Random Interval Arithmetic is Closer to Common Sense: An Observation

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Abstract

From the commonsense viewpoint, if on a bridge whose weight we know with an accuracy of 1 ton, we place a car whose weight we know with an accuracy of 5 kg, then the accuracy with which we know the overall weight of a bridge with a car on it should still be 1 ton. This is what an engineer or a physicist would say. Alas, this is not so in traditional interval arithmetic. In this paper, we show that, in contrast to traditional interval arithmetic, the random interval arithmetic (proposed by the first two authors) actually has this important property.

1 Intuitive Property of Commonsense Arithmetic

From the commonsense viewpoint, if on a bridge whose weight we know with an accuracy of 1 ton, we place a car whose weight we know with an accuracy of 5 kg, then the accuracy with which we know the overall weight of a bridge with a car on it should still be 1 ton. This is what an engineer or a physicist would say.

The problem that we try to solve in this paper can be illustrated by the following joke. A museum guide tells the visitors that a dinosaur that they are looking at is 14,000,005 years old. An impressed visitor asks how scientists can be so accurate in its predictions. "I don't know how they do it, – explains the guide – but 5 years ago, when I started working here, I was told that this dinosaur is 14,000,000 years old, so now it must be 5 years older".

This is clearly a joke, because from the common sense viewpoint, a dinosaur which was approximately 14,000,000 years old 5 years ago is still 14,000,000 years old. In more precise terms, if we add 5 to a number "approximately 14,000,000", we should get the answer "approximately 14,000,000".

Similarly, if the accuracy was "approximately 1 ton" and we add the accuracy "approximately 5 kg", we should get the answer "approximately 1 ton".

In general, if Δ_a is much larger than Δ_b ($\Delta_a \gg \Delta_b$), and we add "uncertainty approximately Δ_b" to "uncertainty approximately Δ_a", we should get "uncertainty approximately Δ_a". It is therefore natural to expect formal systems which formalize commonsense reasoning to have this property.

2 Traditional Interval Arithmetic Does not Have the Desired Property

A natural way of dealing with approximately known values is *interval arithmetic*. In interval arithmetic, the situation in which we know the value $\widetilde{a}$ with an accuracy Δ_a is represented by an interval $[\widetilde{a} - \Delta_a, \widetilde{a} + \Delta_a]$.

In many practical applications, this formalization works well; however, the traditional interval arithmetic does not satisfy the desired intuitive property.

Indeed, let us assume that about a quantity a, we know that it is equal to $\widetilde{a}$ with uncertainty Δ_a (e.g., "with uncertainty 1 ton"), and about a quantity b, we know that b is equal to $\widetilde{b}$ with uncertainty Δ_b (e.g., "with uncertainty 5 kg"). Then, the corresponding intervals are equal to $\mathbf{a} = [\widetilde{a} - \Delta_a, \widetilde{a} + \Delta_a]$ and $\mathbf{b} = [\widetilde{b} - \Delta_b, \widetilde{b} + \Delta_b]$. The set of possible values of $c = a + b$ is an interval

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = [(\widetilde{a} + \widetilde{b}) - (\Delta_a + \Delta_b), (\widetilde{a} + b) + (\Delta_a + \Delta_b)].$$

In accordance with the above interpretation, we thus interpret the sum $\mathbf{a} + \mathbf{b}$ as "$\widetilde{a} + \widetilde{b}$ with uncertainty $\Delta_a + \Delta_b$". So, if we know a with uncertainty 1 ton, and we know b with uncertainty 5 kg, then the resulting uncertainty in $a + b$ is equal not to 1 ton as we would intuitively expect, but to 1.005 ton.

How can we modify interval arithmetic to make sure that the desired property is satisfied, and the uncertainty of the resulting sum is 1 ton?

Comment. A similar problem occurs in a more general case of *fuzzy arithmetic*. Specifically, often, in addition (or instead) the guaranteed bound Δ_a, an expert can provide bounds that contain $\Delta a \stackrel{\text{def}}{a} - \widetilde{a}$ with a certain degree of confidence. Often, we know several such bounding intervals corresponding to different degrees of confidence. Such a nested family of intervals is also called a *fuzzy set*, because it turns out to be equivalent to a more traditional definition of fuzzy set [3, 8, 12, 13, 14] (if a traditional fuzzy set is given, then different intervals from the nested family can be viewed as α-cuts corresponding to different levels of uncertainty α).

A method for solving this problem, both for interval arithmetic and for a more general case of fuzzy arithmetic, has been earlier proposed by the third author [10]. However, in that approach, interval operations such as addition is no longer always easily computable. We need a practical, easy-to-implement approach. This is what we will describe in this paper.

Specifically, we will show that random interval arithmetic [1, 2] proposed by the first two authors actually has the desired property.

3 What Is Random Interval Arithmetic: In Brief

Before we start explaining how random interval arithmetic can help, let us briefly recall what is random interval arithmetic.

Random interval arithmetic is a way of analyzing how the uncertainty in input data and the round-off imprecision of computer operations on real numbers affect the results of the computations. Traditionally, in science and engineering, this analysis have been based on statistical techniques, e.g., analytical statistical techniques similar to sensitivity analysis and Monte-Carlo-type simulation techniques. However, the use of traditional statistical techniques requires that we know the exact probability distribution of the input and round-off errors.

In most real-life situations, we do not know these distributions; at best, we know the upper bounds on these errors – or, more generally, the intervals that are guaranteed to contain the actual (unknown) values of these errors. The need to consider such *interval* uncertainty was realized already in the 1950s. In the late 1950s and the early 1960s, NASA-related problems of space navigation under uncertainty provided a real boost to this area of research. The resulting *interval computations* techniques have been well developed, and they are still actively used in many application areas; see, e.g., [4, 5, 6, 11].

Producing the exact bounds on the inaccuracy of the output is often difficult (this problem is known to be NP-hard [9]). Due to the origin of interval techniques – in NASA-related problems that required high reliability – the emphasis in interval computations has always been on getting the validated results.

Since producing exact bounds is computationally difficult, interval computation techniques usually produce estimates that are guaranteed to contain (enclose) the actual error.

In many applications of such techniques, it is desirable, in addition to guaranteed "overestimates", to produce a reasonable estimate of the size of the actual error, an estimate that may be only valid with a certain probability.

The main idea behind the random interval arithmetic is that for each intermediate computation step $z := x \odot y$, if we know the exact intervals $\mathbf{x}$ and $\mathbf{y}$ of possible values of x and y, then, depending on the relative monotonicity of the x and y relative to inputs, the intervals $\mathbf{z}$ can change from the worst-case situation – when we apply interval arithmetic operation to $\mathbf{x}$ and $\mathbf{y}$ – to the best-case situation when we apply the operations of the so-called *dual (inner) arithmetic*. For example, for addition, when $\mathbf{x} = \underline{x}, \overline{x}]$ and $\mathbf{y} = [\underline{y}, \overline{y}]$, then the resulting interval $\mathbf{z}$ can range from the worst-case situation when $\mathbf{z} = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$ to the best-case situation when

$$\mathbf{z} = [\min(\underline{x} + \overline{y}, \overline{x} + \underline{y}), \max(\underline{x} + \overline{y}, \overline{x} + \underline{y})].$$

When an algorithm consists of numerous computational steps, then it is reasonable to expect that steps in which we have monotonicity in the same direction (and the worst-case interval) are as frequent as cases in which we have monotonicity in different directions (and the best-case interval). To provide a good estimation of the resulting uncertainty, it is therefore reasonable, on each computational step, to consider either traditional, or inner arithmetic with equal probability. This technique – called *random interval arithmetic* – indeed leads to reasonable estimates for the resulting uncertainty.

4 Random Interval Arithmetic Has the Desired Commonsense Property: An Observation

Let us consider addition of two intervals. One can easily see that in the traditional interval arithmetic, the half-width Δc of the sum $\mathbf{a} + \mathbf{b}$ of two intervals is equal to the sum of the corresponding half-widths:

$$\Delta_c^t = \Delta_a + \Delta_b. \tag{1}$$

In dual arithmetic, the half-width is equal to the difference between the larger and the smaller half-widths:

$$\Delta_c^d = \max(\Delta_a, \Delta_b) - \min(\Delta_a, \Delta_b). \tag{2}$$

In random interval arithmetic, we use both operations with equal probability 50%. Therefore, the average width of the resulting interval is equal to

$$\Delta_c^r = \frac{\Delta_c^t + \Delta_c^d}{2}. \tag{3}$$

The formula (1) can be rewritten as

$$\Delta_c^t = \max(\Delta_a, \Delta_b) + \min(\Delta_a, \Delta_b). \tag{4}$$

Substituting (2) and (4) into the formula (3), we conclude that

$$\Delta_c^r = \max(\Delta_a, \Delta_b).$$

This is exactly the intuitive property that we have been trying to formalize.

5 Discussion

What happens when instead of two values, we estimate the sum of several ($n > 2$) different quantities $a = a_1 + \ldots + a_n$? Let us assume that we know each quantity a_i with an accuracy Δ_i. What is the expected value of the resulting accuracy in a?

The sum is computed element by element: first, we compute $a_1 + a_2$; then, we compute $(a_1 + a_2) + a_3$, etc. Let us follow these computations and estimate the uncertainty of all the intermediate results.

At first, we add a_1 and a_2. As we have mentioned, the resulting sum has, on average, the uncertainty $\max(\Delta_1, \Delta_2)$.

To estimate the uncertainty of the next intermediate result $(a_1 + a_2) + a_3$, we can take, as an estimate of the uncertainty in $a_1 + a_2$, the value $\max(\Delta_1, \Delta_2)$. Then, according to the above result, the average uncertainty in $(a_1 + a_2) + a_3$ will be equal to

$$\max(\max(\Delta_1, \Delta_2), \Delta_3) = \max(\Delta_1, \Delta_2, \Delta_3).$$

Similarly, for the intermediate sum $a_1 + a_2 + a_3 + a_4$, we can conclude that the resulting uncertainty is equal to $\max(\Delta_1, \Delta_2, \Delta_3, \Delta_4)$, and that the average uncertainty in the sum $a_1 + \ldots + a_n$ is equal to

$$\max(\Delta_1, \ldots, \Delta_n).$$

An interesting corollary of this formula is that while the process of adding n numbers depends on the order in which we place these numbers, the resulting average uncertainty does not depend on this order, only on the uncertainties Δ_i with which we know the numbers a_i.

What happens if instead of a simple sum, we would like to compute the value of a more complex function $f(a_1, \ldots, a_n)$? For a function of two variables, when the uncertainty is small $\Delta a_i \overset{\text{def}}{a}_{\,i} - \widetilde{a}_i \ll a_i$, we can safely linearize the expression for $f(a_1, a_2)$:

$$f(a_1, a_2) = f(\widetilde{a}_1 + \Delta a_1, \widetilde{a}_2 + \Delta a_2) = f(\widetilde{a}_1, \widetilde{a}_2) + \frac{\partial f}{\partial a_1} \cdot \Delta a_1 + \frac{\partial f}{\partial a_2} \cdot \Delta a_2.$$

So, when $\Delta a_i \in [\Delta_i, \Delta_i]$, the worst-case half-width in $a = f(a_1, a_2)$ is equal to

$$\Delta^t = \left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1 + \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2,$$

while the result of applying dual interval arithmetic is

$$\Delta^d = \left|\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1 - \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2\right|.$$

Thus, the average half-width – corresponding to random interval arithmetic – is equal to

$$\Delta^r = \max\left(\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1, \left|\frac{\partial f}{\partial a_2}\right| \cdot \Delta_2\right).$$

Similarly, for $n > 2$ variables, we conclude that

$$\Delta^r = \max\left(\left|\frac{\partial f}{\partial a_1}\right| \cdot \Delta_1, \ldots, \left|\frac{\partial f}{\partial a_n}\right| \cdot \Delta_n\right). \tag{5}$$

Comment. It is worth mentioning that this same expression appears in a completely different context. In interval computations, when we estimate the range of a function over a box $[\underline{a}_1, \overline{a}_1] \times \ldots \times [\underline{a}_n, \overline{a}_n]$, if a box is not too narrow, the estimates are too wide. To improve the estimates, we can bisect the box along one of the directions and repeat the estimation for each of the two half-boxes. The efficiency of this procedure drastically depends on which side we select for bisecting. It has been shown, both empirically and theoretically, that the optimal direction in a direction a_i in which the product

$$\left|\frac{\partial f}{\partial a_i}\right| \cdot \Delta_i$$

is the largest possible; see, e.g., [7, 15, 16]. The above value (5) is exactly the value of this maximum.

Comment. Our final comment is that the appearance of the max function may lead to one more explanation of why in the most widely used (and most practically successful) version of fuzzy logic, if we know the degree of belief $a = d(A)$ in a statement A and the degree of belief $b = d(B)$ in a statement B, then we estimate the degree of belief $c = d(C)$ in $C \stackrel{\text{def}}{=} A \vee B$ as $\max(a, b)$ – so that if $a \gg b$, we have $c = a$.

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