FLIGHT CESSNA 771 REVISITED: GEOMETRY OF A PLANE RESCUE

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Geometry helped to rescue an airplane: a true story. On December 21, 1978, a Cessna plane got lost over the Pacific Ocean when its navigation instruments malfunctioned. A Cessna plane is not equipped for a water landing, so if the plane was not located and guided to an airport, Jay Parkins, the pilot, could die. Luckily, a large passenger plane happened to fly in the area, navigated by a New Zealand pilot Gordon Vette, and this plane heard the Cessna's radio signal.

Captain Vette used many ideas to locate the plane. The final location breakthrough came from the fact that the Cessna plane was equipped with a radio transmitter which could only be heard within direct visibility. Thus, Captain Vette could only hear this radio when he was within a certain (geometrically easy to compute) distance r ($r \approx 200$ miles) from the plane.

So, he asked the Cessna pilot to circle in place, and flied his plane in a straight line until he lost the radio signal. The point where he lost the signal was exactly r miles from the (unknown) Cessna location. After that, he turned back, and flew in a somewhat different direction until he lost the radio signal again; this way, he found a second point on the circle. Based on these two points on the circle, Captain Vette found the center of this circle – i.e., the location of the missing plane.

Based on this location, Air Traffic Control gave the pilot directions to the nearby airport (directions had to be given in relation to the Sun, since the navigation instruments did not work). The plane was saved, the pilot landed alive.

This dramatic rescue story is described in detail in (Steward 2003); it was even made into a successful TV movie (Young et al. 1993).

Why this story is interesting. This story can be used as a good pedagogical example: that seemingly abstract geometry can actually help in very unusual and drastic situations.

Towards a related geometric problem. With this situation, comes an interesting geometric problem. The crucial aspect of this plane rescue was time: the plane had to be located before it ran out of fuel. So, the passenger plane had to fly at its maximal speed. With this speed, time is proportional to the distance. So, we must choose the shortest of all the trajectories which would allow us to reach the circle.

The most important thing is to reach one point on a circle. Once we have found it, we can easily find nearby points – e.g., by flying in a small circle around that first point. So, the critical question is finding the shortest path which still guarantees that we find a point on a circle.

Captain Vette chose to fly in a straight line until he lost the radio signal. Was this the best possible decision, or could some other trajectory be better?

Intuitively, going in a straight line makes sense because if we are somewhere inside a circular disk, and we follow a straight line in any direction, then eventually, we will reach the circle – the borderline of the disk. However, it is not intuitively clear whether this is indeed the optimal strategy.

Specifically, for each line, we could go in both directions. If we go in one direction and reach a circle after we flew a distance $\leq r$, then this strategy sounds reasonable. However, if we have flown a distance larger than r and we have not yet reached the circle, this means that we were flying in a wrong direction. So, maybe at this point, a reasonable strategy is to change course?

Let us formulate this problem in precise terms. Let us denote the starting point by O. We know that this point O is inside the disk of given radius r; we do not know where is the center of this disk.

Definitions.

- *By a* trajectory*, we mean a planar curve of finite length, i.e., a continuous mapping γ from some interval $[0,T]$ into a plane such that $\gamma(0) = O$ and the overall length of this curve is finite.*
- *Let $r > 0$ be a real number. We say that the trajectory γ* is guaranteed to reach any circle of radius r *if for every disk of radius r which contains the point O, the curve γ has an intersection with its border (i.e., with the corresponding circle).*

Comment. For example, we can restrict ourselves to piece-wise smooth curves. For such curves γ, the length $\ell(\gamma)$ can be described as $\ell(\gamma) = \int \|\dot\gamma\|\,dt$, where $\dot\gamma \stackrel{\rm def}{=} \dfrac{d\gamma}{dt}$, and $\|(a_1,a_2)\| = \sqrt{a_1^2+a_2^2}$ denotes the length of a vector.

Proposition. *For every real number $r > 0$, the following statements hold:*

- *A straight line segment of length $2r$ is guaranteed to reach any circle of radius r.*
- *Every trajectory γ which is guaranteed to reach any circle of radius r has a length $\ell(\gamma) \ge 2r$.*
- *If γ is a trajectory of length $2r$ which is guaranteed to reach any circle of radius r, then γ is a straight line segment.*

Comments. In other words, the only shortest (= fastest) rescue trajectory is a straight line segment.

Proof.

1°. Clearly, the straight line segment of length $2r$ is guaranteed to reach any circle of radius r.

Indeed, inside the circle, the largest distance between the two points is the diameter $2r$. So, once we are inside the disk and we go the distance $2r$, we are outside the disk.

2°. Before we continue with the proof, let us make some remarks about the representation of the curves.

In our definition, we defined a curve as an arbitrary continuous mapping from real numbers to the plane. If we re-scale this curve, i.e., use a function $r(s(t))$, where $s(t)$ is a monotonic function from real numbers to real numbers, then we get the exact same geometric curve. It is convenient to avoid this multiple representation of the same geometric curve by using, e.g., the total length of the path between the point $\gamma(0)$ and $\gamma(t)$ as the new parameter. With this choice of a parameter, the length of a curve from the point $\gamma(0)$ to the point $\gamma(t)$ is equal to t.

In the following text, we will assume that the curve γ is parameterized by length.

3°. Let γ be a curve that is guaranteed to reach a circle of radius r. Let us prove that the length $L = \ell(\gamma)$ of this curve γ is greater than or equal to $2r$.

We will prove this by reduction to a contradiction. Indeed, assume that $L < 2r$. Let us take a circle of radius r with a center in the point $c = \gamma(L/2)$. For every point $\gamma(t)$, the length of the curve between this point and the center is equal to $|t - (L/2)|$. For $0 \le t < L/2$, this value is equal to $(L/2) - t$ and is, thus, $\le L/2$. For $L/2 \le t \le L$, this value is equal to $t - (L/2)$; since $t \le L$, this length is $\le L - (L/2) = L/2$. In both cases, we length is $\le L/2$.

Since the distance is the shortest possible length of a curve connecting two points, we this conclude that the distance $\|\gamma(t) - c\|$ between any point $\gamma(t)$ on this curve and the center c does not exceed the length along the curve and hence, does not exceed $L/2$. Since $L < 2r$, we have $(L/2) < r$, so every point on the curve is at a distance $< r$ from the center c. Hence, none of these points is on the circle, contrary to our assumption that the trajectory is guaranteed to reach any circle of radius r.

This contradiction proves that every trajectory which is guaranteed to reach any circle of radius r has length $\ge 2r$.

4°. To complete the proof, let us prove that if γ is a curve of length $\ell(\gamma) = 2r$ which is guaranteed to reach any circle of radius r, then

γ is a straight line segment.

$4.1°$. Let us first prove that at least one of the half-curves $\gamma([0,r])$ and $\gamma([r,2r])$ is a straight line segment.

We will also prove this by reduction to a contradiction. Let us assume that both half-curves are not straight line segments, and let us prove that under this assumption, the curve γ does not reach a circle of radius r whose center is the midpoint $c = \gamma(r)$.

Indeed, the straight line segment is the only shortest curve connecting two points. Thus, since $\gamma([0,r])$ is not a straight line segment, the distance $\|c - \gamma(0)\|$ between $c = \gamma(r)$ and the point $\gamma(0)$ is smaller than the length r of this half-curve.

For all values $t \in (0,r]$, the distance $\|c - \gamma(t)\|$ between $\gamma(t)$ and $c = \gamma(r)$ is smaller than or equal that the length $r - t$ of the half-curve between these points: $\|c - \gamma(t)\| \leq r - t$. Since $t > 0$, we have $r - t < r$ and hence, $\|c - \gamma(t)\| < r$.

So, for all $t \in [0,r]$, we have $\|c - \gamma(t)\| < r$, i.e., none of the points on this half-curve reaches the circle. Similarly, none of the points on the second half-curve $\gamma([r,2r])$ reaches the circle. This contradicts to our assumption that γ is guaranteed to reach any circle of radius r. This contradiction proves the statement.

$4.2°$. We have just proved that at least one of the half-curves $\gamma([0,r])$ and $\gamma([r,2r])$ is a straight line segment. Without losing generality, let us assume that the first half-curve $\gamma([0,r])$ is a straight line segment (the proof for the case when the second half-curve is a straight line segment is similar).

$4.3°$. Let us now prove, by reduction to a contradiction, that the second half-curve $\gamma([r,2r])$ is also a straight line segment.

Indeed, if it is not a straight line segment, then, as we have mentioned in Part $4.1°$ of this proof, the largest distance d_0 between the midpoint $\gamma(r)$ and points $\gamma(t)$, $r \leq t \leq 2r$, on this half-curve is smaller than r: $d_0 < r$.

Let us now denote the half-difference $(r - d_0)/2$ by δ, and consider a circle of radius r with the center c at the point $\gamma(r - \delta)$. The first half-curve is a straight line segment, so all the points on the first half-curve are located at a distance $\leq r - \delta < r$ from the point c. Thus, none of these points reaches the circle.

For every point $\gamma(t)$ on the second half-curve, we have $\|\gamma(t)-\gamma(r)\| \leq d_0$, hence $\|\gamma(t)-c\| \leq \|\gamma(t)-\gamma(r)\|+\|\gamma(r)-c\| \leq d_0 + \delta$. Since $\delta = (r - d_0)/2 < (r - d_0)$, we have $d_0 + \delta < d_0 + (r - \delta_0) = r$, so $\|\gamma(t) - c\| < r$. Thus, none of the points on this second half-curve reaches the circle either. This contradicts to our assumption that the curve γ is guaranteed to reach any circle of radius r. The contradiction proves that the second half-curve is also a straight line segment.

4.4°. To complete the proof, we must show that the straight line segments $\gamma([0, r])$ and $\gamma([r, 2r])$ continue each other, i.e., that they form a single straight line segment.

Indeed, suppose that they form an angle which is different from 180°. Then, the arc connecting $\gamma(0)$ and $\gamma(2r)$ is not the diameter, hence its half-length ℓ is $< r$. Let us take as the center c of the circle the midpoint $(\gamma(0) + \gamma(2r))/2$ of this arc. For the points $\gamma(0)$ and $\gamma(2r)$, we have $\|c-\gamma(0)\| = \|c-\gamma(2r)\| = \ell < r$. For the midpoint $\gamma(r)$, the distance $\|\gamma(r) - c\|$ is one of the sides of the right triangle $\gamma(0)c\gamma(r)$ of which the side $\gamma(0)\gamma(r)$ is a hypothenuse of length r; hence $\|\gamma(r) - c\| < r$. By convexity of distance, we thus conclude that for all the points on the curve γ, the distance from c is $< r$ – which contradicts to our assumption that the curve γ is guaranteed to reach any circle of radius r. The contradiction proves that the two half-curves from a single straight line segment. The proposition is proven.

Open questions. Our description of the geometric problem was somewhat idealized. The idea of using the second point on a circle in a close vicinity of the first one works if we assume that we can exactly locate the point where we lose the radio signal, i.e., that we can exactly locate the point on an (unknown) circle.

In practice, however, there is always some measurement inaccuracy. If we can only locate a point of intersection with accuracy ε, and we want to find the center of the plane with accuracy δ, what is then the optimal strategy?

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References

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