frequencies and phases; the overall signal $x(t)$ is the sum of the reflections from all these fragments. Due to the central limit theorem, the resulting overall signal $x(t)$ is approximately normally distributed with some mean $E(t)$ and variance $V(t)$.

If a layer only contains fragments, then, due to the independence assumption, $E(t) \approx n(t)\cdot E$ and $V(t) \approx n(t)\cdot V$, where $n(t)$ is the (unknown) number of fragment in layer t, and E and V are the mean and variance corresponding to each fragment. Therefore, for each such layer, $E(t) \approx (E/V)\cdot V(t)$.

For a layer that also contains the core, we have $E(t) \approx E_c + N(t)\cdot E$ and $V(t) \approx N(t)\cdot V$, where E_c is the intensity of the core (since the core is supposed to be not rotating fast, its signal does not change with time, so the corresponding variance is negligible). Thus, for this layer, $E(t) \approx E_c + (E/V)\cdot V(t)$. So, for the core, $E(t)/V(t) \gg E/V$.

Therefore, crudely speaking, our best guess for the core location is the point t for which the ratio $E(t)/V(t)$ is the largest.

This is, of course, a very naive description of the idea. Let us see how this idea can be described in more adequate DSP terms.

3. TOWARDS A STATISTICALLY VALID ALGORITHM

3.1. Motivations for the main distribution formula

The intensity $I_i(t)$ of each fragment i depends on time. Let $a_i = \lim\limits_{T\to\infty} T^{-1}\cdot\int\limits_0^T I_i(t)\,dt$ denote the average intensity over time, and let $b_i = \lim\limits_{T\to\infty} T^{-1}\cdot\int\limits_0^T (I_i(t)-a_i)^2\,dt$.

In the ensemble of fragments, let a_0 be the mean of a_i, let A_0 be the variance of a_i, let b_0 be the mean of b_i, and let B_0 be the mean of a_i. Then, according to the main idea, we can assume that $E(t)$ is normally distributed with the mean $n(t)\cdot a_0$ and the variance $n(t)\cdot A_0$, and $V(t)$ is normally distributed with the mean $n(t)\cdot b_0$ and the variance $n(t)\cdot B_0$.

We assumed the layers to be independent. As a result, we arrive at the following formula for the resulting proba-

bility distribution:

$$\rho = \prod_{t=1}^{N} \frac{1}{\sqrt{2\pi \cdot n(t) \cdot A_0}} \cdot \exp\left(-\frac{(E(t) - n(t) \cdot a_0)^2}{2n(t) \cdot A_0}\right) \times$$

$$\prod_{t=1}^{N} \frac{1}{\sqrt{2\pi \cdot n(t) \cdot B_0}} \cdot \exp\left(-\frac{(V(t) - n(t) \cdot b_0)^2}{2n(t) \cdot B_0}\right),$$

with the proviso that for the layer $t = t_0$ containing the core, we have $E(t) - E_c - n(t) \cdot a_0$ instead of $E(t) - n(t) \cdot a_0$.

Based on the experimental data $E(t)$ and $V(t)$, we must find estimates for the parameters a_0, b_0, A_0, B_0, $n(t)$, t_0, and E_c – and what we are really interested in is t_0. In accordance with the Maximum Likelihood Method (MLM), we must find the values of these parameters for which $\rho \to \max$. As usual in statistics, it is convenient to replace the problem of maximizing ρ with a mathematically equivalent problem of minimizing a simpler function $\psi \stackrel{\text{def}}{=} -\ln(\rho)$, i.e., in our case,

$$\psi = \sum_{t=1}^{N} \frac{(E(t) - n(t) \cdot a_0)^2}{2n(t) \cdot A_0} + \sum_{t=1}^{N} \frac{(V(t) - n(t) \cdot b_0)^2}{2n(t) \cdot B_0} +$$

$$\sum_{t=1}^{N} \ln(n(t)) + \frac{N}{2} \cdot \log(A_0) + \frac{N}{2} \cdot \log(B_0). \qquad (8)$$

3.2. First case: when we know the parameters that characterize fragment distribution

Let us start with the simplest case when we know the values of the parameters a_0, b_0, A_0, and B_0 that describe the distribution of fragments. In this case, differentiating by $n(t)$ and equating the derivative to 0, we conclude that

$$-\frac{1}{2n(t)^2}\left(\frac{E(t)^2}{A_0} + \frac{V(t)^2}{B_0}\right) + \frac{1}{2}\left(\frac{a_0^2}{A_0} + \frac{b_0^2}{B_0}\right) + \frac{1}{n(t)} = 0.$$

The first two terms are approximately independent on the number of fragments $n(t)$, the third term $1/n(t)$ is much smaller (since we have many fragments). So, we can safely ignore the their term and conclude that $n(t) = \|v_t\|/\|v_0\|$, where we denoted

$$v_t \stackrel{\text{def}}{=} \left(\frac{E(t)}{\sqrt{A_0}}, \frac{V(t)}{\sqrt{B_0}}\right); \quad v_0 \stackrel{\text{def}}{=} \left(\frac{a_0}{\sqrt{A_0}}, \frac{b_0}{\sqrt{B_0}}\right),$$

and $\|(v_a, v_b)\| = \sqrt{v_a^2 + v_b^2}$ denotes the length of the vector $v = (v_a, v_b)$. Substituting this expression for $n(t)$ into the corresponding part of (8), i.e., into

$$\psi(t) \stackrel{\text{def}}{=} \frac{(E(t) - n(t) \cdot a_0)^2}{2n(t) \cdot A_0} + \frac{(V(t) - n(t) \cdot b_0)^2}{2n(t) \cdot B_0} +$$

$$\ln(n(t)) = \frac{1}{2n(t)} \cdot \left(\frac{E(t)^2}{A_0} + \frac{V(t)^2}{B_0}\right) -$$

$$\left(\frac{E(t) \cdot a_0}{A_0} + \frac{V(t) \cdot a_0}{A_0}\right) + \frac{n(t)}{2} \cdot \left(\frac{a_0^2}{A_0} + \frac{b_0^2}{B_0}\right) + \ln(n(t)),$$

we conclude that $\psi(t) \approx \psi_0(t)$, where

$$\psi_0(t) \stackrel{\text{def}}{=} \|v_t\| \cdot \|v_0\| - v_t \cdot v_0, \qquad (9)$$

and $v_t \cdot v_0$ denotes the dot (scalar) product. ($\approx$ because we use the approximate value for $n(t)$.)

For $t = t_0$, due to the presence of an additional variable E_c, we get $\psi(t_0) \approx 0$. Thus,

$$\psi = (N/2) \cdot (\log(A_0) + \log(B_0)) + \sum_{t=1}^{N} \psi_0(t) - \psi_0(t_0).$$

Thus, ψ is the smallest if and only if $\psi(t_0)$ is the largest. Therefore, we arrive at the following algorithm for locating the core:

- First, we re-scale the signals $s_k(t)$ into $\widetilde{s}_k(t)$ so that the same value t corresponds to the same fragments.
- For each t, we compute the sample average $E(t)$ and the sample variance $V(t)$ of the values $\widetilde{s}_k(t)$.
- For each t, we compute v_t and $\psi_0(t)$, and find t_0 for which $\psi_0(t_0) = m \stackrel{\text{def}}{=} \max_t \psi_0(t)$.

How reliable is this estimate? We are interested in the value of a single variable t_0, and we know that for one variable, 95% of the values are within 2σ from the mean, and 99.9% are within 3σ. In terms of $\psi = \ln(\rho)$, the mean corresponds to its minimum, the 2σ deviation means difference $(2\sigma)^2/(2\sigma^2) = 2$ from the minimum, and 3σ deviation means the difference of $(3\sigma)^2/(2\sigma^2) = 4.5$ from the minimum. Thus, with reliability 95%, we conclude that the core is among those t for which $\psi_0(t) \geq m - 2$, and that with reliability 99.9%, the core is among those t for which $\psi_0(t) \geq m - 4.5$.

3.3. General case

The value (8) does not change if we re-scale all the parameters: $n(t) \to K \cdot n(t)$, $a_0 \to a_0/K$, $b_0 \to b_0/K$, $A_0 \to A_0/K$, and $B_0 \to B_0/K$, for any $K > 0$. W.l.o.g., we can therefore assume that $a_0 = 1$.

Differentiating (8) by a_0, we conclude that $a_0 = (\sum E(t))/(\sum n(t))$. Similarly, $b_0 = (\sum V(t))/(\sum n(t))$. Since $a_0 = 1$, we thus get $b_0 = (\sum V(t))/(\sum E(t))$. Differentiating by A_0, we conclude that

$$A_0 = \frac{1}{N} \sum_t \frac{(E(t) - n(t) \cdot a_0)^2}{n(t)} =$$

$$\frac{1}{N}\left(\sum_t \frac{E(t)^2}{n(t)} - \sum_t E(t)\right) \qquad (10)$$

and similarly,

$$B_0 = \frac{1}{N}\left(\sum_t \frac{V(t)^2}{n(t)} - b_0 \cdot \sum_t V(t)\right). \qquad (11)$$

If we denote $\lambda \stackrel{\text{def}}{=} A_0/B_0$, then the above formula for $n(t)$ takes the form $n(t)^2 = (E(t)^2 + \lambda \cdot V^2(t))/(1 + \lambda \cdot b_0^2)$. Substituting this expression into (10) and (11) and using the fact that $A_0 = \lambda \cdot B_0$, we conclude that

$$\sum_t \frac{E(t)^2}{\sqrt{E(t)^2 + \lambda \cdot V(t)^2}} \cdot \sqrt{1 + \lambda \cdot b_0^2} - \sum_t E(t) =$$

$$\sum_t \frac{\lambda \cdot V(t)^2}{\sqrt{E(t)^2 + \lambda \cdot V(t)^2}} \cdot \sqrt{1 + \lambda \cdot b_0^2} - b_0 \cdot \left(\sum_t V(t)\right)$$

with the only unknown λ. After we find λ from this equation, we can thus find A_0, B_0, and hence, the desired t_0.

To test our technique, we simulated an explosion with randomly distributed fragments. On this simulation, the above algorithm does detect the core.

4. POSSIBILITY OF PARALLELIZATION

In the above algorithms, processing values corresponding to bin i uses only measurement only from this bin and from the neighboring bins. Therefore, if we have several processors working in parallel (see, e.g., [1]), we can speed up the computations by having each processor process a section of bins. For example, for 2 processors, the first can handle bins 1 to $N/2 + n$, and the second all the bins from $N/2 - n$ to N, where n is the number of neighboring bins that we need to take into consideration.

5. MULTIPLE EXPLOSIONS: CASE OF A VERY ACCURATE RADAR

Sometimes, the observed fragments cloud comes not from a single explosion, but from several consequent explosions. How can we then determine the core?

Let us show that when the radar is accurate enough, so that we can distinguish between individual fragments, the problem of determining the core becomes even easier than in the case of a single explosion.

First, we observe that if the radar is that accurate, then, by making observations at very close moments of time T_1, T_2, etc., we can *trace* individual fragments. Indeed, at the initial moment T_1, we identify fragments by the times $t_1^{(1)} < t_1^{(2)} < \ldots$ at which the corresponding signal $s_1(t)$ is non-zero. At the next moment T_2, we can find the times t_2, t_2', ... corresponding to the fragments as the times t for which $s_2(t) \neq 0$. When the time difference $T_2 - T_1$ is so small that the relative motion of a fragment is smaller than the distance between different fragments, we can identify, for each fragment i, the corresponding time $t_2^{(i)}$ as the closest to $t_1^{(i)}$ among all observed values t_2, t_2', ...

For a single explosion, a linear formula (3) relates $t_2^{(i)}$ and $t_1^{(i)}$; the corresponding slope a_{kl} depends on the moment T_0 of the explosion. If two explosions occurred at moments T_0 and T_0', we get similar linear formulas for the fragments of each explosion, with two slopes $a_{kl} \neq a_{kl}'$. Thus, by plotting the dependence of $t_2^{(i)}$ on $t_1^{(i)}$, we will get two straight lines with different slopes. The core belongs to both families of fragments. Thus, the core can be determined as the fragment i_0 that lies at the intersection of the two corresponding straight lines.

For two explosions, we can determine both lines and easily find the intersection. For numerous explosions, we will have many straight lines, and finding all of them may be computationally difficult; so, we need a different idea.

The dependence of a_k on T_0 is monotonic, so in such situations, the 2-D points $t^{(i)} \stackrel{\text{def}}{=} (t_1^{(i)}, t_2^{(i)})$ occupy a zone between two straight lines with different slope $\underline{a} < \overline{a}$ corresponding to the first and the last explosions; geometrically, it is a 2-D cone with the core's value $t^{(i_0)}$ as the vertex. Since we have numerous explosions, we can conclude that the corresponding pairs fill the entire cone.

Let us show that the core can be determined as the only value i for which

$$\max_{j:\, t_1^{(j)} < t_1^{(i)}} t_j^{(2)} < \min_{j:\, t_1^{(j)} > t_1^{(i)}} t_j^{(2)}. \qquad (12)$$

Let us first consider the case $i = i_0$. For each of the corresponding straight lines, the dependence of $t_2^{(i)}$ on $t_1^{(i)}$ is monotonically increasing; since the core i_0 belongs to all the lines, we can therefore conclude that if $t_1^{(j)} < t_1^{(i_0)}$, then we have $t_2^{(j)} < t_2^{(i_0)}$, and if $t_1^{(j)} > t_1^{(i_0)}$, then we have $t_2^{(j)} > t_2^{(i_0)}$ – which implies (12).

If $t_1^{(i)} > t_1^{(i_0)}$, then the maximum in the left side of the formula (12) corresponds to the largest possible slope $\overline{a}_{kl}$ and is therefore equal to $t_2^{(i_0)} + \overline{a}_{kl} \cdot (t_1^{(i)} - t_1^{(i_0)})$. On the other hand, the minimum in the right side of the formula (12) corresponds to the smallest possible slope slope $\underline{a}_{kl}$ and is therefore equal to $t_2^{(i_0)} + \underline{a}_{kl} \cdot (t_1^{(i)} - t_1^{(i_0)})$ – which is clearly smaller than the maximum in the left side of (12).

Similarly, (12) cannot occur for $t_i^{(i)} < t_1^{(i_0)}$.

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