USE OF DETERMINISTIC TRAFFIC ASSIGNMENT ALGORITHMS IN STOCHASTIC NETWORKS: ANALYSIS OF EQUIVALENT LINK DISUTILITY FUNCTIONS

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ABSTRACT

This paper derives a general equivalent link disutility function for modeling route choice in a stochastic network (in which link travel time is subjected to day-to-day variation). The derived general equivalent link disutility function does not depend on the resolution of network representation, drivers' route choice behavior or travel time distribution and yet enables the traffic assignment problem in a stochastic network to be solved by the conventional solution algorithm as in a deterministic network. We also show that, for risk averse and risk prone route choice behaviors, which are specific cases of the general formulation, the equivalent link disutility functions follow the same form but with constraints on the values of the coefficients. The general equivalent link disutility function can also be applied to more specific cases, e.g., where one assumes that the link travel times follows the Gamma distribution.

Keywords: traffic assignment, route choice, utility function, stochastic network, user equilibrium

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1. Introduction

In traffic assignment, a road network is usually modeled as a set of nodes connected by unidirectional links. A set of vehicles, also known as users, travelers, or drivers, are to be loaded into the network and travel from their origin nodes to the destination nodes. A modeler is always interested in seeking the traffic flow distribution in a network, i.e., the volume of traffic in the links, such that no user can improve his/her travel time by unilaterally changing his/her route. This state of flow distribution is called the User Equilibrium (UE) condition. In assigning traffic flow to a network, among the assumptions made are the characteristics of link travel times and of drivers' knowledge of the link travel times.

Link travel times can be deterministic or stochastic. A network with deterministic link travel times is called a Deterministic Network (DN). A network with stochastic link travel times is known as a Stochastic Network (SN). In a DN, travel time in a link is a deterministic function of the link's characteristics (such as free-flow travel time and link capacity) and link volume. In a SN, travel time in a link is a random variable. The variation in travel time may be due to the difference in vehicle mix, difference in driver reactions, weather, incident conditions, etc. In SN, travel time in a link is usually modeled as a probability distribution, with mean and variance expressed as functions of the link characteristics and link volume.

Drivers' knowledge of a network's link travel times can be perfect or imperfect. A driver with perfect knowledge knows exactly the deterministic or stochastic properties of travel times in all the links in a network. A driver with imperfect knowledge has perception errors on the link travel times. The perception errors are usually modeled as normal distributions with mean equal to zero.

Based on the assumptions in link travel times and drivers' perception on the link travel times, traffic assignment models may be classified into four types: Deterministic Network-Deterministic User Equilibrium (DN-DUE), Deterministic Network-Stochastic User Equilibrium (DN-SUE), Stochastic Network-Deterministic User Equilibrium (SN-DUE), Stochastic Network-Stochastic User Equilibrium (SN-SUE) (Chen and Recker 2000).

The DN-DUE is the simplest, the easiest to understand, and the most widely accepted traffic assignment model. It assumes that drivers have perfect knowledge of the deterministic link travel times (with a given flow distribution) in the network, and they always select the paths that have the shortest travel times between their origins and destinations. This model was originally formulated by Beckman et al. (1956) and can be solved by the Frank-Wolfe algorithm (Sheffi, 1985). In DN-SUE, the network's link travel times are deterministic (with a given flow distribution), but they may be perceived differently by different drivers. Due to the error in travel time perception, drivers will always select what they perceive as the shortest paths but these may not be the actual shortest paths. The DN-SUE model was originally formulated by Daganzo and Sheffi (1977). A popular solution algorithm for the DN-SUE model is the Method of Successive Averages proposed by Sheffi and Powell (1982). The SN-DUE and SN-SUE models were first studied by Mirchandani and Soroush (1987). The SN-DUE assumes that drivers have perfect knowledge of the degree of variation in link travel times, and they factor this variation in their route choice decisions. In DN, the travel time is uniquely determined by the path; a driver selects the path connecting an origin and a destination with the shortest travel time. In SN, the travel time is not uniquely determined by the path; each driver selects the path with the lowest expected value of the disutility. The SN-DUE model can be solved by the Frank-Wolf algorithm simply by replacing the link travel time function with a suitable link disutility function. Chen and Recker (2000) argued that the SN-DUE model may be suitable for modeling of peak hour traffic because regular commuters have a good knowledge of the mean and variance of travel times in the morning commute. The SN-SUE model adds drivers' perception errors into the link travel time variations.

While DN-DUE and DN-SUE models are used by many transportation modelers, only few papers (Mirchandani and Soroush, 1987; Tatineni, et al.,1997; Chen and Recker, 2000; Chen et al. 2000) have been found on the investigations and applications of SN-DUE and SN-SUE models. A difficulty in implementing the SN-DUE and SN-SUE models is the formulation of a suitable equivalent link disutility function which is logically sound and relatively computationally efficient. This paper focuses on the formulation and property of an equivalent link disutility function that can be used in the SN-DUE models.

2. Motivation

Our motivation for this paper is to derive an equivalent link disutility function DU_i (incorporating average link travel time and travel time variation) that can be used to describe drivers' route choice preference in the SN-DUE model. Note that a route *r* is made up of a series of connected links *i*. We would like to assign, to every link *i*, a value DU_i in such a way that the drivers preference is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. It is desirable that the equivalent link disutility function satisfies the following properties:

(P1) It must be mathematically consistent with the route disutility function, in the sense that it leads to the same routing decision (it may however have a different form than the route disutility function).

(P2) If we sub-divide a link into a series of shorter links, the equivalent disutility of the original link must be equal to the sum of the equivalent disutilities of the shorter links.

(P3) The equivalent link disutility function must be a monotonically increasing and continuously differentiable function of link volume.

Property P1 ensures that the equivalent link disutility function describes the same route choice behavior as the original route disutility function. Properties P2 ensures that drivers' route choice and network flow remain the same irrespective of the resolution of network representation. Property P3 ensures that the equivalent link disutility function is consistent with common sense: the higher the link volume, the less preferable it is to the drivers, and small changes in the link volume lead to small changes in the driver's preference.

The consistency in UE flow pattern irrespective of the resolution in network representation is important in many practical applications. Many transportation planning models divide the geographical area to be analyzed into zones, depending on the land-use patterns. The zones in the geographical border (or buffer zones) are usually larger than the zones in the center business district. Naturally, the modeling details are often sized according to the zone dimension. Zones covering larger areas are likely to have longer links. On the other hand, smaller zones are likely to have shorter links and higher node density. Many traffic assignment algorithms use the geographical and topological information of the nodes and links converted from a Geographical Information Systems (GIS) database. To be geographically correct in representing a curve road segment which has a uniform geometry, intermediate nodes are inserted between the two ends of the segment so that it can be represented by a series of piecewise linear links. If the additive property of the link disutility is not preserved, such division of a link into a series of smaller links may produce different UE flow patterns after traffic assignment. A consistent equivalent link disutility function can be placed instead of the deterministic link travel time function in the existing traffic assignment models (such as TransCAD (Caliper, 2005)) and thus enable us to use these models for SN-DUE applications.

3. Illustrating the Desired Properties on the Example of Deterministic Link Travel Time Function

We first use a commonly used deterministic link travel time function to illustrate the concepts of P1, P2 and P3. By far, the most popular deterministic link travel time function used by transportation modelers is the Bureau of Public Road (BPR) function:

$$t_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] \tag{1}$$

where t_i is the travel time in link *i*, t_i^f is the free-flow travel time in link *i*, v_i is the volume in link *i*, c_i is the capacity of link *i*, and α and β are constants. The t_i^f is computed by dividing

l_i, the length of link *i*, by u_i^f, the free-flow speed of link *i*. Typical values of α and β are 0.15 and 4 respectively. For a route *r* which is made up a series of *L* links $(i = 1,...,L)$, the route travel time is

$$t_r = \sum_{i=1}^{L} t_i \tag{2}$$

In short, the route travel time is the arithmetic sum of the link travel times, with the latter represented by the BPR function.

Since α >0 and β >0, t_i is a monotonically increasing and continuously differentiable function of v_i, i.e., the BPR function satisfies P3. In the deterministic case, a driver selects the shortest route, i.e., the route with the smallest value of the sum t_r of the values t_i; thus the deterministic link travel time function t_i satisfies property P1.

We now illustrate the concept of P2. Suppose that we now divide link *i* into *n* consecutive sub-links $\{i_1, i_2,..., i_n\}$, with lengths $\{l_{i_i}, l_{i_2,...}, l_{i_n}\}$. Then, the volume, capacity, and free-flow speed of the sub-links are same as that of link *i*, i.e., $v_{i_1} = v_{i_2} = ... = v_{i_n} = v_i$, $c_{i_1} = c_{i_2} = ... = c_{i_n} = c_i$, and $u_{i_1}^f = u_{i_2}^f = ... = u_{i_n}^f = u_i^f$. The free-flow travel times of the sub-links are thus $\{t_{i_1}^f, t_{i_2}^f,..., t_{i_3}^f\} = \left\{\frac{l_{i_1}}{u_i^f}, \frac{l_{i_2}}{u_i^f},..., \frac{l_{i_n}}{u_i^f}\right\}$. The travel time in link *i*, computed from the sum of the travel times in its sub-links, is

$$t_{i_1} + t_{i_2} + \ldots + t_{i_n} = t_{i_i}^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right] + \ldots + t_{i_n}^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right] = \left(t_{i_1}^f + \ldots + t_{i_n}^f \right) \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right]$$

$$= \left(\frac{l_{i_1} + \ldots + l_{i_n}}{u_i^f} \right) \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right] = \left(\frac{l_i}{u_i^f} \right) \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right] = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^\beta \right] = t_i \qquad (3)$$

Therefore, if we divide an original link into shorter links and compute the travel times of the shorter links, then the sum of the travel times on the shorter links is the same as the original link travel time. Thus, by using the BPR function, the additive property of the link travel time is preserved, and the BPR function satisfies property P3.

4. A General Equivalent Link Disutility Function for Stochastic Networks

Link travel times in a SN are subjected to random variations. Accordingly, the route travel times in a SN are also subjected to random variations. Because of the uncertainty in travel times, a driver will not know exactly when he/she will reach the destination. Therefore, in addition to the expected (average) travel times, drivers will also consider the magnitudes of variation in link or route travel times when making route choice decisions. Therefore the link disutility function for a SN should include travel time variation as a factor.

Let us first recall that in a SN, t_i, the travel time in link *i*, is a random variable. For this random variable t_i, the average travel time $\bar{t}_i$ may be estimated by the BPR function:

$$\bar{t}_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \right] \tag{4}$$

Note that, according to this formula, when v_i=0, we have $\bar{t}_i = t_i^f$. Moreover, in the absence of traffic flow, i.e., when v_i=0, the link travel time t_i should be equal to t_i^f (with probability=1.0). Other than these restriction on the average and on the free-flow travel time, we are not making any other explicit assumption about the distribution of t_i; in this sense, the conclusions of this section are distribution-free.

It is natural to assume that, DU_i, the equivalent disutility of link *i* should depend on the free-flow travel time t_i^f and the relative average delay $r = \left(\bar{t}_i - t_i^f\right)/t_i^f$, i.e.,

$$DU_i = F\left(t_i^f, r\right) \tag{5}$$

where

$$r = \frac{\bar{t}_i - t_i^f}{t_i^f} = \alpha \left(\frac{v_i}{c_i} \right)^{\beta} \tag{6}$$

for some function $F\left(t_i^f, r\right)$. So, to describe an equivalent link disutility function, we must find the appropriate function $F\left(t_i^f, r\right)$.

One would expect a link which has a longer uncongested travel time to have a higher equivalent disutility; so, $F\left(t_i^f, r\right)$ must be an increasing function of t_i^f. One would also expect that as the link becomes more congested, the equivalent disutility would increase; so, $F\left(t_i^f, r\right)$ must also be an increasing function of *r*. In addition, the function $F\left(t_i^f, r\right)$ must satisfies the following conditions:

(i) In the deterministic case, we want our equivalent link disutility function to reduce to the standard link travel time function. We have already mentioned that when $v_i = 0$, then the travel time is deterministically determined $t_i = \bar{t}_i = t_i^f$, therefore

$$F\left(t_i^f, 0\right) = t_i^f \tag{7}$$

(ii) We would like the equivalent link disutility function to satisfy the property P2: If we sub-divide a link into a series of shorter links, the equivalent disutility of the original link must be equal to the sum of the equivalent disutilities of the shorter links. If we sub-divide a link into two sub-links with free-flow travel times $t_{i_1}^f$ and $t_{i_2}^f$ respectively, then $v_{i_1} = v_{i_2} = v_i$, and $c_{i_1} = c_{i_2} = c_i$; so by Equation (6), the relative average delay *r* for both sub-links is the same as for the original link. Thus the desired property P2 takes the following form

$$F\left(t_{i_i}^f + t_{i_2}^f, r\right) = F\left(t_{i_1}^f, r\right) + F\left(t_{i_2}^f, r\right) \tag{8}$$

Let us describe all the functions $F(t_i^f, r)$ which satisfy these conditions. Let us first analyze the Equation (8). Let us fix a value r and introduce an auxiliary function $G(a) = F(a,r)$. In terms of this new function, the Equation (8) takes the form

$$G(a+b) = G(a) + G(b) \tag{9}$$

We also know that $F(t_i^f, r)$ is an increasing function of t_i^f and therefore, $G(a)$ is an increasing function of a. It is known (Aczel, 2006) that every monotonically increasing function $G(a)$ which satisfies Equation (9) has the form $G(a) = k \cdot a$ for some k>0. For different r, the coefficient k may in general be different: $k = k(r)$. Thus we conclude that

$$DU_i = F(t_i^f, r) = t_i^f k(r) \tag{10}$$

From Equation (7), we know that for r=0 we have $F(t_i^f, r) = t_i^f$. Therefore $k(0)$=1.

For typical values of α and β (see Equation (6)), we have $r << 1$. Thus we can use the Taylor series expansion

$$k(r) = 1 + a_1 r + a_2 r^2 + ... \tag{11}$$

and ignore the higher order terms, i.e., use an expression $k(r) = 1 + a_1 r + a_2 r^2$. Substituting the formula for r (Equation (6)) into this expression, we conclude that

$$DU_i \approx t_i^f \left[1 + a_1 \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \tag{12}$$

Equation (12) can also be expressed as

$$DU_i = t_i^f \left[1 + \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + (a_1 - 1) \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right]$$

$$= \bar{t}_i + t_i^f \left[(a_1 - 1) \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2 \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \tag{13}$$

Hence, we may view DU_i as consisting of two components: the deterministic component $\bar{t}_i$ and stochastic component $t_i^f [...]$. The stochastic component is due to the uncertainty in the link travel time in drivers' route choice process. Equation (13) provides a convenient way of solving the SN-DUE model using the Frank-Wolf algorithm, provided that DU_i is a convex function of v_a (property P3). This property of DU_i holds e.g. when $a_1 \geq 0$ and $a_2 \geq 0$; in the following text we will show that this is the case e.g. for risk averse drivers, drivers typical for the morning rush hours. The coefficients a_1 and a_2 can be estimated from user surveys, which will reflect the user behavior in response to uncertain link travel times.

Our derivation (Equations (8) and (9)) has already ensured that the DU_i has property P2. Using the same logic as in Equation (3), we can also show that Equation (13) satisfies property P2. The only remaining property to check is property P1. In terms of our function DU_i, this property means that the driver preferences should be equivalent to selecting a route with the smallest value of the sum

$$du_r = \sum_{i \in r} DU_i \tag{14}$$

In the following section, we will show that this equivalence indeed holds for the standard description of risk averse and risk prone driver behavior. Therefore, we can treat the SN-DUE model like a DN-DUE model simply by replacing the t_i and t_r in the DN-DUE model by DU_i and du_r respectively. In fact, we only need to replace t_i by DU_i in the solution algorithm!

5. Route Choice Behavior in Stochastic Networks

An important consideration in the selection of the link and route disutility functions is how the drivers react to the travel time uncertainty. This is particularly important if drivers have constraints in the time of arrival (e.g., scheduled events, work starting times) with heavy penalties for late arrivals. Mirchandani and Soroush (1987), Tatineni et al. (1997) and Chen and Recker (2000) describe three types of such behavior: risk averse, risk prone and risk neutral. The term risk here refers to the risk of a late arrival at the destination. A risk averse driver prefers a route with longer expected travel time but smaller variation to a route with faster expected travel time but higher variation. That is, he/she would rather use the route with longer travel time (and depart early) to lower the risk of arriving late. On the contrary, a risk prone driver would select the route with a faster travel time but higher variation. A driver with risk neutral behavior does not consider travel time variation in his/her route choice decision.

For the drivers with risk neutral behavior, the route disutility function DU_r is equal to the route travel time t_r. Therefore the expected route disutility $E[DU_r]$ is equal to the average route travel time $\bar{t}_r$. The route travel time t_r is equal to the sum of the link travel times: $t_r = t_1 + ... + t_L$. So, the average route travel time is equal to the sum of the average link travel times: $\bar{t}_r = \bar{t}_1 + ... + \bar{t}_L$. Thus selecting a route with the smallest $E[DU_r]$ is equivalent to selecting a route with the smallest value of the sum of $\bar{t}_i$. Hence, a risk neutral driver can be described by an equivalent link disutility function $DU_i = \bar{t}_i$.

Let us show that a similar representation by an equivalent link disutility function is also possible for risk averse and risk prone behaviors. The exponential form of route disutility function has been used by Tatineni et al. (1997) and Chen and Recker (2000) to represent the risk averse and risk prone behaviors in a SN:

$$DU_r = \begin{cases} b_1[exp(\omega t_r) - 1] & \text{for risk averse drivers} \\ b_2[1 - exp(-\varphi t_r)] & \text{for risk prone drivers} \end{cases} \tag{15}$$

where t_r is the route travel time, and $b_1, b_2, \omega, \varphi$ are positive constants. Given a choice of routes $r \in R$ connecting an origin-destination pair, a driver will select the route r' which has the smallest expected disutility $E[DU_r]$

$$E[DU_{r'}] = \min_{r \in R} \{E[DU_r]\} \tag{16}$$

In this section we will show that both types of behavior can be described by an equivalent link disutility function DU_i -- for which selecting a route with the smallest value of $E[DU_r]$ is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. Following Tatineni et al. (1997) and Chen and Recker (2000) we consider each type of driver separately, and assume that all drivers in a network follow the same route choice behavior.

5.1 Risk Averse Behavior

According to the SN-DUE model, a driver selects a route with the smallest possible value of the expected disutility $E[DU_r]$. If we "rescale" the disutility function, i.e., consider an auxiliary function $A_r = g(E[DU_r])$ for some monotonically increasing function $g(x)$, then minimizing $E[DU_r]$ is equivalent to minimizing A_r. We will use this property to simplify the decision making in the SN-DUE model.

In particular, for risk averse drivers, we have $E[DU_r] = b_1(A_r - 1)$, where

$$A_r = E[exp(\omega t_r)] \quad (17)$$

Therefore, $A_r = g(E[DU_r])$ for $g(x) = (x/b_1) + 1$. Since b_1 >0, the function $g(x)$ is monotonically increasing and therefore, minimizing $E[DU_r]$ is equivalent to minimizing A_r.

The route travel time t_r is composed of link travel times t_i: $t_r = \sum_{i \in r} t_i$. In a SN, link travel times (t_i) are considered to be independent random variables. Thus, the auxiliary expression $A_r = E[exp(\omega t_r)]$ can be expressed as

$$A_r = E[exp(\omega t_r)] = E[exp(\omega(t_1 + t_2 + ... + t_L))] = E[exp(\omega t_1) exp(\omega t_2) ... exp(\omega t_L)]$$

$$= E[exp(\omega t_1)] \cdot E[exp(\omega t_2)] \cdot ... \cdot E[exp(\omega t_L)] \quad (18)$$

Drivers will select the route that minimizes $E[DU_r]$; this is equivalent to minimizing A_r. Since $ln(x)$ is a monotonically increasing function, this choice is, in its turn, equivalent to selecting the route that minimizes $ln(A_r)$. Here

$$ln(A_r) = ln\{E[exp(\omega t_1)]\} + ln\{E[exp(\omega t_2)]\} + ... + ln\{E[exp(\omega t_L)]\} \quad (19)$$

Let us perform one more rescaling, to make this expression similar to that of the DN. A DN can be viewed as a particular case of a SN, in which all travel times t_i and t_r are deterministic. In a DN, the above expression reduces to

$$ln(A_r) = ln[exp(\omega t_1)] + ln[exp(\omega t_2)] + ... + ln[exp(\omega t_L)] = \omega(t_1 + t_2 + ... + t_L) = \omega t_r \qquad (20)$$

In a DN, we select a route with the smallest route travel time t_r. For convenience, let us rescale the objective function $ln(A_r)$ one more time so that for DN, the rescaled objective function will coincide with t_r. Specifically, we consider $du_r = \frac{1}{\omega} ln(A_r)$ instead of $ln(A_r)$. Since $g(x) = \frac{x}{\omega}$ is a monotonically increasing function, selecting a route based on du_r is equivalent to selecting a route based on $ln(A_r)$, and thus equivalent to selecting a route based on $E[DU_r]$. From Equation (19), we conclude that the new objective function du_r can be expressed as $du_r = DU_1 + ... + DU_L$, where $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$. Thus, the drivers preference in SN-DUE is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. So we get the desired equivalence with the equivalent link disutility function $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$. Therefore, selecting a route in a SN is very similar to selecting a route in a DN, but with link disutility $DU_i = \frac{1}{\omega} ln\{E[exp(\omega t_i)]\}$ instead of link travel time.

Let us reformulate this expression for DU_i in terms of mean and variance of t_i. In a SN the actual travel time t_i in link *i* can be expressed as the sum of the mean travel time $\bar{t}_i$ and the deviation from its mean:

$$t_i = \bar{t}_i + (t_i - \bar{t}_i) \qquad (21)$$

It follows that

$$exp(\omega t_i) = exp(\omega \bar{t}_i) exp(\omega(t_i - \bar{t}_i)) \tag{22}$$

Hence

$$E[exp(\omega t_i)] = exp(\omega \bar{t}_i) E[exp(\omega(t_i - \bar{t}_i))] \tag{23}$$

Usually $\omega(t_i - \bar{t}_i)$ is small, so we can expand the exponential function into the Taylor series and only keep the first three terms in this expansion

$$exp(\omega(t_i - \bar{t}_i)) = 1 + \omega(t_i - \bar{t}_i) + \frac{\omega^2 (t_i - \bar{t}_i)^2}{2} + \ldots \tag{24}$$

Therefore

$$E[exp(\omega(t_i - \bar{t}_i))] \approx 1 + \omega E[t_i - \bar{t}_i] + \frac{\omega^2}{2} E\left[(t_i - \bar{t}_i)^2\right] \tag{25}$$

By definition, $E[t_i - \bar{t}_i]$ =0 and $E\left[(t_i - \bar{t}_i)^2\right] = \sigma_{t_i}^2$ which is the variance of t_i . Substituting Equation (25) into Equation (23), we obtain

$$E[exp(\omega t_i)] = exp(\omega\bar{t}_i)\left[1+\frac{\omega^2}{2}\sigma_{t_i}^2\right] \tag{26}$$

The link disutility function thus becomes

$$DU_i = \frac{1}{\omega}ln\{E[exp(\omega t_i)]\} = \frac{1}{\omega}ln\left\{exp(\omega\bar{t}_i)\left[1+\frac{\omega^2}{2}\sigma_{t_i}^2\right]\right\}$$

$$= \bar{t}_i + \frac{1}{\omega}ln\left[1+\frac{\omega^2}{2}\sigma_{t_i}^2\right] \tag{27}$$

Using the Taylor series expansion of $ln(1+z) = z + ...$ we obtain

$$DU_i \approx \bar{t}_i + \frac{\omega}{2}\sigma_{t_i}^2 \tag{28}$$

We have shown that, if the all drivers in a network follow the same risk averse behavior, solving for DUE in a SN is similar to solving for DUE in a DN, except that we replace t_i in a DN with DU_i in a SN. Note that the first term $\bar{t}_i$ in DU_i is the same as Equation (4), the BPR function. Thus, it can be said that, in a SN with risk averse behavior, the additional term in the route choice decision for drivers is the link travel time variance, scaled by a factor ω/2. The magnitude of ω reflects the sensitivity of the drivers in avoiding the risk. Risk averse drivers will avoid links that have high $\sigma_{t_i}^2$. Note that, if σ_{t_i} =0, the SN-DUE model is reduced to a DN-DUE model.

So far, we have shown (i) that Equation (28) satisfies property P3 and (ii) that the link disutility function in the form of Equation (28) is mathematically consistent with the exponential route disutility function in Equation (15) (property P1). What is the functional form of σ_{t_i} so that Equation (28) satisfies properties P2? We have already shown that the property P2 leads to Equation (13). By comparing Equations (13) and (28), one can conclude that

$$\sigma_{t_i}^2 = t_i^f \frac{2}{\omega}\left[(a_1 - 1)\alpha\left(\frac{v_i}{c_i}\right)^{\beta} + a_2\alpha^2\left(\frac{v_i}{c_i}\right)^{2\beta}\right] \tag{29}$$

Therefore, if we assume that the equivalent link disutility function satisfies P2 then the variance $\sigma_{t_i}^2$ must have the form of Equation (29). The variance is always non-negative, $\sigma_{t_i}^2 \geq 0$, so the coefficients a_1 and a_2 must be such that

$$a_1 + a_2\alpha\left(\frac{v_i}{c_i}\right)^{\beta} \geq 1 \tag{30}$$

Note that the final expression for the equivalent link disutility function DU_i in Equation (12) contains only two new parameters a_1 and a_2. These parameters take into account both the dependence of the variance $\sigma_{t_i}^2$ and the parameter ω that describes the drivers' behavior.

5.2 Risk Prone Behavior

According to the SN-DUE model, a driver selects a route with the smallest possible value of the expected disutility $E[DU_r]$

$$E[DU_{r'}] = \min_{r \in R}\{E[DU_r]\} \tag{31}$$

For risk prone drivers, $E[DU_r] = b_2(1 - B_r)$ where

$$B_r = E[exp(-\varphi t_r)] \tag{32}$$

Thus, minimizing $E[DU_r]$ is equivalent to maximizing B_r. Since the link travel times t_i are independent random variables, we conclude that for a route consisting of L links, we have

$$B_r = E[exp(-\varphi t_r)] = E[exp(-\varphi t_1)] \cdot E[exp(-\varphi t_2)] \cdot \ldots \cdot E[exp(-\varphi t_L)] \tag{33}$$

Selecting a route according to Equation (31) is equivalent to selecting a route that maximizes B_r. This choice, in its turn, is equivalent to selecting the route that minimizes $du_r = -\frac{1}{\varphi} ln(B_r)$. Here

$$du_r = -\frac{1}{\varphi} ln\{E[exp(-\varphi t_1)]\} - \frac{1}{\varphi} ln\{E[exp(-\varphi t_2)]\} - \ldots - \frac{1}{\phi} ln\{E[exp(-\varphi t_L)]\} \tag{34}$$

Thus, for risk prone behavior, the drivers preference in SN-DUE is equivalent to selecting a route with the smallest value of the sum $du_r = \sum_{i \in r} DU_i$. So we get the desired equivalence with the equivalent link disutility function $DU_i = -\frac{1}{\varphi} ln\{E[exp(-\varphi t_i)]\}$. Therefore, selecting a route in a SN is very similar to selecting a route in a DN, but with link disutility $DU_i = -\frac{1}{\varphi} ln\{E[exp(-\varphi t_i)]\}$ instead of link travel time.

Let us reformulate this expression for DU_i in terms of mean and variance of t_i. By following the same procedure as in the risk averse case, we can show that

$$-\frac{1}{\varphi} ln\{E[exp(-\varphi t_i)]\} = -\frac{1}{\varphi} ln\left\{exp(-\varphi \bar{t}_i)\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right]\right\} = \bar{t}_i - \frac{1}{\varphi} ln\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right] \quad (35)$$

Therefore, we can write

$$DU_i = \bar{t}_i - \frac{1}{\varphi} ln\left[1+\frac{\varphi^2}{2}\sigma_{t_i}^2\right] \quad (36)$$

Using the Taylor series expansion for $ln(1+x)$, we obtain

$$DU_i \approx \bar{t}_i - \frac{\varphi}{2}\sigma_{t_i}^2 \quad (37)$$

Equation (37) may be interpreted as follows. A risk prone driver will consider the average link travel times $(\bar{t}_i)$ as well as the variance of link travel times $\left(\sigma_{t_i}^2\right)$ in his/her route choice decision. If there are choices of two links with the same average travel time, a risk prone driver prefers the link with the higher variance. The higher the variance, the more favorable the link is to the risk prone driver. Therefore, the link disutility function has the link variance term, weighted by $(-\varphi/2)$. To ensure that Equation (37) satisfies P2 and P3, we compare Equations (37) and (13) and deduce that

$$\sigma_{t_i}^2 = -t_i^f \frac{2}{\varphi}\left[(a_1 - 1)\alpha\left(\frac{v_i}{c_i}\right)^{\beta} + a_2\alpha^2\left(\frac{v_i}{c_i}\right)^{2\beta}\right] \tag{38}$$

To ensure that $\sigma_{t_i}^2 \geq 0$, we must have

$$a_1 + a_2\alpha\left(\frac{v_i}{c_i}\right)^{\beta} \leq 1 \tag{39}$$

Note that, even if the combination of a_1 and a_2 values satisfies Equation (39), it does not guarantee that Equation (37) is monotonically increasing function of v_a. To satisfy P3, the rate of increase in $(\varphi/2)\sigma_{t_i}^2$ term Equation (37) with respect to v_a must be relatively small compared to the rate of increase of $\bar{t}_i$ with respect to v_a.

When the expression for DU_i is convex, we can use the Frank-Wolf algorithm to solve the corresponding traffic assignment problem. For strongly risk prone behavior with large φ, the function DU_i is no longer convex. However, in this paper, following Tatineni et al. (1997) and Chen and Recker (2000), we assume that all drivers in a network follow the same route choice behavior. Under this assumption, the extremely risk prone behavior is highly unlikely and thus the actual expression for DU_i will be convex.

6. Case of Empirical Link Travel Time Distribution in Stochastic Networks

Our derivation of the expression for the equivalent link disutility function (Equation (13)) applies to an arbitrary probability distribution of link travel time t_i. In this section, we will show that this derivation holds for the empirical link travel time distribution.

This empirical distribution was first presented in Tatineni (1996); according to this research t_i follows the Gamma distribution with a lower bound (equal to t_i^f), scale and shape parameters. Based on the assumption of exponential form of DU_r, Tatineni combined the Gamma distribution of t_i with a normal distribution of perception error, and derived a general expression of DU_i for SN-SUE models. In the SN-DUE case, when there is no perception error, the expression for risk averse drivers takes the following form

$$DU_i = \bar{t}_i + c\sigma_{t_i}^2 \left(\frac{1}{2} + \frac{1}{3} c \frac{\sigma_{t_i}^2}{\bar{t}_i - t_i^f} \right) \tag{40}$$

where c is a constant. Let us show that, if the expression for $\sigma_{t_i}^2$ satisfies reasonable properties then Equation (40) is consistent with Equation (13).

It is natural to assume that the variance $\sigma_{t_i}^2$ of the travel time on link i should depend on the free-flow travel time t_i^f and the relative average delay $r = \left(\bar{t}_i - t_i^f\right)/t_i^f$, i.e.,

$$\sigma_{t_i}^2 = f\left(t_i^f, r\right) \tag{41}$$

where

$$r = \frac{\bar{t}_i - t_i^f}{t_i^f} = \alpha\left(\frac{v_i}{c_i}\right)^{\beta} \tag{42}$$

for some function $f\left(t_i^f, r\right)$. So, to describe an expression for the variance, we must find the appropriate function $f\left(t_i^f, r\right)$. As we have mentioned, in SN, it is usually assumed that link travel times are independent random variables. If we sub-divide a link into two sub-links with free-flow travel times $t_{i_1}^f$ and $t_{i_2}^f$ respectively, then $v_{i_1} = v_{i_2} = v_i$, $c_{i_1} = c_{i_2} = c_i$, and the relative average delay r for both sub-links is the same as for the original link. It is known that the variance of the sum $t_i^f = t_{i_1}^f + t_{i_2}^f$ of two independent random variables is equal to the sum of the corresponding variances $\sigma_{t_i}^2 = \sigma_{t_{i_1}}^2 + \sigma_{t_{i_2}}^2$. Thus

$$f\left(t_{i_i}^f + t_{i_2}^f, r\right) = f\left(t_{i_1}^f, r\right) + f\left(t_{i_2}^f, r\right) \tag{43}$$

Similarly to section 4, we thus conclude that

$$\sigma_{t_i}^2 = t_i^f f\left(r\right) \tag{44}$$

for some function $g(r)$. Since $r << 1$, by expanding $g(r)$ into a Taylor series, and ignoring the terms with r^3 and higher order, we get

$$\sigma_{t_i}^2 = t_i^f \left[a_0' + a_1' r + a_2' r^2 + \ldots\right] \approx t_i^f \left[a_0' + a_1' r + a_2' r^2\right] \tag{45}$$

In the absence of traffic flow, when $\bar{t}_i = t_i^f$ and r=0, we have $\sigma_{t_i}^2$=0; therefore we get a_0'=0. Thus

$$\sigma_{t_i}^2 = t_a^f \left[a_1' r + a_2' r^2\right] \tag{46}$$

Substituting this expression into Equation (40), and ignoring the terms of r^3 and higher order, we get

$$DU_i \approx \bar{t}_i + t_i^f \left[\frac{1}{2} a_1' c r + \frac{1}{2} a_2' c r^2 + \frac{1}{3}\left(a_1'\right)^2 c^2 r + \frac{2}{3} a_1' a_2' r^2\right] \tag{47}$$

This expression can be rewritten as

$$DU_i = \bar{t}_i + t_i^f \left[a_1'' r + a_2'' r^2 \right]$$

$$= \bar{t}_i + t_i^f \left[a_1'' \alpha \left(\frac{v_i}{c_i} \right)^{\beta} + a_2'' \alpha^2 \left(\frac{v_i}{c_i} \right)^{2\beta} \right] \qquad (48)$$

We can see that Equation (48) is indeed equivalent to Equation (13) (if one uses the appropriate matching coefficients). So, for risk averse drivers, our derivation of the expression for the equivalent link disutility function (Equation (13)) indeed applies to the empirical (Gamma) link travel time distribution. For risk prone drivers, similar derivation leads to the same conclusion.

7. Summary

In this paper, we have derived a general equivalent link disutility function for route choice in a SN such that the traffic assignment problem in SN-DUE can be solved by the standard DUE algorithm. The derived equivalent link disutility function has two components: a deterministic component that accounts for the average link travel time (based on the well-known BPR function), and a stochastic component which is proportional to the variance of link travel time. The general expression for the equivalent link disutility function depend neither on any assumption on the probability distribution of the link travel time, nor on risk taking behavior of the drivers. It is also independent of the resolution of the network representation.

We also show that, under the standard assumption that the route disutility function is exponential for both risk averse and risk prone behaviors, there exist equivalent link disutility functions which satisfy the abovementioned properties. For risk averse drivers, the stochastic component in the link disutility function is linearly proportional to the link travel time variance. For risk prone drivers, the stochastic component in the link disutility function is linearly proportional to the negative value of the link travel time variance. For these two types of route choice behaviors, we have specified the constraints of the coefficients in the stochastic component of the link disutility function under which the traffic assignment problem can be solved by the DUE algorithm. We present arguments that these constraints are satisfied in real life.

We have further shown that, in the important particular case, when the route disutility function is exponential and link travel time follows a Gamma distribution, the corresponding equivalent link disutility function also has the same general form.

Our work in this paper provides the justification for the use of a general equivalent link disutility function (which can be seen as an extension of the BPR function) so that we can solve the SN-DUE problem using the same approach as in the DN-DUE model.

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