

# Interval Methods: An Introduction

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**Abstract.** This chapter contains selected papers presented at the Minisymposium on Interval Methods of the PARA'04 Workshop "State-of-the-Art in Scientific Computing". The emphasis of the workshop was on high-performance computing (HPC). The ongoing development of ever more advanced computers provides the potential for solving increasingly difficult computational problems. However, given the complexity of modern computer architectures, the task of realizing this potential needs careful attention. A main concern of HPC is the development of software that optimizes the performance of a given computer.

An important characteristic of the computer performance in scientific computing is the accuracy of the computation results. Often, we can estimate this accuracy by using traditional statistical techniques. However, in many practical situations, we do not know the probability distributions of different measurement, estimation, and/or roundoff errors, we only know estimates of the upper bounds on the corresponding measurement errors, i.e., we only know an *interval* of possible values of each such error. The papers from the following chapter contain the description of the corresponding "interval computation" techniques, and the applications of these techniques to various problems of scientific computing.

**Why data processing?** In many real-life situations, we are interested in the value of a physical quantity  $y$  that is difficult or impossible to measure directly. Examples of such quantities are the distance to a star and the amount of oil in a given well. Since we cannot measure  $y$  directly, a natural idea is to measure  $y$  *indirectly*. Specifically, we find some easier-to-measure quantities  $x_1, \dots, x_n$  which are related to  $y$  by a known relation  $y = f(x_1, \dots, x_n)$ ; this relation may be a simple functional transformation, or complex algorithm (e.g., for the amount of oil, numerical solution to an inverse problem). Then, to estimate  $y$ , we first measure the values of the quantities  $x_1, \dots, x_n$ , and then we use the results  $\tilde{x}_1, \dots, \tilde{x}_n$  of these measurements to compute an estimate  $\tilde{y}$  for  $y$  as  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$ .

For example, to find the resistance  $R$ , we measure current  $I$  and voltage  $V$ , and then use the known relation  $R = V/I$  to estimate resistance as  $\tilde{R} = \tilde{V}/\tilde{I}$ .

In this example, the relation between  $x_i$  and  $y$  is known exactly; in many practical situations, we only know an approximate relation  $y \approx \tilde{f}(x_1, \dots, x_n)$

between  $x_i$  and  $y$ . In such situations, the estimate  $\tilde{y}$  for  $y$  is computed as  $\tilde{y} = \tilde{f}(\tilde{x}_1, \dots, \tilde{x}_n)$ .

Computing an estimate for  $y$  based on the results of direct measurements is called *data processing*; data processing is the main reason why computers were invented in the first place, and data processing is still one of the main uses of computers as number crunching devices.

**Why interval computations? From computing to probabilities to intervals.** Measurement are never 100% accurate, so in reality, the actual value  $x_i$  of  $i$ -th measured quantity can differ from the measurement result  $\tilde{x}_i$ . Because of these *measurement errors*  $\Delta x_i \stackrel{\text{def}}{=} \tilde{x}_i - x_i$ , the result  $\tilde{y} = f(\tilde{x}_1, \dots, \tilde{x}_n)$  of data processing is, in general, different from the actual value  $y = f(x_1, \dots, x_n)$  of the desired quantity  $y$  [6].

It is desirable to describe the error  $\Delta y \stackrel{\text{def}}{=} \tilde{y} - y$  of the result of data processing. To do that, we must have some information about the errors of direct measurements.

What do we know about the errors  $\Delta x_i$  of direct measurements? First, the manufacturer of the measuring instrument must supply us with an estimate of the upper bound  $\Delta_i$  on the measurement error. (If no such upper bound is supplied, this means that no accuracy is guaranteed, and the corresponding “measuring instrument” is practically useless.) In this case, once we performed a measurement and got a measurement result  $\tilde{x}_i$ , we know that the actual (unknown) value  $x_i$  of the measured quantity belongs to the interval  $\mathbf{x}_i = [\underline{x}_i, \bar{x}_i]$ , where  $\underline{x}_i = \tilde{x}_i - \Delta_i$  and  $\bar{x}_i = \tilde{x}_i + \Delta_i$ .

In many practical situations, we not only know the interval  $[-\Delta_i, \Delta_i]$  of possible values of the measurement error; we also know the probability of different values  $\Delta x_i$  within this interval. This knowledge underlies the traditional engineering approach to estimating the error of indirect measurement, in which we assume that we know the probability distributions for measurement errors  $\Delta x_i$ .

In practice, we can determine the desired probabilities of different values of  $\Delta x_i$  by comparing the results of measuring with this instrument with the results of measuring the same quantity by a standard (much more accurate) measuring instrument. Since the standard measuring instrument is much more accurate than the current one, the difference between these two measurement results is practically equal to the measurement error; thus, the empirical distribution of this difference is close to the desired probability distribution for measurement error. There are two cases, however, when this determination is not done:

- First is the case of cutting-edge measurements, e.g., measurements in fundamental science. When a Hubble telescope detects the light from a distant galaxy, there is no “standard” (much more accurate) telescope floating nearby that we can use to calibrate the Hubble: the Hubble telescope is the best we have.
- The second case is the case of measurements on the shop floor. In this case, in principle, every sensor can be thoroughly calibrated, but sensor calibration

is so costly – usually costing ten times more than the sensor itself – that manufacturers rarely do it.

In both cases, we have no information about the probabilities of  $\Delta x_i$ ; the only information we have is an estimate on the upper bound on the measurement error.

In this case, after we performed a measurement and got a measurement result  $\tilde{x}_i$ , the only information that we have about the actual value  $x_i$  of the measured quantity is that it belongs to the interval  $\mathbf{x}_i = [\tilde{x}_i - \Delta_i, \tilde{x}_i + \Delta_i]$ . In such situations, the only information that we have about the (unknown) actual value of  $y = f(x_1, \dots, x_n)$  is that  $y$  belongs to the range  $\mathbf{y} = [\underline{y}, \bar{y}]$  of the function  $f$  over the box  $\mathbf{x}_1 \times \dots \times \mathbf{x}_n$ :

$$\mathbf{y} = [\underline{y}, \bar{y}] = \{f(x_1, \dots, x_n) \mid x_1 \in \mathbf{x}_1, \dots, x_n \in \mathbf{x}_n\}.$$

The process of computing this interval range based on the input intervals  $\mathbf{x}_i$  is called *interval computations*; see, e.g., [1–3, 5].

*Comment.* In addition to measurement errors, we also have round-off errors and – in case some parameters are estimated by experts – also uncertainty of expert estimates.

**Interval computations techniques: brief reminder.** Historically the first method for computing the enclosure for the range is the method which is sometimes called “straightforward” interval computations. This method is based on the fact that inside the computer, every algorithm consists of elementary operations (arithmetic operations, min, max, etc.). For each elementary operation  $f(a, b)$ , if we know the intervals  $\mathbf{a}$  and  $\mathbf{b}$  for  $a$  and  $b$ , we can compute the exact range  $f(\mathbf{a}, \mathbf{b})$ . The corresponding formulas form the so-called *interval arithmetic*. For example,

$$\begin{aligned} [\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] &= [\underline{a} + \underline{b}, \bar{a} + \bar{b}]; \quad [\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}]; \\ [\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] &= [\min(\underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b}), \max(\underline{a} \cdot \underline{b}, \underline{a} \cdot \bar{b}, \bar{a} \cdot \underline{b}, \bar{a} \cdot \bar{b})]. \end{aligned}$$

In straightforward interval computations, we repeat the computations forming the program  $f$  step-by-step, replacing each operation with real numbers by the corresponding operation of interval arithmetic. It is known that, as a result, we get an enclosure  $\mathbf{Y} \supseteq \mathbf{y}$  for the desired range.

In some cases, this enclosure is exact. In more complex cases (see examples below), the enclosure has excess width.

There exist more sophisticated techniques for producing a narrower enclosure, e.g., a centered form method. However, for each of these techniques, there are cases when we get an excess width. Reason: as shown in [4, 7], the problem of computing the exact range is known to be NP-hard even for polynomial functions  $f(x_1, \dots, x_n)$  (actually, even for quadratic functions  $f$ ).

**Applications of interval techniques.** The ultimate objective of interval computations has always been to apply these methods to practical problems. The

workshop provided us with a unique opportunity to promote collaboration between interval researchers and researcher interested in applications.

This chapter contains extended versions of selected papers presented at the interval Minisymposium of PARA'04.

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## References

1. Jaulin L., Keiffer M., Didrit O., and Walter E. (2001), "Applied Interval Analysis", Springer-Verlag, Berlin.
2. Kearfott R. B. (1996), "Rigorous Global Search: Continuous Problems", Kluwer, Dordrecht.
3. Kearfott R. B. and Kreinovich V., eds. (1996), "Applications of Interval Computations" (Pardalos. P. M., Hearn, D., "Applied Optimization", Vol. 3), Kluwer, Dordrecht.
4. Kreinovich V., Lakeyev A., Rohn J., and Kahl P. (1997), "Computational Complexity and Feasibility of Data Processing and Interval Computations" (Pardalos. P. M., Hearn, D., "Applied Optimization", Vol. 10), Kluwer, Dordrecht.
5. Moore R. E. (1979), "Methods and Applications of Interval Analysis", SIAM, Philadelphia.
6. Rabinovich S. (1993), "Measurement Errors: Theory and Practice", American Institute of Physics, New York.
7. Vavasis S. A. (1991), "Nonlinear Optimization: Complexity Issues", Oxford University Press, N.Y.