# Which Fuzzy Logic Is the Best: Pragmatic Approach (and Its Theoretical Analysis) ⋆

Vladik Kreinovich *

*Computer Science, University of Texas, El Paso, TX 79968, USA*

Hung T. Nguyen

*Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003, USA*

**Abstract**

In this position paper, we argue that when we are looking for *the best* fuzzy logic, we should specify in what sense the best, and that we get different fuzzy logics as "the best" depending on what optimality criterion we use.

*Key words:* fuzzy logic, intelligent control, sensitivity

## 1 What Is Logic? What Is Fuzzy Logic? A Brief Reminder

In order to elaborate on our viewpoint, let us start with a commonsense pragmatic understanding of what is logic and, in particular, what is fuzzy logic. From the pragmatic viewpoint, logic is an analysis of truth values, logical connectives like "and", "or", and "not", logical deductions, etc.

⋆ This work was supported in part by NASA under cooperative agreement NCC5-209, by NSF grants EAR-0112968, EAR-0225670, and EIA-0321328, by NIH grant 3T34GM008048-20S1, and by the Army Research Lab grant DATM-05-02-C-0046. The authors are very thankful to Vilém Novák for his helpful suggestions and comments.

* Corresponding author

*Email addresses:* vladik@cs.utep.edu (Vladik Kreinovich), hunguyen@nmsu.edu (Hung T. Nguyen).

In fuzzy logic, we start with a set of truth values. This is usually the unit interval $[0,1]$ – or, sometimes, a somewhat more complex set like the set of all subintervals of this interval. Most successful practical applications of fuzzy logic use one of these two sets of truth values – see, e.g., [3]. So, from the pragmatic viewpoint, there is no need to select a set of truth values.

Logical operations are a different story. In fuzzy logic, there are natural analogs of "and", "or", "not", and "implies": t-norms ("and"-operations) $f_{\&}(a,b)$, t-conorms ("or"-operations) $f_{\vee}(a,b)$, negation operations $f_{\neg}(a)$, and implication operations $f_{\rightarrow}(a,b)$. Many different t-norms, t-conorms, and implication operations have been proposed, so there is a pragmatic need to select the best one – especially since it is known that the results of applying fuzzy logic techniques (such as fuzzy control) often change if we use different t-norms and t-conorms (see, e.g., [4,7,12] and references therein).

## 2 Seemingly Natural Selection of the Best Fuzzy Logic – and Why It Does Not Work

A seemingly natural idea is the *axiomatic approach:*

- we first list all the desired properties of the corresponding logical operations,
- and then we look for logical operations that satisfy all these properties.

Alas, this idea does not work. Indeed, this is, in effect, what C. Elkan did in his well-known criticism of fuzzy logic [2]:

- Elkan listed all the logical formulas that these operations should satisfy – which, in effect, included most formulas of traditional propositional logic.
- Then, he showed that the only way to satisfy all these formulas is to use the traditional two-valued logic.

Since we cannot require *all* seemingly natural properties of fuzzy logic operations without sacrificing fuzzy logic, we should select *some* properties. It turns out that the resulting selection of the "best" fuzzy logic depends on which properties we choose.

There has been a lot of research in mathematical fuzzy logics, where it has been explicitly shown how different properties lead to different fuzzy logic; see, e.g., [10].

There has also been an interesting research into describing which operations are the best match for different *semantics* of fuzzy logics; see [10] and especially [11]. This research describes which fuzzy logic operations are the most adequate in representing expert reasoning.

One of the main objectives of fuzzy logic is not only to describe how people reason, but also to develop applied systems that would make automated decisions under uncertainty. It is well known that we humans are not perfect: we make mistakes, we make logical errors, and our reasoning under uncertainty is not always flawless. So, from the *pragmatic* viewpoint, it is desirable to also consider which fuzzy logic operations are the best in terms of different applications. As we will see (not surprisingly), the resulting choice of fuzzy logic operations depends on the goals of the corresponding application.

## 3 First Case Study: The Most Robust Fuzzy Logics

Let us start with an example which can be viewed both as semantic and as pragmatic.

In fuzzy logic, a typical set of truth values is the entire interval $[0, 1]$. So, for each expert's statement $S$, in the fuzzy logic approach, we describe the expert's certainty in this statement by a number from the interval $[0, 1]$. There exist numerous different methods for eliciting a value from an expert (see, e.g., [3]). For example, some methods ask the expert to mark their degrees of certainty on a scale, say, from 0 to 10. If an expert marks a 7, this means that the corresponding fuzzy value is 7/10=0.7.

Different methods lead to somewhat different values. This difference is quite understandable: we are trying to formalize the expert's subjective opinion, and while experts can probably meaningfully distinguish between certainty 0.5 and 0.7, expert normally cannot distinguish between closer values like 0.7 and 0.701.

It is therefore desirable to select logical operations in such a way that they be the least sensitive to the inaccuracy with which we measure the values of the membership functions. If we are looking for the fuzzy logic that is the most robust *in the worst case*, then the best choice is to use $f_\&(a, b) = \min(a, b)$ and $f_\vee(a, b) = \max(a, b)$ [8]; for negation and implication, we similarly get $f_\neg(a) = 1 - a$ and $f_\rightarrow(a, b) = \max(1 - a, b)$.

On the other hand, if we are looking for a fuzzy logic which is the most robust *in the average*, then the best choice is to use $f_\&(a, b) = a \cdot b$ and $f_\vee(a, b) = a + b - a \cdot b$ [8]. For negation and implication, we get $f_\neg(a) = 1 - a$ and $f_\rightarrow(a, b) = 1 - a \cdot (1 - b)$.

It is worth mentioning that the exact same operations result from a slightly different requirement – that, when averaged over all possible pairs of events $A$ and $B$ with different probabilities $a = P(A)$ and $b = P(B)$, the value

$f_{\&}(a,b) = f_{\&}(P(A), P(B))$ be the closest to the corresponding probability $P(A \,\&\, B)$ [11],

## 4 Second Case Study: Fuzzy Logics That Lead to the Best Control

One of the main applications of fuzzy logic is fuzzy control. In most fuzzy control applications, Mamdani-type approach is used – albeit with arbitrary t-norm and t-conorm. Therefore, a reasonable idea is to look for fuzzy logic operations that lead to the best control. It turns out that the resulting fuzzy logic depends on what we want from a control.

If we are looking for a control that is the smoothest (in some precise sense), then we should select $f_{\&}(a,b) = a \cdot b$ and $f_{\vee}(a,b) = \max(a,b)$ [7,12]. On the other hand, if we are looking for a control that is the most stable – i.e., a control that, once a system is perturbed, leads to the fastest return to the original trajectory – then we should select $f_{\&}(a,b) = \min(a,b)$ and $f_{\vee}(a,b) = a+b-a\cdot b$ [7,12].

For applications in which the result needs to be computed as fast as possible, an important feature of a fuzzy logic is the speed with which we can compute the corresponding logical operations. It turns out that the computationally simplest fuzzy logic is $f_{\&}(a,b) = \min(a,b)$ and $f_{\vee}(a,b) = \max(a,b)$ [7,12], and similarly, $f_{\neg}(a) = 1 - a$ and $f_{\rightarrow}(a,b) = \max(1-a, b)$.

It is also possible to describe other reasonable criteria – e.g., based on the amount of information – that lead to different selection of the best fuzzy logic (see [4,7] and references therein).

## 5 That In Different Situations, Different Fuzzy Logic Are The Best Should Not Be That Surprising

The question of which "and" and "or" operations are the best did not start with fuzzy logic. The designers of the world's first successful expert system MYCIN [1] – a system for diagnosing rare blood diseases – spend quite some time trying to find the logical operations that best match the reasoning of medical doctors. After MYCIN's success, they thought that they have uncovered the law of human reasoning about uncertainty, so they designed an expert system shell eMYCIN (empty Mycin) with the purpose of using the same logical operations in different application areas.

To their surprise, in the first first area where they tried – in oil exploration –

the "perfect" MYCIN's "and" operations did not work well at all. The post-fact analysis provided a very clear explanation for this difference:

- medical doctors must exercise extreme caution and, e.g., do not start surgery until they are absolutely sure that this surgery will not harm the patient;
- on the other hand, in oil exploration, the best strategy is to dig a well if there is a reasonable chance of oil, and if, say, 20% of wells come out dry, this is an acceptable risk.

Not surprisingly, the reasoning behind cautious and high-risk behavior leads to different "and"-operations. In short, for different problems, we get different operations.

## 6 Does This All Mean That Anything Goes in Fuzzy Logic? Group-Theoretic Approach to Optimization Under Uncertainty

A reader may interpret what we are saying as a claim that any "and", "or", and implication operations maybe the best under appropriate circumstances. Actually, while different operations are possible, for *reasonable* optimality criteria, only operations from a certain family – namely, piecewise fractionally-linear operations – can be the best; see [6].

The motivation for this result is that, as we have mentioned, the numerical values describing the expert uncertainty are only uniquely determined. In addition to the above-mentioned subjective uncertainty, there is also a possibility of using different *scales*. For example, we can determine the truth value $\mu$ of a statement $S$ by polling $N$ experts and computing $\mu$ as $M/N$, where $M$ is the number of experts who believe $S$ to be true. To improve the accuracy of this estimate, we can add, to our top $N$ experts, several ($N'$) additional experts. The additional experts may be shy in the presence of the top ones, then the new value $\mu'$ will be $M/(N+N')$; they may follow the majority, then we get $(M+N')/(N+N')$; we may have intermediate cases. In all these cases, we get a (piece-wise) fractional-linear transformation $\mu \to \mu'$.

It is desirable, when selecting the best fuzzy logic, to use optimality criteria which are invariant relative to the group of all such re-scalings. This idea, described in detail in [6], explains the formulas for the empirically best fuzzy logics – specifically, it explains pragmatically best logics (as described above), semantically best logics (as in [11]), and specific empirically useful fuzzy logics that come from the mathematical analysis of fuzzy logics [10].

The invariance idea also explains the formulas for the empirically best neural networks, etc. [6]. The whole idea of invariance relative to symmetry groups is also in good accordance with modern physics, where group-theoretic methods are one of the main tools.

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