

WHY PRODUCT OF PROBABILITIES (MASSES) FOR INDEPENDENT EVENTS? A REMARK

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Abstract: For independent events A and B , the probability $P(A \& B)$ is equal to the product of the corresponding probabilities: $P(A \& B) = P(A) \cdot P(B)$. It is well known that the product $f(a, b) = a \cdot b$ has the following property: once $\sum_{i=1}^n P(A_i) = 1$ and $\sum_{j=1}^m P(B_j) = 1$, the probabilities $P(A_i \& B_j) = f(P(A_i), P(B_j))$ also add to 1: $\sum_{i=1}^n \sum_{j=1}^m f(P(A_i), P(B_j)) = 1$.

In 1986, D. Dubois, H. Prade, and R. Giles proved that the product is the only continuous function that satisfies this property, i.e., that if, vice versa, this property holds for some continuous function $f(a, b)$, then this function f is the product. This result provided an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

In this paper, we strengthen this result by showing that it holds for arbitrary (not necessarily continuous) functions $f(a, b)$.

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1. Product is Normally Used as a Combination Rule for Independent Events

For independent events A and B , the probability $P(A \& B)$ is equal to the product of the corresponding probabilities: $P(A \& B) = f(P(A), P(B))$, where the combination function is the product $f(a, b) = a \cdot b$; see, e.g., [6].

Similarly, in Dempster-Shafer theory (see, e.g., [3], [7]), one of the ways to combine the masses from two independent knowledge bases is to multiply them.

2. A Reasonable Property of the Combination Rule

Due to the additivity property of probability, if the events $A_1, \dots, A_n$ form a partition of the universal set, i.e., if one of these events always occurs and no two can occur at the same time, then $\sum_{i=1}^n P(A_i) = 1$. If the events A_i form a partition and the events B_j form a partition, then their combinations $A_i \& B_j$ also form a partition; indeed:

- since A_i and B_j form a partition, any situation belongs to one of A_i and to one of B_j , thus, for this situation, the corresponding event $A_i \& B_j$ holds;
- similarly, since the events A_i are mutually exclusive and the events B_j are mutually exclusive, the combinations $A_i \& B_j$ are also mutually exclusive.

It is therefore reasonable to expect that if the events A_i form a partition, i.e., $\sum_{i=1}^n P(A_i) = 1$, and if events B_j form a partition, i.e., $\sum_{j=1}^m P(B_j) = 1$, then the events $A_i \& B_j$ should also form a partition, i.e., $\sum_{i=1}^n \sum_{j=1}^m f(P(A_i), P(B_j)) = 1$.

In formal terms, the function $f : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that describes the combination rule should satisfy the following property:

For every two finite sequences

of non-negative real numbers $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$, (1)

$$\text{if } \sum_{i=1}^n a_i = 1 \text{ and } \sum_{j=1}^m b_j = 1, \text{ then } \sum_{i=1}^n \sum_{j=1}^m f(a_i, b_j) = 1.$$

3. What Is Known

It is well known that the product function $f(a, b) = a \cdot b$ satisfies the property (1). It is also known that many other possible combination functions, e.g., many t-norms that are different from the product (see, e.g., [4], [5]), do not satisfy this property.

D. Dubois, H. Prade, and R. Giles proved [2] that among *continuous* functions f , the product function is the only function that satisfies the above property.

This result provides an additional explanation of why for independent events, we multiply probabilities (or, in the Dempster-Shafer case, masses).

4. Main Result

In this paper, we strengthen the result from [2] by showing that it holds for arbitrary (not necessarily continuous) functions $f(a, b)$.

We also extend this result to the case when we combine more than two events.

Theorem 1. *If a function $f : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies the property (1), then this function is the product: $f(a, b) = a \cdot b$ for all a and b .*

5. Case of Several Events

Let $k \geq 2$ be an integer, and let $f : [0, 1]^k \rightarrow [0, 1]$ be a function of k variables. For such functions, we will consider the following property:

For every k finite sequences

of non-negative real numbers $(a_1^{(1)}, \dots, a_{n_1}^{(1)}), \dots, (a_1^{(k)}, \dots, a_{n_k}^{(k)})$,

$$\text{if } \sum_{i_1=1}^{n_1} a_{i_1}^{(1)} = 1 \text{ and } \dots \text{ and } \sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1, \quad (2)$$

$$\text{then } \sum_{i_1=1}^{n_1} \dots \sum_{i_k=1}^{n_k} f(a_{i_1}^{(1)}, \dots, a_{i_k}^{(k)}) = 1.$$

Theorem 2. *If a function $f : [0, 1]^k \rightarrow [0, 1]$ satisfies the property (2), then this function is the product: $f(a_1, \dots, a_k) = a_1 \cdot \dots \cdot a_k$ for all $a_1, \dots, a_k$.*

6. Proofs

The proof of Theorems 1 and 2 is based on the following Lemma:

Lemma. *Let a function $g : [0, 1] \rightarrow R_0^+ \stackrel{\text{def}}{=} [0, \infty)$ satisfy the following property:*

For every finite sequence of non-negative real numbers $(a_1, \dots, a_n)$,

$$\text{if } \sum_{i=1}^n a_i = 1, \text{ then } \sum_{i=1}^n g(a_i) = 1. \quad (3)$$

Then, $g(a) = a$ for every real number a .

Proof of the Lemma. Let us first consider the case when $n = 2$. In this case, the condition of the Lemma means that $a_1 + a_2 = 1$ implies $g(a_1) + g(a_2) = 1$, i.e., that $g(a_2) = 1 - g(a_1)$. The equality $a_1 + a_2 = 1$ means that $a_2 = 1 - a_1$, so the condition of the Lemma means that

$$g(1 - a_1) = 1 - g(a_1) \quad (4)$$

for all $a_1 \in [0, 1]$.

For $n = 3$, we similarly conclude that $g(a_1) + g(a_2) + g(1 - (a_1 + a_2)) = 1$ for all $a_1 \geq 0$ and $a_2 \geq 0$ for which $a_1 + a_2 \leq 1$. Therefore, $g(a_1) + g(a_2) = 1 - g(1 - (a_1 + a_2))$. Due to (4), we have $1 - g(1 - (a_1 + a_2)) = g(a_1 + a_2)$, so the above property reads $g(a_1 + a_2) = g(a_1) + g(a_2)$. It is known (see, e.g., [1]) that every function g whose values are non-negative and which satisfies the above *additivity* property is linear, i.e., $g(a) = k \cdot a$ for some real number k . Substituting this expression for $g(a)$ into both sides of the formula (4), we conclude that $k = 1$, i.e., that $g(a) = a$. The Lemma is proven.

Proofs of Theorems 1 and 2. Let us first prove Theorem 1. Let b_j be a sequence for which $\sum_{j=1}^m b_j = 1$. For this sequence, let us introduce an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum_{j=1}^m f(a, b_j)$. In terms of this function, the double sum in (1) takes the form $\sum_{i=1}^n g(a_i)$, so the property (1) takes the form (3).

Since the values of the function f are non-negative, the new auxiliary function $g(a)$ has non-negative values as well. Due to Lemma, we now conclude that $g(a) = a$, i.e., that for every a , we have

$$\sum_{j=1}^m f(a, b_j) = a. \quad (5)$$

When $a = 0$, then, from the fact that $f(a, b) \geq 0$ for all b , we conclude that $f(a, b_j) = 0$ for all j – since the only way for a sum of non-negative numbers to be 0 is when each of these numbers is equal to 0. Thus, we conclude that $f(0, b) = 0$ for all b , i.e., that $f(a, b) = a \cdot b$ for $a = 0$.

When $a > 0$, then we can divide both sides of the formula (5) by a and get the following formula:

$$\sum_{j=1}^m \frac{f(a, b_j)}{a} = 1.$$

So, for every $a > 0$, the new auxiliary function $g(b) \stackrel{\text{def}}{=} \frac{f(a, b)}{a}$ satisfies the following property:

For every finite sequence of non-negative real numbers $(b_1, \dots, b_m)$,

$$\text{if } \sum_{j=1}^m b_j = 1, \text{ then } \sum_{j=1}^m g(b_j) = 1.$$

This is exactly the property (3), so, due to Lemma, $g(b) = b$ for every real number b . Since $g(a) = f(a, b)/a$, we conclude that $f(a, b) = a \cdot b$ for all a and b .

Theorem 2 can be now proved by induction over k . We have already proven this theorem for $k = 2$ – this case corresponds exactly to Theorem 1. Let us now assume that we have proved this result for $k - 1$, let us show how to prove it for k . For that, we first fix $k - 1$ sequences $(a_1^{(2)}, \dots, a_{n_2}^{(2)}), \dots, (a_1^{(k)}, \dots, a_{n_k}^{(k)})$, and consider an auxiliary function $g(a) \stackrel{\text{def}}{=} \sum_{i_2=1}^{n_2} \dots \sum_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \dots, a_{i_k}^{(k)})$. For this function, the condition (2) turns into (3), so, due to Lemma, we conclude that $g(a) =$

$\sum_{i_2=1}^{n_2} \dots \sum_{i_k=1}^{n_k} f(a, a_{i_2}^{(2)}, \dots, a_{i_k}^{(k)}) = a$ for all a . Thus, for every a , the new function $f'(a_2, \dots, a_k) \stackrel{\text{def}}{=} f(a, a_2, \dots, a_k)a$ of $k-1$ variables satisfies the following property:

For every $k-1$ finite sequences

of non-negative real numbers $(a_1^{(2)}, \dots, a_{n_2}^{(2)}), \dots, (a_1^{(k)}, \dots, a_{n_k}^{(k)})$,

if $\sum_{i_2=1}^{n_2} a_{i_2}^{(2)} = 1$ and $\dots$ and $\sum_{i_k=1}^{n_k} a_{i_k}^{(k)} = 1$,

then $\sum_{i_2=1}^{n_2} \dots \sum_{i_k=1}^{n_k} f'(a_{i_2}^{(2)}, \dots, a_{i_k}^{(k)}) = 1$.

This is exactly the property (2) for $k-1$, so, due to induction assumption, we conclude that $f'(a_2, \dots, a_k) = a_2 \cdot \dots \cdot a_k$. Since $f'(a_2, \dots, a_k) = f(a, a_2, \dots, a_k)/a$, we thus conclude that $f(a, a_2, \dots, a_k) = a \cdot f'(a_2, \dots, a_k) = a \cdot a_2 \cdot \dots \cdot a_k$. The induction step is proven, and so is the theorem.

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