Why Product of Probabilities (Masses) for Independent Events? A Theorem

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Abstract

For independent events A and B, the probability $P(A \,\&\, B)$ is equal to the product of the corresponding probabilities: $P(A\&B) = P(A) \cdot P(B)$. It is well known that the product $f(a,b) = a \cdot b$ has the following property: once $\sum\limits_{i=1}^{n} P(A_i) = 1$ and $\sum\limits_{j=1}^{m} P(B_j) = 1$, the probabilities $P(A_i \,\&\, B_j) = f(P(A_i), P(B_j))$ also add to 1: $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} f(P(A_i), P(B_j)) = 1$. We prove that the product is the only function that satisfies this property, i.e., that if, vice versa, this property holds for some function $f(a,b)$, then this function f is the product. Thus, we provide an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

Product is normally used as a combination rule for independent events. For independent events A and B, the probability $P(A \,\&\, B)$ is equal to the product of the corresponding probabilities: $P(A\&B) = f(P(A), P(B))$, where the combination function is the product $f(a,b) = a \cdot b$; see, e.g., [5].

Similarly, in Dempster-Shafer theory (see, e.g., [2, 6]) one of the ways to combine the masses from two independent knowledge bases is to multiply them.

A reasonable property of the combination rule. Due to the additivity property of probability, if the events $A_1, \ldots, A_n$ form a partition of the universal set, i.e., if one of these events always occurs and no two can occur at the same time, then $\sum\limits_{i=1}^{n} P(A_i) = 1$. If the events A_i form a partition and the events B_j

form a partition, then their combinations $A_i \,\&\, B_j$ also form a partition; indeed:

- since A_i and B_j form a partition, any situation belongs to one of A_i and to one of B_j, thus, for this situation, the corresponding event $A_i \,\&\, B_j$ holds;
- similarly, since the events A_i are mutually exclusive and the events B_j are mutually exclusive, the combinations $A_i \,\&\, B_j$ are also mutually exclusive.

It is therefore reasonable to expect that if the events A_i form a partition, i.e., $\sum\limits_{i=1}^{n} P(A_i) = 1$, and if events B_j form a partition, i.e., $\sum\limits_{j=1}^{m} P(B_j) = 1$, then the events $A_i \,\&\, B_j$ should also form a partition, i.e., $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} f(P(A_i), P(B_j)) = 1$.

In formal terms, the function $f : [0,1] \times [0,1] \to [0,1]$ that describes the combination rule should satisfy the following property:

$$\begin{gathered}\text{For every two finite sequences}\\ \text{of non-negative real numbers } (a_1, \ldots, a_n) \text{ and } (b_1, \ldots, b_m), \\ \text{if } \sum_{i=1}^{n} a_i = 1 \text{ and } \sum_{j=1}^{m} b_j = 1, \text{ then } \sum_{i=1}^{n} \sum_{j=1}^{m} f(a_i, b_j) = 1.\end{gathered} \tag{1}$$

What is known. It is well known that the product function $f(a,b) = a \cdot b$ satisfies the property (1). It is also known that many other possible combination functions, e.g., many t-norms that are different from the product (see, e.g., [3, 4]), do not satisfy this property.

What we will prove. In this paper, we prove that the product function is the only function that satisfies the above property. We will also prove a similar result for the case when we combine more than two events.

Theorem 1. *If a function $f : [0,1] \times [0,1] \to [0,1]$ satisfies the property* (1)*, then this function is the product: $f(a,b) = a \cdot b$ for all a and b.*

Conclusion. Thus, we provide an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

Case of several events. Let $k \geq 2$ be an integer, and let $f : [0,1]^k \to [0,1]$ be a function of k variables. For such functions, we will consider the following property:

For every k finite sequences

of non-negative real numbers $(a_1^{(1)}, \ldots, a_{n_1}^{(1)}), \ldots, (a_1^{(k)}, \ldots, a_{n_k}^{(k)})$,

$$\text{if } \sum_{i_1=1}^{n_1} a_{i_1}^{(1)} = 1 \text{ and } \ldots \text{ and } \sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1, \tag{2}$$

$$\text{then } \sum_{i_1=1}^{n_1} \ldots \sum_{i_k=1}^{n_k} f(a_{i_1}^{(1)}, \ldots, a_{i_k}^{(k)}) = 1.$$

Theorem 2. *If a function $f : [0,1]^k \to [0,1]$ satisfies the property* (2)*, then this function is the product: $f(a_1, \ldots, a_k) = a_1 \cdot \ldots \cdot a_k$ for all $a_1, \ldots, a_k$.*

Proof of the Theorems. The proof of Theorems 1 and 2 is based on the following Lemma:

Lemma. *Let a function $g : [0,1] \to R_0^+ \stackrel{\text{def}}{=} [0, \infty)$ satisfy the following property:*

For every finite sequence of non-negative real numbers $(a_1, \ldots, a_n)$,

$$\textit{if } \sum_{i=1}^{n} a_i = 1, \textit{ then } \sum_{i=1}^{n} g(a_i) = 1. \tag{3}$$

Then, $g(a) = a$ for every real number a.

Proof of the Lemma. Let us first consider the case when $n = 2$. In this case, the condition of the Lemma means that $a_1 + a_2 = 1$ implies $g(a_1) + g(a_2) = 1$, i.e., that $g(a_2) = 1 - g(a_1)$. The equality $a_1 + a_2 = 1$ means that $a_2 = 1 - a_1$, so the condition of the Lemma means that

$$g(1 - a_1) = 1 - g(a_1) \tag{4}$$

for all $a_1 \in [0,1]$.

For $n = 3$, we similarly conclude that $g(a_1) + g(a_2) + g(1 - (a_1 + a_2)) = 1$ for all $a_1 \geq 0$ and $a_2 \geq 0$ for which $a_1 + a_2 \leq 1$. Therefore, $g(a_1) + g(a_2) = 1 - g(1 - (a_1 + a_2))$. Due to (4), we have $1 - g(1 - (a_1 + a_2)) = g(a_1 + a_2)$, so the above property reads $g(a_1 + a_2) = g(a_1) + g(a_2)$. It is known (see, e.g., [1]) that every function g whose values are non-negative and which satisfies the above *additivity* property is linear, i.e., $g(a) = k \cdot a$ for some real number k. Substituting this expression for $g(a)$ into both sides of the formula (4), we conclude that $k = 1$, i.e., that $g(a) = a$. The Lemma is proven.

Completing the proof. Let us first prove Theorem 1. Let b_j be a sequence for which $\sum\limits_{j=1}^{m} b_j = 1$. For this sequence, let us introduce an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum\limits_{j=1}^{m} f(a, b_j)$. In terms of this function, the double sum in (1) takes the form $\sum\limits_{i=1}^{n} g(a_i)$, so the property (1) takes the form (3).

Since the values of the function f are non-negative, the new auxiliary function $g(a)$ has non-negative values as well. Due to Lemma, we now conclude that $g(a) = a$, i.e., that for every a, we have

$$\sum_{j=1}^{m} f(a, b_j) = a. \tag{5}$$

When $a = 0$, then, from the fact that $f(a, b) \geq 0$ for all b, we conclude that $f(a, b_j) = 0$ for all j – since the only way for a sum of non-negative numbers to be 0 is when each of these numbers is equal to 0. Thus, we conclude that $f(0, b) = 0$ for all b, i.e., that $f(a, b) = a \cdot b$ for $a = 0$.

When $a > 0$, then we can divide both sides of the formula (5) by a and get the following formula:

$$\sum_{j=1}^{m} \frac{f(a, b_j)}{a} = 1.$$

So, for every $a > 0$, the new auxiliary function $g(b) \stackrel{\text{def}}{=} \dfrac{f(a, b)}{a}$ satisfies the following property:

For every finite sequence of non-negative real numbers $(b_1, \ldots, b_m)$,

$$\text{if } \sum_{j=1}^{m} b_j = 1, \text{ then } \sum_{j=1}^{m} g(b_j) = 1.$$

This is exactly the property (3), so, due to Lemma, $g(b) = b$ for every real number b. Since $g(a) = f(a, b)/a$, we conclude that $f(a, b) = a \cdot b$ for all a and b.

Theorem 2 can be now proved by induction over k. We have already proven this theorem for $k = 2$ – this case corresponds exactly to Theorem 1. Let us now assume that we have proved this result for $k - 1$, let us show how to prove it for k. For that, we first fix $k - 1$ sequences $(a_1^{(2)}, \ldots, a_{n_2}^{(2)}), \ldots, (a_1^{(k)}, \ldots, a_{n_k}^{(k)})$, and consider an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum\limits_{i_2=1}^{n_2} \ldots \sum\limits_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \ldots, a_{i_k}^{(k)})$. For this function, the condition (2) turns into (3), so, due to Lemma, we conclude that $g(a) = \sum\limits_{i_2=1}^{n_2} \ldots \sum\limits_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \ldots, a_{i_k}^{(k)}) = a$ for all a. Thus, for every a, the

new function $f'(a_2, \ldots, a_k) \stackrel{\text{def}}{=} f(a, a_2, \ldots, a_k)a$ of $k-1$ variables satisfies the following property:

$$\text{For every } k-1 \text{ finite sequences}$$

$$\text{of non-negative real numbers } (a_1^{(2)}, \ldots, a_{n_2}^{(2)}), \ldots, (a_1^{(k)}, \ldots, a_{n_k}^{(k)}),$$

$$\text{if } \sum_{i_2=1}^{n_2} a_{i_1}^{(2)} = 1 \text{ and } \ldots \text{ and } \sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1,$$

$$\text{then } \sum_{i_2=1}^{n_2} \ldots \sum_{i_k=1}^{n_k} f'(a_{i_2}^{(2)}, \ldots, a_{i_k}^{(k)}) = 1.$$

This is exactly the property (2) for $k-1$, so, due to induction assumption, we conclude that $f'(a_2, \ldots, a_k) = a_2 \cdot \ldots \cdot a_k$. Since $f'(a_2, \ldots, a_k) = f(a, a_2, \ldots, a_k)/a$, we thus conclude that $f(a, a_2, \ldots, a_n) = a \cdot f'(a_2, \ldots, a_k) = a \cdot a_2 \cdot \ldots \cdot a_k$. The induction step is proven, and so is the theorem.

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References

[1] J. Aczél, *Lectures on functional equations and their applications,* Academic Press, N.Y., London, 1966.

[2] S. A. Ferson, V. Kreinovich, L. Ginzburg, D. S. Myers, and K. Sentz, *Constructing Probability Boxes and Dempster-Shafer Structures*, Sandia National Laboratories, Report SAND2002-4015, January 2003.

[3] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[4] H. T. Nguyen and E. Walker, *First Course in Fuzzy Logic*, CRC Press, Boca Raton, Florida, 1999.

[5] H. M. Wadsworth Jr., *Handbook of statistical methods for engineers and scientists*, McGraw-Hill, N.Y., 1990.

[6] R. R. Yager, J. Kacprzyk, and M. Pedrizzi (eds.), *Advances in the Dempster-Shafer Theory of Evidence*, Wiley, N.Y., 1994