Exponential Disutility Functions in Transportation Problems: A New Theoretical Justification ⋆

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Abstract

In modeling drivers' route choice in stochastic networks, several researchers have successfully used exponential disutility functions. A usual justification for these functions is that they are consistent with common sense and that they lead to simpler computations than several other alternative disutility functions. In principle, such a justification leaves open a possibility that there is some other (yet un-tried) disutility function which is also consistent with common sense and for which the computations are even simpler than for the exponential function. In this paper, we prove that exponential disutility functions are the only ones that are consistent with the (appropriately formalized) common sense and the only ones for which computations can be simplified.

Key words: utility functions, route selection, stochastic uncertainty

1 Formulation of the Problem

Transportation problem: traditional deterministic approach. As population grows in an area, the existing networks become more and more congested, so an expansion becomes necessary. Usually several alternative plans are proposed for such an expansion. To select a plan, we need to know to

⋆ This work was supported in part by NSF grants EAR-0225670 and DMS-0532645 and by Texas Department of Transportation grant No. 0-5453. The authors are thankful to Yi-Chang Chiu from the University of Arizona for helpful discussions.
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what extent the implementation of each plan will help in easing the congestion problem.

At present, such estimates are usually performed within an (approximate) deterministic traffic assignment model, in which we assume that the travel time t_i along each road link is uniquely determined by the flow on this link and by the capacity of this link; see e.g., [6]. Thus, once we know the flow and capacity of each link, then for each path, we can find the travel time t along this path as the sum of the travel times over all its links: $t = \sum t_i$. We then assume that each driver who needs to go from point A to point B – and who can choose several possible paths – selects the fastest of these paths (i.e., the path with smallest overall travel time t).

More realistic stochastic approach, and the need to use utility or disutility functions. In real life, travel times are non-deterministic (*stochastic*): on each link, for the same capacity and flow, we may have somewhat different travel times [6].

In other words, for each link, the travel time t_i is no longer a uniquely determined real number, it is a *random variable* whose characteristics may depend on the capacity and flow along this link. As a result, the overall travel time t is also a random variable.

If we take this uncertainty into account, then it is no longer easy to predict which path will be selected: if we have two alternative paths, then it often happens that with some probability, the time along the first path is smaller, but with some other probability, the first path may turn out to be longer. How can we describe decision making under such uncertainty?

In decision making theory, it is proven that under certain reasonable assumption, a person's preferences are defined by his or her *utility function* $U(x)$ which assigns to each possible outcome x a real number $U(x)$ called *utility*; see, e.g., [3,5]. In many real-life situations, a person's choice s does not determine the outcome uniquely, we may have different outcomes $x_1, \ldots, x_n$ with probabilities, correspondingly, $p_1, \ldots, p_n$. For example, when a driver selects a path s, the travel time is often not uniquely determined: we may have different travel times $x_1, \ldots, x_n$ with corresponding probabilities $p_1, \ldots, p_n$. For such a choice, we can describe the utility $U(s)$ associated with this choice as the expected value of the utility of outcomes: $U(s) = E[U(x)] = p_1 \cdot U(x_1) + \ldots + p_n \cdot U(x_n)$. Among several possible choices, a user selects the one for which the utility is the largest: a possible choice s is preferred to a possible choice s' (denoted $s > s'$) if and only if $U(s) > U(s')$.

For the applications presented in this paper, it is important to emphasize that the utility function is not uniquely determined by the preference relation.

Namely, for every two real numbers $a > 0$ and b, if we replace the original utility function $U(x)$ with the new one $V(x) \stackrel{\text{def}}{=} a \cdot U(x) + b$, then for each choice s, we will have

$$V(s) = E[a \cdot U(x) + b] = a \cdot E[U(x)] + b = a \cdot U(s) + b$$

and thus, $V(s) > V(s')$ if and only if $U(s) > U(s')$.

In transportation, the main concern is travel time t, so the utility depends on time: $U = U(t)$. Of course, all else being equal, the longer it takes to travel, the less preferable the choice of a path; so, the utility function $U(t)$ must be strictly increasing: if $t < t'$, then $U(t) > U(t')$.

In general, decision making is formulated in terms of *maximizing* a utility function $U(x)$. In traditional (deterministic) transportation problems, however, decision making is formulated in terms of *minimization*: we select a route with the smallest possible travel time. Thus, when people apply decision making theory in transportation problems, they reformulate these problems in terms of a *disutility* function $u(x) \stackrel{\text{def}}{=} -U(x)$. Clearly, for every choice s, we have

$$u(s) \stackrel{\text{def}}{=} E[u(x)] = E[-U(x)] = -E[U(x)] = -U(s).$$

So, selecting the route with the *largest* value of expected utility $U(s)$ is equivalent to selecting the route with the *smallest* value of expected disutility $u(s)$. In line with this usage, in this paper, we will talk about disutility functions.

Since a disutility function $U(t)$ is strictly decreasing, the corresponding utility function $u(t) = -U(t)$ must be strictly increasing: if $t < t'$ then $u(t) < u(t')$.

Disutility functions traditionallly used in transportation: description and reasons. In transportation, traditionally, three types of disutility functions are used; see, e.g., [4,7,8].

First, we can use *linear* disutility functions $u(t) = a \cdot t + b$, with $a > 0$. As we have mentioned, multiplication by a constant $a > 0$ and addition of a constant b does not change the preferences, so we can safely assume that the utility function simply conincides with the travel time $u(t) = t$.

Second, we can use *risk-prone exponential* disutility functions

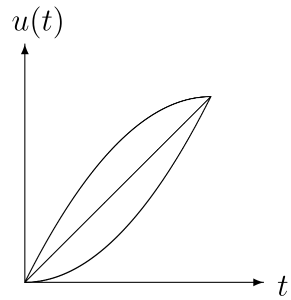
$$u(t) = -a \cdot \exp(-c \cdot t) + b$$

for some $a > 0$ and $c > 0$. This is equivalent to using $u(t) = -\exp(-c \cdot t)$.

Third, we can us*e risk-averse exponential* disutility functions

$$u(t) = a \cdot \exp(c \cdot t) + b$$

for some $a > 0$ and $c > 0$. This is equivalent to using $u(t) = \exp(c \cdot t)$.



Several other possible disutility functions have been proposed, e.g., quadratic functions $u(t) = t + c \cdot t^2$; see, e.g., [4].

In practice, mostly linear and exponential functions are used. Actually, a linear function can be viewed as a limit of exponential functions:

$$t = \lim_{\alpha \to 0} \frac{1}{\alpha} \cdot (\exp(\alpha \cdot t) - 1),$$

so we can say that mostly exponential functions are used.

There are two reasons for using exponential disutility functions. First, these functions are in accordance with common sense [4,8]. Indeed:

- functions $-\exp(-c \cdot t)$ indeed lead to risk-prone behavior, i.e., crudely speaking, a behavior in which a person, when choosing between two paths, one with a deterministic time t_1 and another with a stochastic time t_2, prefers the second path if there is a large enough probability that $t_2 < t_1$ – even when the average time of the second path may be larger $\bar{t}_2 > t_1$;
- functions $\exp(c \cdot t)$ indeed lead to risk-averse behavior, i.e., crudely speaking, a behavior in which a person, when choosing between two paths, one with a deterministic time t_1 and another with a stochastic time t_2, prefers the first path if there is a reasonable probability that $t_2 > t_1$ – even when the average time of the second path may be smaller: $\bar{t}_2 > t_1$.

This accordance, however, does not limit us to only exponential functions: e.g., quadratic functions are also in reasonably good accordance with common sense.

However, there is another justification for using linear and exponential disutility functions, a justification that excludes quadratic functions and several other alternatives. This justification is that linear and exponential disutility functions simplify computations; see, e.g., [4,7].

Indeed, in the deterministic case, the selection of a route is based on the travel time $t = t_1 + \ldots + t_n$ along this route. So, to describe the "quality" of a route, it is sufficient to know the values t_i which characterize the "quality" of all of its links.

In the more realistic (stochastic) case, as we have mentioned, the selection of a route is based on the expected value of the utility along this route, i.e., on the number $E[u(t)] = E[u(t_1 + t_2 + \ldots + t_n)]$.

For a linear disutility function $u(t) = t$, the disutility of a route is equal to

$$E[t_1 + \ldots + t_n] = E[t_1] + \ldots + E[t_n].$$

Thus, to find the disutility $E[t]$ of the route, it is sufficient to know the disutilities $E[t_i]$ of all the links. Same argument holds for $u(t) = a \cdot t + b$: we have

$$u(t) = a\cdot t+b = a\cdot(t_1+\ldots+t_n)+b = (a\cdot t_1+b)+\ldots+(a\cdot t_n+b)-(n-1)\cdot b =$$

$$u(t_1) + \ldots + u(t_n) - (n-1)\cdot b,$$

and therefore,

$$E[u(t)] = E[u(t_1)] + \ldots + E[u(t_n)] - (n-1)\cdot b.$$

Similarly, for an exponential disutility function $u(t) = \varepsilon \cdot \exp(\alpha \cdot t)$, where $\varepsilon = \pm 1$, the disutility of a route is equal to

$$E[\varepsilon \cdot \exp(\alpha \cdot t)] = E[\varepsilon \cdot \exp(\alpha \cdot (t_1 + \ldots + t_n))].$$

For the exponential function,

$$\exp(\alpha \cdot (t_1 + \ldots + t_n) = \exp(\alpha \cdot t_1) \cdot \ldots \cdot \exp(\alpha \cdot t_n).$$

The difference between the actual travel time and the average travel time is usually caused by factors local for this link; thus, it is reasonable to assume that the random variables t_i corresponding to different links are independent. Thus, the variables $\exp(\alpha \cdot t_i)$ are also independent, hence

$$E[\exp(\alpha \cdot t)] = E[\exp(\alpha \cdot (t_1 + \ldots + t_n))] = E[\exp(\alpha \cdot t_1)] \cdot \ldots \cdot E[\exp(\alpha \cdot t_n)].$$

So, for the exponential disutility function of the type $u(t) = \exp(\alpha \cdot t)$, to find the "quality"

$$E[u(t)] = E[\exp(\alpha \cdot t)]$$

of a route, it is sufficient to know the qualities $E[u(t_i)] = E[\exp(\alpha \cdot t_i)]$ of different links along this route.

For the exponential disutility function $u(t) = -\exp(\alpha \cdot t)$, we similarly have

$$u(t) = -\exp(\alpha \cdot t) = -\exp(\alpha \cdot t_1) \cdot \ldots \cdot \exp(\alpha \cdot t_n) =$$

$$(-1)^{n+1} \cdot (-\exp(\alpha \cdot t_1)) \cdot \ldots \cdot (-\exp(\alpha \cdot t_n)) = (-1)^{n+1} \cdot u(t_1) \cdot \ldots \cdot u(t_n),$$

and thus,

$$E[u(t)] = (-1)^{n+1} \cdot E[u(t_1)] \cdot \ldots \cdot E[u(t_n)].$$

So, for the exponential disutility function of the type $u(t) = -\exp(\alpha \cdot t)$, to find the "quality"

$$E[u(t)]$$

of a route, it is also sufficient to know the qualities $E[u(t_i)]$ of different links along this route.

This reduction is no longer true, e.g., for quadratic disutility functions. For example, for a route consisting of two links $t = t_1 + t_2$ and for a disutility function $u(t) = t^2$, we have $u(t) = u(t_1 + t_2) = (t_1 + t_2)^2 = t_1^2 + t_2^2 + 2 \cdot t_1 \cdot t_2$. Thus,

$$E[u(t)] = E[t_1^2] + E[t_2^2] + 2 \cdot E[t_1] \cdot E[t_2] = E[u(t_1)] + E[u(t_2)] + 2 \cdot E[t_1] \cdot E[t_2].$$

In this case, in addition to knowing the quality $E[u(t_i)]$ of each link, we also need to know another characteristic of each link – its average time $E[t_i]$. Thus, for quadratic disutility functions, it is no longer sufficient to compute one characteristic of each link, we need at least two – and this makes corresponding computations more complex.

Remaining open problem. The above justification of linear and exponential disutility functions is that they are consistent with common sense and that they lead to simpler computations than several other alternative disutility functions.

A natural question arises: are these disutility functions really the only ones with these properties – or there exists some other (yet un-tried) disutility function which is also consistent with common sense and for which the computations are even simpler than for the exponential function.

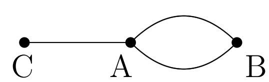
What we do in this paper. In this paper, we prove that exponential disutility functions are the only ones that are consistent with the (appropriately formalized) common sense and the only ones for which computations can be simplified.

2 Exponential Disutility Functions Are the Only Functions Which Are Consistent with Common Sense: A Theorem

A common sense assumption about the driver's preferences. Let us present an example that, in our opinion, captures some common sense meaning of decision making in transportation problems.

Let us assume that we have several routes going from point A to point B, and a driver selected one of these routes as the best for him/her. For example, A may be a place at the entrance to the driver's department, and B is a similar department at another university located in a nearby town.

Let us now imagine a similar situation, in which the driver is also interested in reaching the point B, but this time, the driver starts at some prior point C. At this point C, there is only one possible way, and it leads to the point A; after A, we still have several possible routes. We can also assume that the time t_0 that it takes to get from C to A is deterministic. For example, C may be a place in the parking garage from where there is only one exit.

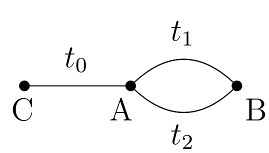


It is reasonable to assume that if the road conditions did not change, then, after getting to the point A, the driver will select the exact same route as last time, when this driver started at A.

Comment. Similarly, if two routes from A to B were equally preferable to the driver, then both routes should be equally preferable after we add a deterministic link from C to A to both routes.

In the deterministic case, this assumption is automatically satisfied. In the deterministic case, the travel time along each route is deterministic, and the driver selects a route with the shortest travel time.

Let us assume when going from A to B, the drive prefers the first route because its travel time t_1 is smaller than the travel time t_2 of the second route: $t_1 < t_2$. In this case, next time, when the travel starts from the point C, we have time $t_1 + t_0$ along the first route and $t_2 + t_0$ along the second route. Since we had $t_1 < t_2$, we thus have $t_1 + t_0 < t_2 + t_0$ – and therefore, the driver will still select the first route.



In the stochastic case, this assumption is not necessarily automatically satisfied. In the stochastic case, when going from A to B, the driver selects the first route if $E[u(t_1)] < E[u(t_2)]$, where $u(t)$ is the corresponding disutility function.

Next time, when the driver goes from C to B, the choice between the two routes depends on comparing different expected values: $E[u(t_1+t_0)]$ and $E[u(t_2+t_0)]$, where t_0 is the (deterministic) time of traveling from C to A. In principle, it may be possible that $E[u(t_1)] < E[u(t_2)]$ but

$$E[u(t_1+t_0)] > E[u(t_2+t_0)].$$

Let us describe a simple numerical example when this counter-intuitive phenomenon happens. In this example, we will use a simple non-linear disutility function: namely, the quadratic function $u(t) = t^2$. Let us assume that the first route from A to B is deterministic, with $t_1 = 7$, and the second route from A to B is highly stochastic: with equal probability 0.5, we may have $t_2 = 1$ and $t_2 = 10$. In this case, $E[u(t_1)] = t_1^2 = 49$ and

$$E[u(t_2)] = E[t_2^2] = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 10^2 = 0.5 + 50 = 50.5.$$

Here, $E[u(t_1)] < E[u(t_2)]$, so the driver will prefer the first route.

However, if we add the same constant time $t_0 = 1$ for going from C to A to both routes, then in the first route, we will have $t_1 + t_0 = 7 + 1 = 8$, while in the second route, we will have $t_2 + t_0 = 1 + 1 = 2$ and $t_2 + t_0 = 10 + 1 = 11$ with equal probability 0.5. In this case,

$$E[u(t_1+t_0)] = (t_1+t_0)^2 = 8^2 = 64,$$

while

$$E[u(t_2+t_0)] = \frac{1}{2} \cdot 2^2 + \frac{1}{2} \cdot 11^2 = 2 + 60.5 = 62.5.$$

We see that here, $E[u(t_2+t_0)] < E[u(t_1+t_0)]$, i.e., the drive will select the second route instead of the first one.

This counter-intuitive phenomenon does not happen for linear or exponential disutility functions. Indeed, for a linear disutility function $u(t) = t$, we have $u(t_1+t_0) = t_1+t_0 = u(t_1)+t_0$; therefore, $E[u(t_1+t_0)] = E[u(t_1)] + t_0$ and similarly, $E[u(t_2+t_0)] = E[u(t_2)] + t_0$. Thus, if the driver selected the first route, i.e., if $E[u(t_1)] < E[u(t_2)]$, then by adding t_0 to both sides of this inequality, we can conclude that $E[u(t_1+t_0)] < E[u(t_2+t_0)]$ – i.e., that, in accordance with common sense, the same route will be selected if we start at the point C.

For the exponential disutility function $u(t) = \exp(\alpha \cdot t)$, we have $u(t_1+t_0) = \exp(\alpha \cdot (t_1+t_0)) = \exp(\alpha \cdot t_1) \cdot \exp(\alpha \cdot t_0)$ and therefore, $u(t_1+t_0) = u(t_1) \cdot \exp(\alpha \cdot t_0)$. Similarly, for the exponential disutility function $u(t) = -\exp(\alpha \cdot t)$, we have $u(t_1 + t_0) = -\exp(\alpha \cdot (t_1 + t_0)) = -\exp(\alpha \cdot t_1) \cdot \exp(\alpha \cdot t_0)$ and thus, $u(t_1 + t_0) = u(t_1) \cdot \exp(\alpha \cdot t_0)$;

For both types of exponential disutility function, we have $E[u(t_1 + t_0)] = \exp(\alpha \cdot t_0) \cdot E[u(t_1)]$ and similarly, $E[u(t_2 + t_0)] = \exp(\alpha \cdot t_0) \cdot E[u(t_2)]$. Thus, if the driver selected the first route, i.e., if $E[u(t_1)] < E[u(t_2)]$, then by multiplying both sides of this inequality by the same constant $\exp(\alpha \cdot t_0)$, we can conclude that $E[u(t_1 + t_0)] < E[u(t_2 + t_0)]$ – i.e., that, in accordance with common sense, the same route will be selected if we start at the point C.

Our first result. Our first result is that linear and exponential disutility functions are the only ones which are consistent with the above common sense requirement – for every other disutility function, a paradoxical counter-intuitive situation like the one described above is quite possible.

Let us describe this result in precise terms.

Definition 1 *By a* disutility function*, we mean a strictly increasing function $u(t)$ from non-negative real numbers to real numbers.*

Definition 2 *We say that two disutility functions $u(t)$ and $v(t)$ are* equivalent *if there exist real numbers $a > 0$ and b such that $v(t) = a \cdot u(t) + b$ for all t.*

Definition 3 *We say that a disutility function is* consistent with common sense *if it has the following property: let t_1 and t_2 be random variables with non-negative values, and let t_0 be an arbitrary (deterministic) non-negative real number; then,*

- *if $E[u(t_1)] < E[u(t_2)]$, then $E[u(t_1 + t_0)] < E[u(t_2 + t_0)]$;*
- *if $E[u(t_1)] = E[u(t_2)]$, then $E[u(t_1 + t_0)] = E[u(t_2 + t_0)]$.*

Theorem 1 *A disutility function is consistent with common sense if and only if it is equivalent to either the linear function $u(t) = t$, or to an exponential function $u(t) = \exp(c \cdot t)$ or $-\exp(-c \cdot t)$.*

Comment. For reader's convenience, all the proofs are placed in a special mathematical Appendix.

3 Exponential Disutility Functions Are the Only Functions Which Allow Simplified Computations: A Theorem

As we have mentioned in the first section, the computational advantage of linear and exponential disutility functions $u(t)$ is that for these functions, the expected value $E[u(t_1+t_2)]$ characterizing the route is uniquely determined by the expected values $E[u(t_1)]$ and $E[u(t_2)]$ characterizing the links. Let us formulate this property in precise terms.

Definition 4 *We say that a disutility function is* computationally simple *if it has the following property: let t_1, t_2, t'_1, and t'_2 be independent random variables for which $E[u(t_1)] = E[u(t'_1)]$ and $E[u(t_2)] = E[u(t'_2)]$, then $E[u(t_1+t_2)] = E[u(t'_1+t'_2)]$.*

Comment. We have already mentioned that the linear disutility function $u(t) = t$ and the exponential disutility functions $u(t) = \exp(c\cdot t)$ and $u(t) = -\exp(-c\cdot t)$ are computationally simple in this sense. Indeed, for the linear disutility function, $E[u(t_1+t_2)] = E[u(t_1)] + E[u(t_2)]$, hence

$$E[u(t_1+t_2)] = E[u(t_1)] + E[u(t_2)] = E[u(t'_1)] + E[u(t'_2)] = E[u(t'_1+t'_2)].$$

For an exponential disutility function, $E[u(t_1+t_2)] = E[u(t_1)]\cdot E[u(t_2)]$, so $E[u(t_1)] = E[u(t'_1)]$ and $E[u(t_2)] = E[u(t'_2)]$ imply that

$$E[u(t_1+t_2)] = E[u(t_1)]\cdot E[u(t_2)] = E[u(t'_1)]\cdot E[u(t'_2)] = E[u(t'_1+t'_2)].$$

It turns out that these two functions are the only computationally simple ones.

Theorem 2 *A disutility function is computationally simple if and only if it is equivalent to either the linear function $u(t) = t$, or to an exponential function $u(t) = \exp(c\cdot t)$ or $-\exp(-c\cdot t)$.*

Comment. The term "computationally simple" should be, of course, understood in the *relative* sense – transportation networks are huge, and their analysis often requires a lot of computation time.

4 Conclusion

In modeling drivers' route choice in stochastic networks, researchers have been actively using exponential disutility functions. A usual justification for these functions is that, in contrast to many other possible functions, they lead to simpler computations.

In this paper, we provide a stronger justification for these functions: namely, we prove that the linear and exponential disutility functions are the only ones which are consistent with common sense. We also formally prove that they are the only ones which allow for simplified computations.

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A Proofs

A.1 Proof of Theorem 1

1°. We already know that linear and exponential disutility functions are consistent with common sense in the sense of Definition 3. It is therefore sufficient to prove that every disutility function $u(t)$ which is consistent with common sense is equivalent either to a linear one or to an exponential one.

2°. Let $u(t)$ be a disutility function which is consistent with common sense. By definition of computational simplicity, for every random variables t_1, once we know the values $u_1 = E[u(t_1)]$ and t_0, we can uniquely determine the value $E[u(t_1 + t_0)]$. Let us denote the value $E[u(t_1 + t_0)]$ corresponding to u_1 and t_0 by $F(u_1, t_0)$.

3°. Let t'_1 be a non-negative number. For the case when $t_1 = t'_1$ with probability 1, we have $u'_1 = E[u(t_1)] = u(t'_1)$. In this case, $t_1 + t_0 = t'_1 + t_0$ with probability 1, so $E[u(t_1 + t_0)] = u(t'_1 + t_0)$. Thus, in this case, $u(t'_1 + t_0) = F(u'_1, t_0)$, where $u'_1 = u(t'_1)$.

4°. Let us now consider the case when t_1 is equal to t'_1 with some probability $p'_1 \in [0, 1]$, and to some smaller value $t''_1 < t'_1$ with the remaining probability $p''_1 = 1 - p'_1$. In this case,

$$u_1 = E[u(t_1)] = p'_1 \cdot u(t'_1) + (1 - p'_1) \cdot u(t''_1).$$

We have already denoted $u(t'_1)$ by u'_1; so, if we denote $u''_1 \stackrel{\text{def}}{=} u(t''_1)$, we can rewrite the above expression as

$$u_1 = p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1.$$

In this situation, $t_1 + t_0$ is equal to $t'_1 + t_0$ with probability p'_1 and to $t''_1 + t_0$ with probability $1 - p'_1$. Thus,

$$E[u(t_1 + t_0)] = p'_1 \cdot u(t'_1 + t_0) + (1 - p'_1) \cdot u(t''_1 + t_0).$$

We already know that $u(t'_1 + t_0) = F(u'_1, t_0)$ and $u(t''_1 + t_0) = F(u''_1, t_0)$. So, we can conclude that

$$E[u(t_1 + t_0)] = p'_1 \cdot F(u'_1, t_0) + (1 - p'_1) \cdot F(u''_1, t_0). \tag{A.1}$$

On the other hand, by the definition of the function F as $F(u_1, t_0) = E[u(t_1 + t_0)]$, we conclude that

$$E[u(t_1 + t_0)] = F(u_1, t_0),$$

i.e.,

$$E[u(t_1 + t_0)] = F(p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1, t_0). \tag{A.2}$$

Comparing the expressions (A.1) and (A.2) for $E[u(t_1 + t_0)]$, we conclude that

$$F(p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1, t_0) = p'_1 \cdot F(u'_1, t_0) + (1 - p'_1) \cdot F(u''_1, t_0).$$

Let us analyze this formula. For every value $u_1 \in [u''_1, u'_1]$, we can find the probability p'_1 for which $u_1 = p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1$: namely, the desired equation means that $u_1 = p'_1 \cdot u'_1 + u''_1 - p'_1 \cdot u''_1$; rearranging the terms, we get $u_1 - u''_1 = p'_1 \cdot (u'_1 - u''_1)$ and hence, the value $p'_1 = \dfrac{u_1 - u''_1}{u'_1 - u''_1}$. Substituting this expression into the above formula, we conclude that for a fixed t_0, the function $F(u_1, t_0)$ is a linear function of u_1:

$$F(u_1, t_0) = A(t_0) \cdot u_1 + B(t_0)$$

for some constants $A(t_0)$ and $B(t_0)$ which, in general, depend on t_0.

5°. We have already shown, in Part 3 of this proof, that $u(t'_1 + t_0) = F(u'_1, t_0)$. Thus, we conclude that for every $t'_1 \geq 0$ and $t_0 \geq 0$, we have

$$u(t'_1 + t_0) = A(t_0) \cdot u(t'_1) + B(t_0).$$

6°. For an arbitrary function $u(t)$, by introducing an appropriate constant $b = -u(0)$, we can always find an equivalent function $v(t)$ for which $v(0) = 0$. So, without losing generality, we can assume that $u(0) = 0$ for our original disutility function $u(t)$.

Since the disutility function is strictly increasing, we have $u(t) > 0$ for all $t > 0$.

For $t'_1 = 0$, the above formula takes the form $u(t_0) = B(t_0)$. Substituting this expression for $B(t_0)$ into the above formula, we conclude that

$$u(t'_1 + t_0) = A(t_0) \cdot u(t'_1) + u(t_0).$$

7°. The above property has to be true to arbitrary values of $t'_1 \geq 0$ and $t_0 \geq 0$. Swapping these values, we conclude that

$$u(t_0 + t'_1) = A(t'_1) \cdot u(t_0) + u(t'_1).$$

Since $t'_1 + t_0 = t_0 + t'_1$, we have $u(t'_1 + t_0) = u(t_0 + t'_1)$, hence

$$A(t_0) \cdot u(t'_1) + u(t_0) = A(t'_1) \cdot u(t_0) + u(t'_1).$$

Moving terms proportional to $u(t'_1)$ to the left hand side and terms proportional to $u(t_0)$ to the right hand side, we conclude that

$$(A(t_0) - 1) \cdot u(t'_1) = (A(t'_1) - 1) \cdot u(t_0). \qquad \text{(A.3)}$$

In the following text, we will consider two possible situations:

- the first situation is when $A(t_0) = 1$ for some $t_0 > 0$;

- the second situation is when $A(t_0) \neq 1$ for all $t_0 > 0$.

In the first situation, $A(t_0) = 1$ for some $t_0 > 0$. For this t_0, the equation (A.3) takes the form $(A(t_1') - 1) \cdot u(t_0) = 0$ for all t_1'. Since $u(t_0) > 0$ for $t_0 > 0$, we conclude that $A(t_1') - 1 = 0$ for every real number $t_1' \geq 0$, i.e., that the function $A(t)$ is identical to a constant function 1.

So, we have two possible situations:

- the first situation is when $A(t_0) = 1$ for some $t_0 > 0$; we have just shown that in this case, $A(t) = 1$ for all t; in the following text, we will show that in this situation, the disutility function $u(t)$ is linear;
- the second situation is when $A(t_0) \neq 1$ for all $t_0 > 0$; we will show that in this situation, the disutility function $u(t)$ is exponential.

8°. Let us first consider the situation in which $A(t)$ is always equal to 1. In this case, the above equation takes the form

$$u(t_0 + t_1') = u(t_0) + u(t_1').$$

In other words, in this case,

$$u(t_1 + t_2) = u(t_1) + u(t_2)$$

for all possible values $t_1 > 0$ and $t_2 > 0$.

In particular, for every $t_0 > 0$, we get:

- first, $u(2t_0) = u(t_0) + u(t_0) = 2u(t_0)$,
- then $u(3t_0) = u(2t_0) + u(t_0) = 2u(t_0) + u(t_0) = 3u(t_0)$, and,
- in general, $u(k \cdot t_0) = k \cdot u(t_0)$ for all integers k.

For every integer n and for $t_0 = 1/n$, we have $u(n \cdot t_0) = u(1) = n \cdot u(1/n)$, hence $u(1/n) = u(1)/n$. Then, for an arbitrary non-negative rational number k/n, we get

$$u(k/n) = u(k \cdot (1/n)) = k \cdot u(1/n) = k \cdot (1/n) \cdot u(1) = k/n \cdot u(1).$$

In other words, for every rational number $r = k/n$, we have $u(r) = r \cdot u(1)$.

Every real value t can be bounded, with arbitrary accuracy, by rational numbers k_n/n and $(k_n + 1)/n$: $k_n/n \leq t \leq (k_n + 1)/n$, where $k_n/n \to t$ and $(k_n + 1)/n \to t$ as $n \to \infty$. Since the disutility function $u(t)$ is strictly increasing, we conclude that $u(k_n/n) \leq u(t) \leq u((k_n + 1)/n)$. We already know that for rational values r, we have $u(r) = r \cdot u(1)$, so we have

$$k_n/n \cdot u(1) \leq u(t) \leq (k_n + 1)/n \cdot u(1).$$

In the limit $n \to \infty$, both sides of this inequality converge to $t \cdot u(1)$, hence $u(t) = t \cdot u(1)$.

So, in this case, we get a linear disutility function.

9°. Let us now analyze the case when $A(t) \neq 1$ for all $t > 0$. Since the values $u(t)$ are positive for all $t > 0$, we can divide both sides of the equality

$$(A(t_0) - 1) \cdot u(t'_1) = (A(t'_1) - 1) \cdot u(t_0)$$

by $u(t_0)$ and $u(t'_1)$, and conclude that

$$\frac{A(t_0) - 1}{u(t_0)} = \frac{A(t'_1) - 1}{u(t'_1)}.$$

The ratio $\dfrac{A(t) - 1}{u(t)}$ has the same value for arbitrary two numbers $t = t_0$ and $t = t'_1$; thus, this ratio is a constant. Let us denote this constant by k; then, $A(t) - 1 = k \cdot u(t)$ for all $t > 0$. Since $A(t) \neq 1$, this constant k is different from 0.

Substituting the resulting expression $A(t) = 1 + k \cdot u(t)$ into the formula $u(t'_1 + t_0) = A(t_0) \cdot u(t'_1) + u(t_0)$, we conclude that

$$u(t'_1 + t_0) = u(t_0) + u(t'_1) + k \cdot u(t_0) \cdot u(t'_1),$$

i.e., that

$$u(t_1 + t_2) = u(t_1) + u(t_2) + k \cdot u(t_1) \cdot u(t_2)$$

for arbitrary numbers $t_1 > 0$ and $t_2 > 0$.

10°. Let us now consider a re-scaled function $v(t) \stackrel{\text{def}}{=} 1 + k \cdot u(t)$.

For this function $v(t)$, from the above formula, we conclude that

$$v(t_1 + t_2) = 1 + k \cdot u(t_1 + t_2) = 1 + k \cdot (u(t_1) + u(t_2)) + k^2 \cdot u(t_1) \cdot u(t_2).$$

On the other hand, we have

$$v(t_1) \cdot v(t_2) = (1 + k \cdot u(t_1)) \cdot (1 + k \cdot u(t_2)) =$$

$$1 + k \cdot (u(t_1) + u(t_2)) + k^2 \cdot u(t_1) \cdot u(t_2).$$

The expression for $v(t_1 + t_2)$ and for $v(t_1) \cdot v(t_2)$ coincide, so we conclude that

$$v(t_1 + t_2) = v(t_1) \cdot v(t_2)$$

for all possible values $t_1 > 0$ and $t_2 > 0$.

11°. When $k > 0$, then the new function $v(t)$ is an equivalent disutility function. We know that $u(0) = 0$ hence $v(0) = 1 + k \cdot 0 = 1$. Since $v(t)$ is a strictly increasing function, we thus conclude that $v(t) \ge v(0) > 0$ for all $t \ge 0$.

Thus, we can take a logarithm of all the values, and for the new function $w(t) \stackrel{\rm def}{=} \ln(v(t))$, get an equation

$$w(t_1+t_2) = \ln(v(t_1+t_2)) = \ln(v(t_1)\cdot v(t_2)) = \ln(v(t_1))+\ln(v(t_2)) = w(t_1)+w(t_2),$$

i.e., $w(t_1+t_2) = w(t_1) + w(t_2)$ for all t_1 and t_2. The function $w(t)$ is increasing – as the logarithm of an increasing function. Thus, as we have already shown, $w(t) = c \cdot t$ for some $c > 0$.

From the logarithm $w(t) = \ln(v(t))$, we can reconstruct the original disutility function $v(t)$ as $v(t) = \exp(w(t))$. Since $w(t) = c \cdot t$, we conclude that the disuility function $v(t)$ has the desired risk-averse exponential form

$$v(t) = \exp(c \cdot t).$$

12°. When $k < 0$, the new function is strictly decreasing (and is thus not a disutility function; its opposite $-v(t)$ is a disutility function).

For the function $v(t)$, we cannot have $v(t_0) = 0$ for any t_0 – because otherwise we would have

$$v(t) = v(t_0 + (t - t_0)) = v(t_0) \cdot v(t - t_0) = 0$$

for all $t \ge t_0$ which contradicts to our conclusion that the function $v(t)$ should be strictly decreasing.

13°. For the function $v(t)$, we cannot have $v(t_0) < 0$ for any $t_0 > 0$ – because otherwise, we would have $v(2t_0) = v(t_0)^2 > 0$ hence $v(2t_0) > v(t_0)$ – which, since $2t_0 > t_0$, also contradicts to our conclusion that the function $v(t)$ should be strictly decreasing.

We thus conclude that $v(t) > 0$ for all t.

14°. Thus, we can take a logarithm of all the values, and for the new function $w(t) \stackrel{\rm def}{=} \ln(v(t))$, get the equation $w(t_1 + t_2) = w(t_1) + w(t_2)$ for all t_1 and t_2. The function $w(t)$ is decreasing – as the logarithm of a decreasing function. Thus, $w(t) = -c \cdot t$ for some $c > 0$.

From the logarithm $w(t) = \ln(v(t))$, we can reconstruct the original function $v(t)$ as $v(t) = \exp(w(t)) = \exp(-c \cdot t)$, and the disutility function $u(t)$ as $-v(t) = -\exp(-c \cdot t)$.

So, we conclude that the disuility function $v(t)$ has the desired risk-prone exponential form $v(t) = -\exp(-c \cdot t)$.

The theorem is proven.

A.2 Proof of Theorem 2

1°. We already know that linear and exponential disutility functions are computationally simple in the sense of Definition 4. It is therefore sufficient to prove that every computationally simple disutility function $u(t)$ is equivalent either to a linear one or to an exponential one.

2°. Let $u(t)$ be a computationally simple disutility function. By definition of computational simplicity, for every two independent random variables t_1 and t_2, once we know the values $u_1 = E[u(t_1)]$ and $u_2 = E[u(t_2)]$, we can uniquely determine the value $E[u(t_1+t_2)]$. Let us denote the value $E[u(t_1+t_2)]$ corresponding to u_1 and u_2 by $F(u_1, u_2)$.

3°. Since $t_1 + t_2 = t_2 + t_1$, we should have $E[u(t_1 + t_2)] = E[u(t_2 + t_1)]$. So, the value of $E[u(t_1 + t_2)]$ should not depend on the order in which we add the random variables t_1 and t_2, and we should have $F(u_1, u_2) = F(u_2, u_1)$; in other words, the function $F(u_1, u_2)$ is symmetric.

4°. In the following sections 5° and 6°, we will consider the case when t_1 may be non-deterministic but t_2 is deterministic, i.e., when t_1 may take different values t_1', t_1'', ..., but t_2 is equal to the same real number t_2' with probability 1. Since we have different possible values t_1, we will have different values of the disutility function; we will denote these values by $u_1' \stackrel{\rm def}{=} u(t_1')$, $u_1'' \stackrel{\rm def}{=} u(t_1'')$. For t_2, there is only one possible value $t_2 = t_2'$; we will therefore denote the corresponding value of disutility simply by $u_2 \stackrel{\rm def}{=} u(t_2')$.

5°. Let t_1', t_2' be two non-negative numbers. For the case when $t_1 = t_1'$ with probability 1 and $t_2 = t_2'$ with probability 1, we have $u_1' = E[u(t_1)] = u(t_1')$, $u_2 = E[u(t_2)] = u(t_2')$. In this case, $t_1 + t_2 = t_1' + t_2'$ with probability 1, so $E[u(t_1 + t_2)] = u(t_1' + t_2')$. Thus, in this case, $u(t_1' + t_2') = F(u_1', u_2)$, where $u_1' = u(t_1')$ and $u_2 = u(t_2')$.

Similarly, for a different value $t_1'' < t_1'$, we conclude that $u(t_1''+t_2') = F(u_1'', u_2)$, where $u_1'' \stackrel{\rm def}{=} u(t_1'')$. Since the function $u(t)$ is strictly increasing, we have $u_1'' < u_1'$.

6°. Let us now consider the case when t_2 is still a constant (i.e., $t_2 = t_2'$

with probability 1), but t_1 is already non-deterministic: it is equal to t'_1 with some probability $p'_1 \in [0,1]$, and to $t''_1 < t'_1$ with the remaining probability $p''_1 = 1 - p'_1$. In this case,

$$u_1 = E[u(t_1)] = p'_1 \cdot u(t'_1) + (1 - p'_1) \cdot u(t''_1) = p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1. \quad (A.4)$$

In this situation, $t_1 + t_2$ is equal to $t'_1 + t'_2$ with probability p'_1 and to $t''_1 + t'_2$ with probability $1 - p'_1$. Thus,

$$E[u(t_1 + t_2)] = p'_1 \cdot u(t'_1 + t'_2) + (1 - p'_1) \cdot u(t''_1 + t'_2). \quad (A.5)$$

We already know that $u(t'_1 + t'_2) = F(u'_1, u_2)$ and $u(t''_1 + t'_2) = F(u''_1, u_2)$, so we conclude that

$$E[u(t_1 + t_2)] = p'_1 \cdot F(u'_1, u_2) + (1 - p'_1) \cdot F(u''_1, u_2).$$

On the other hand, by the definition of the function F, we conclude that

$$E[u(t_1 + t_2)] = F(u_1, u_2), \quad (A.6)$$

i.e.,

$$E[u(t_1 + t_2)] = F(p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1, u_2). \quad (A.7)$$

Comparing the displayed expressions (A.5) and (A.7) for $E[u(t_1 + t_2)]$, we conclude that

$$F(p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1, u_2) = p'_1 \cdot F(u'_1, u_2) + (1 - p'_1) \cdot F(u''_1, u_2). \quad (A.8)$$

For every value $u_1 \in [u''_1, u'_1]$, we can find the probability p'_1 for which $u_1 = p'_1 \cdot u'_1 + (1 - p'_1) \cdot u''_1$: namely, the value $p'_1 = \dfrac{u_1 - u''_1}{u'_1 - u''_1}$. Substituting this expression into the above formula, we conclude that for a fixed u_2, the function $F(u_1, u_2)$ is a linear function of u_1.

7°. Similarly, for fixed $u_1 = u'_1$ and $u_1 = u''_1$, the functions $F(u'_1, u_2)$ and $F(u''_1, u_2)$ are linear functions of u_2. Thus, the expression $F(u_1, u_2)$ is a bilinear function of u_1 and u_2.

In general, a bilinear function has the form

$$F(u_1, u_2) = a_0 + a_1 \cdot u_1 + a_2 \cdot u_2 + a_{12} \cdot u_1 \cdot u_2.$$

Since we know that our function $F(u_1, u_2)$ is symmetric, we conclude that $a_1 = a_2$, hence

$$F(u_1, u_2) = a_0 + a_1 \cdot (u_1 + u_2) + a_{12} \cdot u_1 \cdot u_2.$$

8°. To complete our proof, we will consider three cases:

- the case when $a_{12} = 0$ (in which case, we will get a linear disutility function),
- the case when $a_{12} > 0$ (in which case, we will get a risk-averse exponential disutility function), and
- the case when $a_{12} < 0$ (in which case, we will get a risk-prone exponential disutility function).

9°. Let us first assume that $a_{12} = 0$ and $F(u_1, u_2) = a_0 + a_1 \cdot (u_1 + u_2)$. Let us prove that in this case, the disutility function is linear.

9.1°. According to Definition 2, two disutility functions $u(t)$ and $v(t)$ are equivalent if there exist real numbers $a > 0$ and b such that $v(t) = a \cdot u(t) + b$ for all t. We would like to use this definition to simplify the disutility function. Namely, we would like to find the values a and b for which the equivalent disutility function $v(t)$ has the property that $v(0) = 0$ and $v(1) = 1$.

For these two properties to hold, we must have $a \cdot u(0) + b = 0$ and $a \cdot u(1) + b = 1$. Subtracting the first equation from the second one, we get $a \cdot (u(1) - u(0)) = 1$ hence $a = \dfrac{1}{u(1) - u(0)}$. Due to Definition 1, a disutility function $u(t)$ is strictly increasing hence $u(0) < u(1)$. Therefore, the above value a is indeed positive. To satisfy the first equation, we now take $b = -a \cdot u(0)$. For these values $a > 0$ and b, we indeed have $v(0) = 0$ and $v(1) = 1$.

So, without losing generality, we can assume that $u(0) = 0$ and $u(1) = 1$ for our original disutility function $u(t)$.

9.2°. If we add $t_1 = 0$ to an arbitrary value t_2, then we get $t_1 + t_2 = t_2$. Thus, in this case, $u(t_1 + t_2) = u(t_2)$, and $E[u(t_1 + t_2)] = E[u(t_2)]$, i.e., due to (A.5), $F(u_1, u_2) = u_2$ for arbitrary u_2.

For $t_1 = 0$, we have $u(t_1) = u(0) = 0$, hence $u_1 = E[u(t_1)] = 0$. So, we conclude that

$$F(0, u_2) = u_2$$

for an arbitrary u_2. Substituting the expression $F(u_1, u_2) = a_0 + a_1 \cdot (u_1 + u_2)$ into the formula $F(0, u_2) = u_2$, we conclude that $a_0 + a_1 \cdot u_2 = u_2$ for all real numbers u_2. Thus, $a_0 = 0$, $a_1 = 1$, and we have $F(u_1, u_2) = u_1 + u_2$.

9.3°. For deterministic values t_1 and t_2, in general, we have $u_1 = u(t_1)$, $u_2 = u(t_2)$, and hence, $u(t_1+t_2) = F(u_1, u_2) = F(u(t_1), u(t_2))$. So, for our specific function $F(u_1, u_2) = u_1 + u_2$, we conclude that

$$u(t_1 + t_2) = u(t_1) + u(t_2)$$

for all possible values t_1 and t_2. We already know (from the proof of Theorem 1) that this condition leads to the linear disutility function.

So, for $a_{12} = 0$, we indeed get a linear disutility function.

10°. Let us now prove that for $a_{12} > 0$, we get an risk-averse exponential disutility function.

10.1°. Similarly to Part 9.1 of this proof, let us use the definition of an equivalent disutility function $v(t) = a \cdot u(t) + b$ to simplify the expression for $u(t)$. Specifically, we want to have an equivalent disutility function for which $v(0) = 1$. This can be achieved by taking $a = 1$ and $b = 1 - u(0)$.

Thus, without losing generality, we can assume that $u(0) = 1$ for our original disutility function $u(t)$.

10.2°. Similarly to the previous case, for $t_1 = 0$, we have $F(u_1, u_2) = u_2$ for arbitrary u_2.

For $t_1 = 0$, we have $u(t_1) = u(0) = 1$, hence $u_1 = E[u(t_1)] = 1$. So, we conclude that

$$F(1, u_2) = u_2$$

for an arbitrary u_2. Substituting the expression $F(u_1, u_2) = a_0 + a_1 \cdot (u_1 + u_2) + a_{12} \cdot u_1 \cdot u_2$ into this formula, we conclude that $a_0 + a_1 + a_1 \cdot u_2 + a_{12} \cdot u_2 = u_2$ for all real numbers u_2. Thus, $a_0 + a_1 = 0$ and $a_1 + a_{12} = 1$. If we denote $\beta = a_{12}$, then we have $a_1 = 1 - \beta$, $a_0 = -a_1 = -(1 - \beta)$, and hence, $F(u_1, u_2) = -(1 - \beta) + (1 - \beta) \cdot (u_1 + u_2) + \beta \cdot u_1 \cdot u_2$.

10.3°. For deterministic values t_1 and t_2, in general, we have $u_1 = u(t_1)$, $u_2 = u(t_2)$, and hence, $u(t_1 + t_2) = F(u_1, u_2) = F(u(t_1), u(t_2))$. So, for our specific function $F(u_1, u_2) = u_1 + u_2$, we conclude that

$$u(t_1 + t_2) = -(1 - \beta) + (1 - \beta) \cdot (u(t_1) + u(t_2)) + \beta \cdot u(t_1) \cdot u(t_2) \quad \text{(A.9)}$$

for all possible values t_1 and t_2.

10.4°. Let us now consider a re-scaled (equivalent) disutility function

$$v(t) \stackrel{\text{def}}{=} (1 - \beta) + \beta \cdot u(t). \quad \text{(A.10)}$$

For $t = 0$, we have $v(0) = (1-\beta) + \beta \cdot u(0)$. Since we assumed that $u(0) = 1$, we conclude that $v(0) = (1-\beta) + \beta = 1$.

For this function $v(t)$, for $t = t_1 + t_2$, we conclude that

$$v(t_1 + t_2) = (1-\beta) + \beta \cdot u(t_1 + t_2).$$

Using the expression (A.9) for $u(t_1 + t_2)$, we conclude that

$$v(t_1+t_2) = (1-\beta) - \beta\cdot(1-\beta) + \beta\cdot(1-\beta)\cdot(u(t_1)+u(t_2)) + \beta^2\cdot u(t_1)\cdot u(t_2) =$$

$$(1-\beta)^2 + \beta\cdot(1-\beta)\cdot(u(t_1)+u(t_2)) + \beta^2\cdot u(t_1)\cdot u(t_2). \tag{A.11}$$

On the other hand, from the definition (A.10) of the function $v(t)$, we have

$$v(t_1)\cdot v(t_2) = ((1-\beta)+\beta\cdot u(t_1))\cdot((1-\beta)+\beta\cdot u(t_2)) =$$

$$(1-\beta)^2 + \beta\cdot(1-\beta)\cdot(u(t_1)+u(t_2)) + \beta^2\cdot u(t_1)\cdot u(t_2). \tag{A.12}$$

The expressions (A.11) for $v(t_1 + t_2)$ and (A.12) for $v(t_1)\cdot v(t_2)$ coincide, so we conclude that

$$v(t_1 + t_2) = v(t_1)\cdot v(t_2)$$

for all possible values t_1 and t_2.

10.5°. We know that $v(0) = 1$. Since $v(t)$ is a strictly increasing function, we thus conclude that $v(t) \ge v(0) > 0$ for all t. Thus, similarly to the proof of Theorem 1, we can take a logarithm $w(t) \stackrel{\text{def}}{=} \ln(v(t))$, and conclude that $w(t) = c\cdot t$, and that the disuility function $v(t)$ has the desired risk-averse exponential form

$$v(t) = \exp(c\cdot t).$$

11°. Let us now prove that for $a_{12} < 0$, we get a risk-prone exponential disutility function.

Indeed, similarly to the previous case, we can assume that $u(0) = 1$. In this case, we can describe a similar expression $v(t) \stackrel{\text{def}}{=} (1-\beta)+\beta\cdot u(t)$, with $\beta = a_{12}$, for which $v(t_1\cdot t_2) = v(t_1)\cdot v(t_2)$. The only difference is that since $\beta = a_{12} < 0$, this new function is strictly decreasing (and is thus not a disutility function; its opposite $-v(t)$ is a disutility function).

Similarly to the proof of Theorem 1, we can now conclude that $v(t) = \exp(w(t)) = \exp(-c\cdot t)$, and so, the disutility function $u(t) = -v(t)$ has the desired risk-prone exponential form $u(t) = -\exp(-c\cdot t)$.

The theorem is proven.