

Growth Rates under Interval Uncertainty

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Abstract. For many real-life systems ranging from financial to population-related to medical, dynamics is described by a system of linear equations. For such systems, the growth rate λ can be determined as the largest eigenvalue of the corresponding matrix a_{ij} . In many practical situations, we only know the components of the matrix a_{ij} with interval (or fuzzy) uncertainty. In such situations, it is desirable to find the range of possible values of λ . In this paper, we propose an efficient algorithm for computing λ for a practically important case when all the components a_{ij} of the matrix are non-negative.

1 Growth Rates: A Linear Approximation to the Description of a General System

General description. In general, the state of a real-life complex system can be described by listing the current values of its parameters x_1, \dots, x_n .

- For continuous-time systems, their dynamic can be described as

$$\dot{x}_i = f_i(x_1, \dots, x_n)$$

for some functions f_1, \dots, f_n .

- For discrete-time systems, their dynamic can be described as

$$x_i(t+1) = f_i(x_1(t), \dots, x_n(t))$$

for some functions f_1, \dots, f_n .

Linearized description. The dependencies $f_i(x_1, \dots, x_n)$ are usually smooth, so within a reasonable range of values x_i , we can approximate each of these functions by a linear expression:

$$f_i(x_1, \dots, x_n) = b_i + \sum_{j=1}^n a_{ij} \cdot x_j.$$

By applying an appropriate shift $x_i \rightarrow x_i - s_i$, we can simplify this system even further, into

$$f_i(x_1, \dots, x_n) = \sum_{j=1}^n a_{ij} \cdot x_j.$$

Thus:

- the dynamic of a continuous-time system can be described by the equation

$$\dot{x}_i = \sum_{j=1}^n a_{ij} \cdot x_j;$$

- the dynamic of a discrete-time systems can be described by the equation

$$x_i(t+1) = \sum_{j=1}^n a_{ij} \cdot x_j(t).$$

The notion of a growth rate. By using eigenvectors of the matrix $A = (a_{ij})$ as a new base, we get a yet simpler expression for the new variables y_i – the coefficients in the expansion of $x_i(t)$ in this new base.

In the generic case when all eigenvalues are different, the dynamic equations take the simplest possible form:

- for a continuous-time system,

$$\dot{y}_i = \lambda_i \cdot y_i,$$

where λ_i is the corresponding eigenvalue;

- for a discrete-time system system,

$$y_i(t+1) = \lambda_i \cdot y_i(t).$$

These equations have an explicit solution:

- for a continuous-time system, we get

$$y_i(t) = y_i(0) \cdot \exp(\lambda_i \cdot t);$$

- for a discrete-time system, we get

$$y_i(t) = y_i(0) \cdot \lambda_i^t.$$

This decomposition into simple solutions $y_i(t)$ is one of the main ideas behind the Principle Component Analysis.

A general solution $x_i(t)$ is a linear combination of such terms. Thus, in the general case, asymptotically,

- for a continuous-time system,

$$x_i(t) \sim \exp(\lambda \cdot t),$$

where λ is the largest of these eigenvalues;

- for a discrete-time system,

$$x_i(t) \sim \lambda^t.$$

When the largest eigenvalue is degenerate, we have $x_i(t) \sim x^k \cdot \exp(\lambda \cdot t)$ or $x_i(t) \sim x_k \cdot \lambda^t$ for some integer k , i.e., modulo polynomial terms, still the same asymptotic.

Because of this fact, the largest eigenvalue λ is called the *growth rate* of a system.

This formula indeed describes a growth rate. This asymptotic behavior well describes different types of growth (see, e.g., [3]):

- the population growth,
- the growth in animals and plants,
- the growth rates of number of affected people under an epidemic,
- financial growth,
- etc.

For example, in population growth, different variables x_i describe the number of people of i -th age group, etc.

2 Growth under Interval Uncertainty: A Computational Problem

Computing the growth rate is important. In view of the above applications, it is important to compute the growth rate for a given system.

Idealized case: exactly known coefficients. In general, we never know the *exact* values of parameters of real-life systems, these parameters are always known with some *uncertainty*.

In many real-life situations, however, this uncertainty is small. In such situations, we can safely assume that we know the exact values a_{ij} of all the coefficients. In such situations, we can use known algorithms to find the eigenvalues [3, 7, 14] and thus, find the largest of these eigenvalues – the growth rate.

Often, we cannot ignore the uncertainty. In many real-life situations, however, we cannot ignore the uncertainty. In such situations, we have to take into account the fact that the coefficients a_{ij} are only known with uncertainty.

Case of interval uncertainty. Often, in addition to the approximate values \tilde{a}_{ij} of the corresponding coefficients, we also know the upper bounds Δ_{ij} on the approximation error $|\tilde{a}_{ij} - a_{ij}|$. In such situations, we know that the actual (unknown) value of each coefficient a_{ij} belongs to the interval

$$\mathbf{a}_{ij} = [a_{ij}, \bar{a}_{ij}] \stackrel{\text{def}}{=} [\tilde{a}_{ij} - \Delta_{ij}, \tilde{a}_{ij} + \Delta_{ij}].$$

Computing the growth rate under interval uncertainty: a computational problem. In case of interval uncertainty, different values $a_{ij} \in \mathbf{a}_{ij}$ lead to different growth rates λ . In such situations, it is desirable to find the interval $[\underline{\lambda}, \bar{\lambda}]$ of possible values of λ – or at least an interval that guarantees to contain this interval.

Of special importance is the upper endpoint $\bar{\lambda}$ of the desired interval, because this upper endpoint indicates how fast a population can grow, or how fast a disease can spread.

The need for computing such an interval has been known for a few decades; see, e.g., [2].

Case of small uncertainty: sensitivity analysis. When the uncertainty is relatively small, i.e., when the uncertainty $\Delta a_{ij} \stackrel{\text{def}}{=} \tilde{a}_{ij} - a_{ij}$ is much smaller than the approximate value \tilde{a}_{ij} , we can linearize the equations for describing the eigenvalues in terms of a_{ij} and use the sensitivity analysis techniques to get reasonable estimates for $[\underline{\lambda}, \bar{\lambda}]$; see, e.g., [2].

General case: the problem is computationally intractable (NP-hard). In many real-life situations, e.g., in many financial and biological systems, the uncertainty is not small, so we can no longer use the linearized techniques to find $\bar{\lambda}$.

In general, we thus face a problem of finding the range of possible values of λ for all matrices a_{ij} within a given interval matrix \mathbf{a}_{ij} , i.e., for all matrices for which $a_{ij} \in \mathbf{a}_{ij}$. It is known that in general, this problem is NP-hard; see [9] and references therein. This means, crudely speaking, that it is not possible to have an algorithm that would always compute the desired range for λ in physically reasonable time.

Moreover, it is also known [9] that even the problem of computing the eigenvalues with a given accuracy is NP-hard.

This means that while there exist computationally efficient methods of computing an enclosure for the desired interval $[\underline{\lambda}, \bar{\lambda}]$, but these methods sometimes lead to a drastic excess width.

Important case: a non-negative matrix. In many real-life situations, the matrix a_{ij} is *non-negative* in the sense that all its coefficients are non-negative. Such non-negative matrices describe population growth, spread of disease, financial situations, etc. [3].

What we propose. We propose a new algorithm that, for non-negative matrices, exactly computes the upper bound $\bar{\lambda}$ on λ in feasible computation time.

Comment. In this paper, we concentrated on the computation of the largest eigenvalue, a practically useful characteristic of an interval matrix. If, in addition to describing the asymptotic growth rate, we want to find a more detailed description of a growth, then we need to find not only the largest eigenvalue, but also other eigenvalues and the corresponding eigenvectors. Algorithms for solving this problem under interval uncertainty are presented, e.g., in [10].

3 New Algorithm

This algorithm is based on known algorithms for the case of the exact matrix. Our algorithm assumes that already have an algorithm \mathcal{A} for computing the largest eigenvalue $\lambda(A)$ of a given non-negative matrix $A = \|a_{ij}\|$. Such algorithms are described, e.g., in [3, 7, 14].

Input to the new algorithm. Let us assume that instead of the exact non-negative matrix $A = \|a_{ij}\|$, we are given the interval-valued matrix $\mathbf{A} = \|\mathbf{a}_{ij}\|$, where $\mathbf{a}_{ij} = [\underline{a}_{ij}, \bar{a}_{ij}]$.

In other words, for each i and each j , instead of a single value a_{ij} , we have an interval $[\underline{a}_{ij}, \bar{a}_{ij}]$ – i.e., in effect, two values \underline{a}_{ij} and \bar{a}_{ij} . We can alternatively describe this situation by saying that:

- for each i and j , we know the lower endpoint \underline{a}_{ij} , and
- for each i and j , we know the upper endpoint \bar{a}_{ij} .

In other words, instead of a single matrix $A = \|a_{ij}\|$, we are given two matrices: $\underline{A} \stackrel{\text{def}}{=} \|\underline{a}_{ij}\|$ and $\bar{A} \stackrel{\text{def}}{=} \|\bar{a}_{ij}\|$.

Description of the new algorithm.

- First, we apply the algorithm \mathcal{A} for computing $\lambda(A)$ to the matrix \underline{A} ; the resulting value is returned as $\underline{\lambda}$.
- Then, we apply the algorithm \mathcal{A} for computing $\lambda(A)$ to the matrix \bar{A} ; the resulting value is returned as $\bar{\lambda}$.

Comment. This simple idea does not work for general interval matrices, only for non-negative ones. In the following text, we provide a proof that for non-negative matrices, this algorithm indeed works well.

Justification of the new algorithm. In order to provide the desired justification, let us introduce some notations. We have defined a matrix A to be non-negative if all its components are non-negative, i.e., as $a_{ij} \geq 0$ for all i and j . It is natural to denote this non-negativity in the usual way, as $A \geq 0$.

We can similarly define a vector $x = (x_1, \dots, x_n)$ to be non-negative if all its components are non-negative, i.e., if $x_i \geq 0$ for all i . This relation will also be denoted by $x \geq 0$.

We can also define the relation $A \leq B$ for two matrices A and B as $B - A \geq 0$. In other words, for matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$, the order $A \leq B$ is defined component-wise, as $a_{ij} \leq b_{ij}$ for all i and j .

Now, we are ready for the justification. This justification is based on the two known facts (described below in detail):

- the representation of the largest eigenvalue as a solution to an auxiliary optimization problem, and
- the Perron-Frobenius Theorem about the eigenvectors of non-negative matrices.

The first known fact is that the largest eigenvalue $\lambda(A)$ of a matrix A can be described as

$$\lambda(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

where

$$\|x\|_2 \stackrel{\text{def}}{=} \sqrt{x_1^2 + \dots + x_n^2}$$

denotes the length of a vector x ; see, e.g., [3, 7, 14].

For non-negative matrices A , the Perron-Frobenius Theorem [3, 4, 7, 14] states that at least one of the eigenvectors $x = (x_1, \dots, x_n)$ corresponding to the largest eigenvalue $\lambda(A)$ is also non-negative: $x \geq 0$. Thus, the maximum in the above definition of $\lambda(A)$ is attained on a non-negative vector. Therefore, when computing this maximum, we can restrict ourselves only to non-negative vectors:

$$\lambda(A) = \max_{x \geq 0 \ \& \ x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

When $0 \leq A$ and $x \geq 0$, then, by definition of the matrix multiplication, we get $Ax \geq 0$. If $0 \leq A \leq B$ and $x \geq 0$, then similarly $0 \leq Ax \leq Bx$. When we have two vectors a and b for which $0 \leq a_i \leq b_i$ for all i , then, of course,

$$a_1^2 + \dots + a_n^2 \leq b_1^2 + \dots + b_n^2,$$

hence $\|a\|_2 \leq \|b\|_2$. Thus, if $0 \leq A \leq B$, then for every vector $x \geq 0$, we get $\|Ax\|_2 \leq \|Bx\|_2$ and therefore,

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \frac{\|Bx\|_2}{\|x\|_2}.$$

Since this inequality holds for every vector $x \neq 0$, the maximum $\lambda(A)$ of its left-hand side is smaller than or equal than the maximum $\lambda(B)$ of its right-hand side. In other words, if $0 \leq A \leq B$, then $\lambda(A) \leq \lambda(B)$.

By definition of an interval-valued matrix, all possible matrices $A \in [\underline{A}, \overline{A}]$ satisfy the inequality $\underline{A} \leq A \leq \overline{A}$. Since we assumed that our matrices are non-negative, we conclude that $0 \leq \underline{A} \leq A \leq \overline{A}$. Thus, for every possible matrix $A \in [\underline{A}, \overline{A}]$, we get $\lambda(\underline{A}) \leq \lambda(A) \leq \lambda(\overline{A})$.

Hence, all the values $\lambda(A)$ lie within the interval $[\lambda(\underline{A}), \lambda(\overline{A})]$. Since both endpoints of this interval are attained for some matrices from the matrix interval $[\underline{A}, \overline{A}]$, we thus conclude that the interval $[\lambda(\underline{A}), \lambda(\overline{A})]$ is the actual range of $\lambda(A)$. Thus, for non-negative interval matrices, the above algorithm is indeed justified.

Practical applications. In [5, 6], we apply our ideas to the dynamics of real-life ecological systems for which we only know the components a_{ij} with interval uncertainty.

4 Case of Fuzzy Uncertainty

Case of fuzzy uncertainty. Often, knowledge comes in terms of uncertain expert estimates. To describe this uncertainty, for each possible value of a_{ij} , we describe the degree $\mu_{ij}(a_{ij}) \in [0, 1]$ to which this value is possible. These degrees form a *fuzzy set*.

Processing fuzzy uncertainty can be reduced to processing interval uncertainty. For each degree of certainty α , we can determine the set of values of a_{ij} that are possible with at least this degree of certainty – the α -cut

$$\{a_{ij} \mid \mu_{ij}(a_{ij}) \geq \alpha\}$$

of the original fuzzy set. In most cases, this α -cut is an interval.

Vice versa, if we know α -cuts for every α , then, for each value a_{ij} , we can determine the degree of possibility that a_{ij} belongs to the original fuzzy set [1, 8, 11–13]. A fuzzy set can be thus viewed as a nested family of its α -cuts.

So, if instead of a (crisp) interval \mathbf{a}_{ij} of possible values of the component a_{ij} , we have a fuzzy set $\mu_{ij}(a_{ij})$ of possible values, then we can view this information as a family of nested interval matrices $\mathbf{a}_{ij}(\alpha)$ – α -cuts of the given fuzzy-valued matrices.

Computation under fuzzy uncertainty. Let us consider the case when instead of a (crisp) interval \mathbf{a}_{ij} of possible values of the components, we have a fuzzy set $\mu_{ij}(a_{ij})$ of possible values. In this case, we can view this information as a family of nested interval-valued matrices $\mathbf{a}_{ij}(\alpha)$ – α -cuts of the given fuzzy sets.

Our objective is then to compute the fuzzy number λ – the largest eigenvalue of this fuzzy-valued matrix. In this case, for each level α , to compute the α -cut of this fuzzy number, we can apply interval computations to the α -cuts $\mathbf{a}_{ij}(\alpha)$ of the corresponding fuzzy-valued matrix. The resulting nested intervals form the desired fuzzy set for λ .

So, e.g., if we want to describe 10 different levels of uncertainty, then we must solve 10 interval computation problems – i.e., apply the above algorithm 10 times.

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