How to Reconstruct the Original Shape of a Radar Signal?

Matthew G. Averill, Gang Xiang, Vladik Kreinovich,
G. Randy Keller, and Scott A. Starks
Pan-American Center for Earth and Environmental Studies
University of Texas at El Paso, El Paso, TX 79968, USA
averill@geo.utep.edu, gxiang@utep.edu, vladik@cs.utep.edu,
keller@utep.edu, sstarks@utep.edu

Patrick S. Debroux and
James Boehm
Army Research Laboratory
SLAD

Abstract— The shape of the radar signal can provide us with the additional information about the reflecting surface. However, to decrease the noise, radars use filtering, and filtering changes the shapes of the radar signal. It is therefore necessary to reconstruct the original shape of the radar signal.

I. Radars are Important

Radar measurements are used in many areas of science and engineering. Historically the first use of radars was in tracing airplanes and missiles; this is still one of the main uses of radars.

However, radars are used more and more in geosciences as well. The information provided by airborne radars nicely supplements other remote sensing information – e.g., radar beams can go below the leaves, to the actual Earth surface and even deeper than the surface; see, e.g., [2], [3], [5], [6], [7], [8], [11], [12], [13], [14], [21], [22], [23].

II. Main Use of Radars: Localization

The main idea behind a radar is simple:
- we send a pulse-like radio signal,
- this signal gets reflected by the target, and
- we measure the reflected signal.

The main information that we can get from the radar is the travel time. Based on the travel time, we can find the distance between the radar and the target. If we use several radars, we can thus get an exact location of the target.

This is how radars determine the exact position of the planes in the vicinity of an airport. This is how radars produce high-accuracy digital elevation maps that is so important in geophysics.

III. Radars Provide Additional Information

If the targets were points, then after sending a pulse signal, we would get a pulse back, and the only information we would be able to get is the distance from the radar to the point target. In reality, the target is not a point. As a result, even if we send a pulse signal, this pulse is reflected from different points on a target and therefore, we get a continuous signal back.

The shape of this signal can provide us with the additional information about the reflecting surface.

IV. It Is Desirable to Determine the Probability Distribution of the Reflected Signal

In an airborne geophysical radar, pulses are sent one after another, so individual reflections get entangled. We can still measure the probability distribution of the values of the reflected signal and try to extract the information about the reflecting surface from this distribution.

V. Filtering Makes This Determination Difficult

The trouble with the above idea is that the reflected signals are weak and covered with noise. To decrease the noise, we apply filtering – usually, linear filtering, when instead of the original signal $x(t)$, we consider a linear combination of this signal and the signals ate the previous moments of time:

$$y(t) = \sum_s a(s) \cdot x(t - s).$$

This filtering decreases the noise and makes the distance measurement very accurate. On the other hand, it replaces the original possibly non-Gaussian signal $x(t)$ with a linear combination of such signals. It is well known that as we increase the number of terms in a linear combination of several small random variables, the resulting distribution of a sum tends to Gaussian – this Central Limit Theorem is one of the main reasons why Gaussian distribution is so frequent in practice; see, e.g., [25]. So, after filtering, we get a distribution that is close to Gaussian.

The problem is now as follows:
- we have a probability distribution for y, and
- we want to reconstruct the original distribution for x.

VI. Main Idea

Our main idea is to describe both distribution in terms of logarithmic moments.

A. What Are Logarithmic Moments?

Namely, for a random variable ξ, we can define its characteristic function as

$$\chi_\xi(\omega) \stackrel{\text{def}}{=} E[\exp(\mathrm{i} \cdot \omega \cdot \xi)].$$

For the sum $\xi = \xi_1 + \xi_2$ of two independent random variables,

$$E[\exp(i \cdot \omega \cdot \xi)] = E[\exp(i \cdot \omega \cdot \xi_1) \cdot \exp(i \cdot \omega \cdot \xi_2)].$$

Since ξ_1 and ξ_2 are independent, we get

$$E[\exp(i \cdot \omega \cdot \xi)] = E[\exp(i \cdot \omega \cdot \xi_1)] \cdot E[\exp(i \cdot \omega \cdot \xi_2)],$$

i.e.,

$$\chi_\xi(\omega) = \chi_{\xi_1}(\omega) \cdot \chi_{\xi_2}(\omega).$$

Hence, for the logarithms, we get the formula

$$\ln(\chi_\xi(\omega)) = \ln(\chi_{\xi_1}(\omega)) + \ln(\chi_{\xi_2}(\omega)).$$

So, if we define n-the logarithmic moment as

$$L_n(\xi) \stackrel{\text{def}}{=} \frac{1}{i^n} \cdot \frac{d^n \chi_\xi(\omega)}{d\omega^n}\Big|_{\omega=0},$$

we conclude that

$$L_n(\xi) = L_n(\xi_1) + L_n(\xi_2).$$

Comment. The factor $\dfrac{1}{i^n}$ is added to make the moments real numbers.

B. Why Logarithmic Moments?

In general, from the formula that relates x and y, we conclude that

$$L_n(y) = \left(\sum_s (a(s))^n \right) \cdot L_n(x).$$

So, we can reconstruct the n-the logarithmic moment of x by using the formula

$$L_n(x) = \frac{L_n(y)}{\sum_s (a(s))^n}.$$

C. Reconstructing x From y: Main Idea

In the ideal non-noise case, if we know the exact distribution for y, we can reconstruct the desired distribution for x as follows:

- first, we compute the logarithmic moments $L_n(y)$ of the signal y;
- then, we use the above formula to compute all the logarithmic moments $L_n(x)$ of the original distribution x;
- finally, we use the Taylor series to reconstruct the logarithm of the characteristics function as

$$\ln(\chi_x(\omega)) = L_1 \cdot i \cdot \omega + L_2 \cdot i^2 \cdot \omega^2 + L_3 \cdot i^3 \cdot \omega^3 + \dots$$

So, in the ideal case, we can determine the characteristic function $\chi_x(\omega)$ of the original distribution x.

D. From Characteristic Function to, e.g., Probability Density Function

It is known that the characteristic function uniquely determines the distribution. For example, from its definition, we can describe its relation to the probability density function $\rho(x)$ as follows:

$$\chi(\omega) = E[\exp(i \cdot \omega \cdot \xi)] = \int \exp(i \cdot \omega \cdot x) \cdot \rho(x) \, dx.$$

So, $\chi(x)$ is a Fourier transform of the probability density function. Hence, the original probability density function $\rho(x)$ can be determined as the inverse Fourier transform of the characteristics function

$$\rho(x) = \frac{1}{2\pi} \cdot \int \exp(-i \cdot \omega \cdot x) \cdot \chi(\omega) \, d\omega.$$

VII. COMPUTATIONS RELATED TO DIFFERENT FILTERS

To implement the above idea, for a given filter and for every n, we need to compute the value $\sum_s (a(s))^n$.

These computations may be computationally intensive, but we only need to do them once for each filter; after that, we can simply use the resulting values when reconstructing the logarithmic moments of x.

VIII. PROBLEM: WE CAN ONLY DETERMINE FINITELY MANY MOMENTS

The above description referred to the idealized no-noise case. In reality, the noise is always present: the whole purpose of the filter was to decrease this noise.

Because of the noise, in practice, we can only reconstruct a few first logarithmic moments $L_1(x)$, $L_2(x)$, …, $L_n(x)$.

These moments do not determine the distribution uniquely, there exist several different distributions with the same values of the first moments. We must therefore select a distribution with given values of these logarithmic models. How can we do that?

IX. FROM LOGARITHMIC MOMENTS TO TRADITIONAL MOMENTS

The problem of reconstructing a distribution from the logarithmic moments is reasonably new, but, as we will see, this problem is closely related to the well-studied problem of reconstructing a distribution from the standard moments

$$M_n \stackrel{\text{def}}{=} E[\xi^n] = \int x^n \cdot \rho(x) \, dx.$$

Indeed, let us show that knowing the first n logarithmic moments $L_1, \dots, L_2$ is equivalent to knowing the first moments $M_1, \dots, M_n$.

Let us start with the relation between L_1 and M_1. By definition,

$$L_1 = \frac{1}{i} \cdot \frac{\partial \ln(\chi)}{\partial \omega}\Big|_{\omega=0}.$$

Using the chain rule for differentiation, we conclude that

$$\frac{\partial \ln(\chi)}{\partial \omega} = \frac{1}{\chi} \cdot \frac{\partial \chi}{\partial \omega}.$$

For $\omega = 0$, we have $\chi(0) = E[\exp(\mathrm{i} \cdot 0 \cdot \omega)] = 1$, and

$$\frac{\partial \chi}{\partial \omega}_{|\omega=0} = \left(\frac{\partial}{\partial \omega} E[\exp(\mathrm{i} \cdot \omega \cdot \xi)]\right)_{|\omega=0} =$$

$$E\left[\left(\frac{\partial}{\partial \omega} \exp(\mathrm{i} \cdot \omega \cdot \xi)\right)_{|\omega=0}\right] = E[\mathrm{i} \cdot \xi] = \mathrm{i} \cdot M_1.$$

Therefore,

$$L_1 = \frac{1}{\mathrm{i}} \cdot \mathrm{i} \cdot M_1,$$

i.e., $L_1 = M_1$. In other words, the first logarithmic moment L_1 simply coincides with the first moment M_1.

For the second logarithmic moments, we have

$$L_2 = \frac{1}{\mathrm{i}^2} \cdot \frac{\partial^2 \ln(\chi)}{\partial \omega^2}_{|\omega=0}.$$

Differentiating the expression for the first derivative of χ once again, we conclude that

$$\frac{\partial^2 \ln(\chi)}{\partial \omega^2} = \frac{1}{\chi} \cdot \frac{\partial^2 \chi}{\partial \omega^2} - \frac{1}{\chi^2} \cdot \left(\frac{\partial \chi}{\partial \omega}\right)^2.$$

For $\omega = 0$, we have

$$\frac{\partial^2 \chi}{\partial \omega^2}_{|\omega=0} = \mathrm{i}^2 \cdot M_2,$$

hence

$$L_2 = M_2 - M_1^2.$$

In other words, the second logarithmic moment L_2 is the variance.

It is worth mentioning that the variance does not change with shift; this shift-invariance can be deduced directly from the definition of L_n and is, therefore, a general feature of all logarithmic moments L_n with $n \geq 2$. Indeed, if we shift the starting point of measuring ξ, i.e., replace ξ with $\xi' = \xi + x_0$ for some real number x_0, then

$$\exp(\mathrm{i}\cdot\omega\cdot\xi') = \exp(\mathrm{i}\cdot\omega\cdot(\xi+x_0)) = \exp(\mathrm{i}\cdot\omega\cdot\xi)\cdot\exp(\mathrm{i}\cdot\omega\cdot x_0).$$

Therefore,

$$E[\exp(\mathrm{i} \cdot \omega \cdot \xi')] = E[\exp(\mathrm{i} \cdot \omega \cdot \xi)] \cdot \exp(\mathrm{i} \cdot \omega \cdot x_0),$$

i.e.,

$$\chi'(\omega) = \chi(\omega) \cdot \exp(\mathrm{i} \cdot \omega \cdot x_0)$$

and hence,

$$\ln(\chi'(\omega)) = \ln(\chi(\omega)) + \mathrm{i} \cdot \omega \cdot x_0.$$

The additional term is linear in ω, so for $n \geq 2$, its n-th derivative is 0 – hence, $L'_n = L_n$, i.e., n-th logarithmic moment indeed does not change with shift.

For L_3, we have

$$\frac{\partial^3 \ln(\chi)}{\partial \omega^3} = \frac{1}{\chi} \cdot \frac{\partial^3 \chi}{\partial \omega^3} + \frac{2}{\chi^3} \cdot \left(\frac{\partial \chi}{\partial \omega}\right)^3 - \frac{3}{\chi^2} \cdot \frac{\partial \chi}{\partial \omega} \cdot \frac{\partial^2 \chi}{\partial \omega^2},$$

so $L_3 = M_3 + 2M_1^3 - 2M_1 \cdot M_2$.

Similarly, we can conclude that for every $n > 1$, $L_n = M_n$ + terms depending on the moments of smaller orders $M_1, \ldots, M_{n-1}$. (Actually, since L_n does not change with shift, we can compute the moment w.r.t. the average M_1, and thus conclude that L_n depends only on central moments of the distribution.)

Thus, in general,

- once we know the moments $M_1, \ldots, M_n$, we can easily compute the corresponding logarithmic moments, and
- vice versa, once we know the logarithmic moments, we can determine the corresponding moments $M_1, \ldots, M_n$ one by one.

So, in general, the problem of reconstructing a probability distribution from its logarithmic moments can be reduced to a well-researched problem of reconstructing a distribution from its moments $M_1, \ldots, M_n$.

X. MAXIMUM ENTROPY APPROACH

Of course, there are many possible probability distributions with the given values of the moments $M_1, \ldots, M_n$; among such distributions, we must select the most "reasonable" one.

We have mentioned that in many cases, the actual distribution is Gaussian – e.g. (due to the Central Limit theorem), when the random variable is a sum of several small independent components. It is well known that a Gaussian distribution can be uniquely determined by its first two moments (or, in multi-D case, by its moments of the first and second orders). It is therefore reasonable to require that when we only know the first two moments, we should get the Gaussian distribution.

This is achieved, e.g., if among all possible distributions, we select a distribution for which the entropy

$$-\int \rho(x) \cdot \ln(\rho(x)) \, dx$$

attains its largest possible value. Indeed, if we know M_1 and M_2, this means that the unknown probability distribution $\rho(x)$ satisfies the constraints

$$\int \rho(x) \, dx = 1,$$

$$\int x \cdot \rho(x) \, dx = M_1,$$

$$\int x^2 \cdot \rho(x) \, dx = M_2.$$

To maximize the entropy under these three constraints, we can use the Lagrange multiplier approach, in which we replace the original constrained optimization problem with an unconstrained problem of optimizing a new objective function

$$-\int \rho(x) \cdot \ln(\rho(x)) \, dx +$$

$$\lambda_0 \cdot \int \rho(x) \, dx + \lambda_1 \cdot \int x \cdot \rho(x) \, dx + \lambda_2 \cdot \int x^2 \cdot \rho(x) \, dx$$

for some real numbers (Lagrange multipliers) λ_i.

To find the maximum of the corresponding function, we can simply equate its derivative w.r.t. $\rho(x)$ to 0, hence

$$-\ln(\rho(x)) - 1 + \lambda_0 + \lambda_1 \cdot x + \lambda_2 \cdot x^2 = 0,$$

i.e.,

$$\rho(x) = C \cdot \exp(-\lambda_1 \cdot x - \lambda_2 \cdot x^2)$$

for some constant C. (The values C and λ_i can then be determined from the fact that the overall probability should be 1, and the first two moments are M_1 and M_2.)

This is exactly the formula for the Gaussian distribution. There are many other arguments in favor of selecting a distribution corresponding to the maximum entropy; see, e.g., [15], [16]. So, when we know $n > 2$ moments $M_1, \ldots, M_n$, it is also reasonable to look for a probability distribution $\rho(x)$ for which the entropy is the largest among all the distributions for which

$$\int \rho(x)\,dx = 1,$$

$$\int x \cdot \rho(x)\,dx = M_1,$$

$$\cdots$$

$$\int x^n \cdot \rho(x)\,dx = M_n.$$

For this constraint optimization problem, the Lagrange multiplier method leads to the unconstrained optimization of the functional

$$-\int \rho(x) \cdot \ln(\rho(x))\,dx +$$

$$\lambda_0 \cdot \int \rho(x)\,dx + \lambda_1 \cdot \int x \cdot \rho(x)\,dx + \ldots + \lambda_n \cdot \int x^n \cdot \rho(x)\,dx$$

Differentiating this expression w.r.t. $\rho(x)$ and equating the corresponding derivatives to 0, we conclude that

$$-\ln(\rho(x)) - 1 + \lambda_0 + \lambda_1 \cdot x + \ldots + \lambda_n \cdot x^n = 0,$$

hence

$$\rho(x) = C \cdot \exp(-\lambda_1 \cdot x - \ldots - \lambda_n \cdot x^n).$$

The values C and $\lambda_1, \ldots, \lambda_n$ can then be determined from the fact that the overall probability should be 1, and the first n moments are equal to $M_1, \ldots, M_n$.

XI. PROBLEM: THE MAXIMUM ENTROPY APPROACH IS RATHER COMPUTATIONALLY INTENSIVE

According to the Maximum Entropy approach, to find $n+1$ parameters C, λ_1, $\ldots$, λ_n, we must solve a system of $n + 1$ equations with $n + 1$ unknowns. This system is highly non-linear, so solving the corresponding system is not easy.

Good news is that in practice, we can only determine a few moments, so the number of unknowns $(n + 1)$ is small. Therefore, the existing algorithms for solving systems of non-linear equations can be effectively applied here. Still, solving a system of non-linear equations is much more computationally intensive than computations for the Gaussian case, where the values C and λ_i can be explicitly computed from the moments.

XII. FIRST ALTERNATIVE TO THE MAXIMUM ENTROPY APPROACH THAT ENABLES US TO SPEED UP COMPUTATIONS

Since the Maximum Entropy approach is computationally intensive, it is reasonable to look for alternative approaches that would require fewer computations.

If we still want this approach to lead to Gaussian distributions for the case when we know only the first two moments, then it is reasonable to look for distributions that can be obtained from the Gaussian distribution after an appropriate rescaling. What are the most reasonable rescalings? In [19], we have shown that w.r.t. reasonable optimality criteria, the optimal rescalings are power laws. So, in the symmetric case (when $L_3 = 0$), we get Weibull-type distributions with the probability density $\text{const} \cdot \exp(-k \cdot |x - a|^p)$ with p possibly different from 2.

It turns out that these distributions indeed well describe measurement errors [20]; in particular, these distributions well describe the errors related to geophysical measurements and estimates [10], [24].

For this distribution, once we know the 4th central moment M_4 and the variance M_2, we can then find the value p by solving a single non-linear equation with only one unknown [20]:

$$\varepsilon = \frac{\Gamma(1/p) \cdot \Gamma(5/p)}{\Gamma(3/p)^2},$$

where $\Gamma(x)$ is the gamma-function and

$$\varepsilon \overset{\text{def}}{=} \frac{M_4}{M_2^2}.$$

If we want to make computations even faster, we can use the following approximate formula [20]:

$$p = \frac{1.46}{\ln(\varepsilon - 2/9 - 10.7/\varepsilon^7) - 0.289}.$$

For asymmetric distributions, the general result from [19] leads to two different scalings for $x > 0$ and $x < 0$; as a result, we get a probability distribution for which:

- $\rho(x) = \text{const}_- \cdot \exp(-k_- \cdot |x - a|^p)$ for $x \leq a$ and
- $\rho(x) = \text{const}_+ \cdot \exp(-k_+ \cdot |x - a|^p)$ for $x \geq a$.

In this manner, we can get match arbitrary values of the first 4 moments.

Here also, we can produce explicit formulas for the moments of this distribution in terms of the gamma-function; so, while we still need to solve a non-linear systems of equations, this non-linearity is easier than in the Maximum Entropy approach – because this non-linearity is described not by difficult-to-compute integrals, but rather by a known special function (gamma function).

XIII. OTHER ALTERNATIVES TO THE MAXIMUM ENTROPY APPROACH

For the first four moments, we can get even faster computations if we do not require that for the first two moments, we always get Gaussian distribution.

Several such families have been proposed; see, e.g., [1] and references therein. The computationally simplest family, for which there are explicit formulas relating the first four moments with the parameters of the distribution, is the family of *Generalized Lambda* distributions, in which the quantile function $Q(u)$ – inverse to the cumulative distribution function $F(t)$ – has the form

$$Q(u) = \lambda_1 + \frac{1}{\lambda_2} \cdot \left[\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1 - u)^{\lambda_4} - 1}{\lambda_4} \right].$$

For this distribution, finding the four parameters requires solving a system of two non-linear equations with two unknowns, equations in which non-linearity is described by another class of known special functions: by beta functions.

XIV. THE USE OF EXPERT KNOWLEDGE

Often, in addition to the four (or more) moments, we also have some expert knowledge about the unknown probability distribution $\rho(t)$. This expert knowledge usually comes in terms of words from natural language, so it is natural to use fuzzy techniques to transform this expert knowledge into an exact formula that describes, for each distribution ρ, the degree $\mu(\rho)$ with which this distribution is consistent with the expert knowledge.

Then, it is reasonable to select, among all the distributions with the given values of the first n moments $L_1, \ldots, L_n$, the distribution ρ for which this degree $\mu(\rho)$ is the largest possible. This idea is similar to other uses of fuzzy techniques in geosciences; see, e.g., [4], [9]

In the absence of additional expert information, this approach leads either to the Maximum Entropy formulas [17], [18] – or to a more general situation in which we optimize the *generalized entropy*

$$\int \rho(x)^\alpha \, dx$$

for some real number α. In this case, if we know the first n moments, then the Lagrange multiplier methods leads to the following probability density function:

$$\rho(x) = (\lambda_0 + \lambda_1 \cdot x + \ldots + \lambda_n \cdot x^n)^{-\beta}$$

for some real number $\beta > 0$.

In the presence of additional expert knowledge, we can get more specific criteria hence more specific distributions.

ACKNOWLEDGMENTS

REFERENCES

[1] S. W. M. Au-Yeung, *Finding Probability Distributions from Moments*, Master's Thesis, Imperial College, London, 2003.

[2] A. P. Annan, *Ground Penetrating Radar Workshop Notes*, Sensors and Software Inc., Mississauga, Ontario, Canada, 1992.

[3] Atlantis Scientific, Inc., *Theory of Synthetic Aperture Radar*, http://www.geo.unizh.ch/fpaul/sar-theory.html, 2005.

[4] G. Bardossy and J. Fodor, *Evaluation of Uncertainties and Risks in Geology*, Springer Verlag, Berlin, 2004.

[5] J. J. Daniels, "Fundamentals of ground penetrating radar", *Proceedings of the Symposium on the Application of Geophysics to Engineering and Environmental Problems*, Colorado School of Mines, Golden, Colorado, 1989, pp. 62-142.

[6] J. J. Daniels, J. Bower, and F. Baumgartner, "High resolution GPR at Brookhaven National Lab to delineate complex subsurface structure", *Journal Environmental and Engineering Geophysics*, 1989, Vol. 3, No. 1, pp. 1–6.

[7] J. J. Daniels, D. L. Grumman, and M. Vendl, "Coincident antenna three dimensional GPR", *Journal of Environmental and Engineering Geophysics*, 1997, Vol. 2.

[8] J. L. Davis and A. P. Annan, "Ground penetrating radar for soil and rock stratigraphy", *Geophysical Prospecting*, 1997, Vol. 37, pp. 531–551.

[9] R. Demicco and G. Klir (eds.), *Fuzzy Logic in Geology*, Academic Press, 2003.

[10] D. I. Doser, K. D. Crain, M. R. Baker, V. Kreinovich, and M. C. Gerstenberger "Estimating uncertainties for geophysical tomography", *Reliable Computing*, 1998, Vol. 4, No. 3, pp. 241–268.

[11] E. Fisher, G. A. McMechan, and A. P. Annan, "Acquisition and processing of wide-aperture ground-penetrating radar data", *Geophysics*, 1992, Vol. 57, No. 3, pp. 495–504.

[12] E. D. Guy, J. J. Daniels, S. J. Radzevicius, and M. A. Vendl, "Demonstration of using crossed dipole GPR antenna for site characterization", *Geophysical Research Letters*, 1999, Vol. 26, No. 22. pp. 3421–3424.

[13] F. P. Haeni, "Use of ground-penetrating radar and continuous seismic-reflection on surface-water bodies in environmental and engineering studies", *Journal of Environmental & Engineering Geophysics*, 1996, Vol. 1, No. 1, pp. 27–36.

[14] J. Holt, M. Vendl, F. Baumgartner, S. Radziviscius, and J. J. Daniels, "Brownfields site-characterization using geophysics: A case history from East Chicago", *Transactions of the Symposium on the Application of Geophysics to Engineering and Environmental Problems: Soc. Eng. and Min. Expl. Geophys.*, April 26–29, 1998, Chicago, Il.

[15] E. T. Jaynes, "Where do we Stand on Maximum Entropy?", In: in R. D. Levine and M. Tribus (eds.), *The Maximum Entropy Formalism*, M.I.T. Press, Cambridge, MA, 1979, pp. 15–118.

[16] E. T. Jaynes, *Probability Theory: The Logic of Science*, Cambridge University Press, Cambridge, Massachusetts, 2003.

[17] O. Kosheleva, "Symmetry-group justification of maximum entropy method and generalized maximum entropy methods in image processing", In: G. J. Erickson, J. T. Rychert, and C. R. Smith (eds.), *Maximum Entropy and Bayesian Methods*, Kluwer, Dordrecht, 1998, pp. 101–113.

[18] O. Kosheleva, "On the Optimal Choice of Quality Metric In Image Compression: A Soft Computing Approach", *Soft Computing*, 2004, Vol. 8, No. 4, pp. 268–273.

[19] H. T. Nguyen and V. Kreinovich, *Applications of continuous mathematics to computer science*, Kluwer, Dordrecht, 1997.

[20] P. V. Novitskii and I. A. Zograph, *Estimating the Measurement Errors*, Energoatomizdat, Leningrad, 1991 (in Russian).

[21] G. R. Olhoeft, "Electrical, magnetic, and geometric properties that determine ground penetrating performance", In: *Proc. Seventh International Conference on Ground Penetrating Radar*, May 27–30, 1998, the University of Kansas, Lawrence, KS, 1998.

[22] *Proceedings of the 10th International Conference on Ground Penetrating Radar GPR'2004*, Delft, The Netherlands, June 21–24, 2004.

[23] S. Srikrishnan, R. Araiza, V. Kreinovich, S. A. Starks, and H. Xie, "Automatic Referencing of Satellite and Radar Images", *Proceedings of the 2001 IEEE Systems, Man, and Cybernetics Conference*, Tucson, Arizona, October 7–10, 2001, pp. 2176–2181.

[24] A. Tarantola, *Inverse Problem Theory: Methods for Data Fitting and Model Parameter Estimation*, Elsevier, Amsterdam, 1987.

[25] H. M. Wadsworth Jr., *Handbook of statistical methods for engineers and scientists*, McGraw-Hill, N.Y., 1990.