# Towards a General Description of Physical Invariance in Category Theory

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**Abstract**

Invariance is one of the most important notions in applications of mathematics. It is one of the key concepts in modern physics, is a computational tool that helps in solving complex equations, etc. In view of its importance, it is desirable to come up with a definition of invariance which is as general as possible. In this paper, we describe how to formulate a general physically meaningful (e.g., unit-invariant) notion of physical invariance in categorial terms.

**Invariance is important.** Invariance is one of the most important concepts in applications of mathematics. In addition to its role as a computational tool in the solution of complex equations (see, e.g., [3, 6]), invariance (symmetry) is perhaps the most important notion in the conceptual foundations of modern physics; see, e.g., [5, 12]. Invariance has a central role in contemporary metaphysics insofar as it relates to the problem of individuation. For example, Robert Nozick describes objectivity in terms of invariance under transformation and describes necessary truths as those which are invariant in all possible worlds [9]. While this paper treats invariance only in physical contexts, our analysis is conducted with an eye to basic metaphysical questions of the type that Nozick informally addressed.

**It is important to provide a general definition of invariance.** Since the notion of invariance plays such a central role in foundational research, it is desirable to provide a formal definition of invariance which is as general as possible.

In mathematics, such general definitions are usually provided by category theory; see, e.g., [1, 8]. In this paper, we will therefore attempt to describe how to formulate a general notion of invariance in categorial terms.

Before outlining the problem, let us first briefly review the main notions of category theory.

**What are categories: motivation.** In mathematical theories, we usually have a class of objects and corresponding mappings. For example, objects of set theory are sets, and mapping are functions (mappings) between these sets. In topology, objects are topological spaces, and natural mapping are continuous functions. In order theory, mappings are ordered sets, and natural mappings are monotonic functions. In linear algebra, linear spaces are objects, and linear functions are natural mappings, etc.

In all these cases, the identity mapping $f(x) = x$ (that maps each element $x$ into itself) is a natural mapping. Also, a composition $f(g(x))$ of two natural mappings is also natural: e.g., a composition of two continuous functions is continuous, a composition of two monotonic functions is monotonic, etc.

**Categories: a formal definition.** In precise terms, a *category* consists of *objects* $A$, $B$, $C$, ..., and *morphisms* (also called *arrows*) $f$, $g$, $h$, ...

For every arrow $f$, there are given objects $A$ (called *domain* of $f$) and $B$ (called *codomain* of $f$). This is usually denoted by $f : A \to B$.

For every two arrows $f : A \to B$ and $g : B \to C$, there is an arrow $g \circ f : A \to C$ called a *composition* of $f$ and $g$. Composition is *associative* in the sense that $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$, $g : B \to C$, and $h : C \to D$.

For every object $A$, there is a special *identity morphism* $1_A : A \to A$ for which $f \circ 1_A = f = 1_B \circ f$ for all $f : A \to B$.

***Comment.*** While category theory is generally introduced by reference to sets and functions, it is important to recognize that categories have more general bearing and that there are useful categories in which morphisms are *not* functions; see, e.g., [1, 8]. In this sense, category theory addresses the most general features of morphisms.

**From the intuitive notion of invariance towards a formal category definition: analysis of the problem.** Informally, invariance means that after we perform some transformation, the result of a certain operation remains unchanged. For example, invariance of energy means that after we apply the corresponding transformation (e.g., rotate and/or shift a configuration of electric charges) the energy remains the same. In physics, transformations which preserve certain quantities are often called *symmetries.* It is known that symmetries form a *group* in the sense that a composition of two symmetries is a symmetry, and an inverse transformation to a symmetry is also a symmetry. Such a group is usually called a *symmetry group.* There is also a known deeper relation between invariants and symmetry groups – provided by Noether's theorem; see, e.g., [5]. Let us describe the situation of invariance in terms of sets, in such a way that we will be able to reformulate this description categorially.

In set theoretic terms, we have a set $S$ of possible states. A *transformation* can be naturally described as a mapping $t$ that transform each state $s \in S$ into a new state $t(s) \in S$.

To describe the quantity (such as energy) which remains invariant under this transformation, we must describe the set $V$ of values of this quantity, and we must be able to assign, to each state $s \in S$, the corresponding value $v(s)$ of this quantity at the state $s$. In other words, we need to describe a mapping $v$ that maps every state $s \in S$ into a value $v(s) \in V$.

In these terms, invariance can be described as follows. For each state $s$, prior to the transformation, the analyzed quantity had the value $v(s)$. After the transformation, we have a new state $s' = t(s)$ in which this quantity has the value $v(s') = v(t(s))$. Invariance means that the value of the quantity does not change after the transformation, i.e., that $v(t(s)) = v(s)$ for all $s \in S$.

The expression $v(t(s))$ is a composition of $v$ and $t$; so, invariance means that the composition of $v$ and $t$ coincides with $v$. So, we arrive at the following definition:

**Formal definition of invariance: first try.** We say that a morphism $v : S \to V$ is *invariant* under the morphism $t : S \to S$ if $v \circ t = v$.

**Limitations of this definition.** From a purely *mathematical* perspective, the above definition may seem to capture the intuitive notion of invariance perfectly.

However, as we will show, from a *physical* perspective, this definition is far from perfect. Consider for instance invariance of energy. We can certainly define energy as a mapping $v$ from states to real numbers. However, this mapping does not necessarily capture all significant features of the notion of energy and as such should not be taken as representing energy itself.

Clearly, for example, we can use different units to measure energy. If we use a different unit for energy (e.g., joules from SI instead of ergs in the old SGS system of units), then the numerical value of energy will change. So, if $v(s)$ denotes energy as expressed in the original units and $v'(s)$ denoted energy as expressed in the new units, then $v(s) \neq v'(s)$. Thus, from the mathematical (and categorial) viewpoint, we have two *different* functions $v(s)$ and $v'(s)$. By contrast, from the physical viewpoint, both mappings represent the same physical quantity – energy.

In other words, our intial definition of invariance required that we *fix* a unit for the preserved quantity (e.g., for energy) – while from the physical viewpoint, the notion of invariance does not depend on what unit we choose for representing the preserved property. We are interested in capturing the invariant features of the physical properties themselves apart from the details of the choices of representational artifacts.

It is not only the choice of units, we may have more complex choices of different scales. For example, the energy of a noise can be described in absolute units, and it can be also be described in *logarithmic* scale of decibels. Consequently, in order to generate a meaningful categorial definition of physical invariance it is desirable to modify the above definition in such a way that it should not depend on the choice of units (or, more generally, on the choice of a scale) or on the choice of a coordinate system.

**Towards a more physically adequate definition.** How can we improve upon our initial definition? In order to control for choice of unit and scale in our definition, let us first describe the notion of re-scaling in more general terms. Let $v(s) \in V$ be the value of the quantity in the original scale.

A change of a measuring unit means that we go from the original value $v = v(s)$ to the new value $v' = \lambda \cdot v$, where $\lambda$ is the ratio between the two measuring units. For example, going from meters to centimeters (a new unit which is 100 times smaller than the original one) means that all the numerical values are multiplied by 100. A logarithmic re-scaling means that we go from $v$ to $v' = \log(v)$. In general, we go from the original value $v$ to the new value $r(v)$, where $r$ is the new function which represents the re-scaling procedure.

In the above two examples, re-scaling goes from the set of values to the same set of values. However, it is possible that we have different ranges. For example, when we change the unit of measuring angle from degree to a radian, we also changes the range: originally, we had $V = [0, 360]$, now, we have $V' = [0, 2\pi]$.

So, in general, a re-scaling can be described as a mapping $r : V \to V'$. If originally, we had a quantity $v = v(s)$, then after re-scaling, we have a new quantity $v'(s) = r(v(s))$. In category theory terms, this means that $v' = r \circ v$.

In these terms, a reasonable description of invariance means not only that the *original* quantity $v$ is invariant relative to the transformation $t$, but also that *all* re-scaled expressions of this quantity must also be invariant.

How do we describe this class of possible re-scaled transformations of a quantity? The only reasonable requirement is that if a morphism $v : S \to V$ belongs to this class, then for every re-scaling (i.e., for every mapping) $r : V \to V'$, the composition $r \circ v$ should also belong to this same class. Classes with this property are known in category theory as *left ideals* [7, 11]. Thus, we arrive at the following formal definition.

**Formal definition of invariance: second try.** By a *left ideal*, we mean a class $\mathcal{V}$ of morphisms such that if a morphism $v : S \to V$ belongs to this class, then for every $r : V \to V'$, the composition $r \circ v$ also belongs to this class.

We say that a left ideal $\mathcal{V}$ is *invariant* under the morphism $t : S \to S$ if $v \circ t = v$ for all $v \in \mathcal{V}$.

**Limitations of this definition.** The above definition is intuitively reasonable if we consider transformations like shift or rotation that transform the state of an object into a different state of the same object. However, the notion of invariance in physics goes well beyond such transformations. For example, physicists talk about C-symmetry which maps a particle (such as an electron $e^-$) into the corresponding anti-particle (e.g., positron $e^+$).

In such examples, it often makes sense to talk about invariance – in the sense, e.g., that in similar situations, the electron and positron will have the same energy. However, the "transformation" of the state of an electron into the corresponding state of a positron is no longer a physically possible transformation.

In other words, in this situation, we no longer have a single set of states $S$: we have a set of states $S_1$ of an electron, we have a set $S_2$ of states of a positron, and the transformation is a mapping $t : S_1 \to S_2$. Instead of a single mapping $v$, energy can now be described by two different mappings $v_1 : S_1 \to V$ and $v_2 : S_2 \to V$, and invariance means that $v_2(t(s)) = v_1(s)$ for all $s$.

The above definition does not capture this meaning. How can we capture it?

**Towards a more physically adequate definition.** We would like to describe the fact that even if we fix a single scale, still a quantity (such as energy) does not correspond to a *single* mapping $v : S \to V$, but rather to *several* mappings $v_1 : S_1 \to V$, $v_2 : S_2 \to V$ from several different objects $S_1$, $S_2$, ...

Of course, once we fix the object (= set of states) $S_i$, the energy should be uniquely defined for all states $s \in S_i$. So, for every object $S_i$, we can have at most one function $v_i : S_i \to V$.

Such a construction also exists in applications of category theory – namely, the notion of a *local section*. This notion is usually defined in the context of fiber bundles [2, 10], but it can also be applied to more general cases. For instance, if we have a mapping $\pi : E \to B$, then its *section* is a mapping $f : b \to E$ from some subset $b \subseteq B$ into $E$ such that $\pi(f(x)) = x$ for all $x \in b$. In categorial terms, this condition can be described as $\pi \circ f = 1_b$.

This is directly related to the notion of an inverse morphism. Namely, in a category, a mapping $f : A \to B$ is called *inverse* to a mapping $\pi : B \to A$ if $\pi \circ f = 1_A$ and $f \circ \pi = 1_B$. In the above definition, only one of these two requirements is postulated, and only locally (i.e., for a subset $b \subseteq B$, so we can call a section a *local right inverse*.

In our case, in every category, we have a mapping which maps every arrow $v : S \to V$ into its domain $S$. What we want is a *local section* that assigns, to some objects $S$, a morphism $v_S : S \to V$. In these terms, if we have a state $s \in S$, then the value of the desired quantity in this state can be described as $v_S(s)$.

How can we describe invariance in these terms? Suppose that we have a transformation $t : S \to S'$. Originally, the energy of a state $s \in S$ is $v_S(s)$; after the transformation, we have the new state $t(s) \in S'$, and the new value of energy $v_{S'}(s') = v_{S'}(t(s))$. Invariance means that the new value of energy is the same as the old value, i.e., that $v_{S'}(t(s)) = v_S(s)$ for all $s \in S$. In category terms, this means that $v_{S'} \circ t = v_S$.

In this definition, we did not take re-scaling into account. To take into account, instead of individual local sections, we should consider *left ideals* of local sections – defined similarly to left ideals of morphisms. Thus, we arrive at the following definition.

**Formal definition of invariance: our final result.** Let $V$ be an object. By a *$V$-local section* $v$, we mean a mapping which assigns to some objects $S$ from the category, a mapping $v_S : S \to V$. By a *local section*, we mean a $V$-local section corresponding to some codomain $V$.

We say that a local section $v$ is *invariant* under the morphism $t : S \to S'$ if $v$ is defined for both $S$ and $S'$ and $v_{S'} \circ t = v_S$.

For every $V$-local section $v$ and a morphism $r : V \to V'$, we can define a *composition* $v' \stackrel{\text{def}}{=} r \circ v$ as a $V'$-local section which assign to an object $S$ a mapping $v'_S = r \circ v_S : S \to V'$.

By a *left ideal of local sections*, we mean a class $\mathcal{V}$ of local sections such that if a $V$-local section $v$ belongs to this class, then for every $r : V \to V'$, the composition $r \circ v$ also belongs to this class.

We say that a left ideal of local sections $\mathcal{V}$ is *invariant* under the morphism $t : S \to S'$ if all local section $v \in \mathcal{V}$ are invariant under this morphism.

*Comment.* In this paper, we use category theory to provide a new physically meaningful definition of an *invariance* with respect to a transformation – or with respect to a transformation group. A similar category approach has been used to show that a more natural definition of a class of transformations is not a *group* (a class in which composition is well defined for every two transformations) but a *groupoid* – crudely speaking, a class in which composition is defined only for *some* pairs of transformations (as in a category); see, e.g., [4]. It is desirable to analyze the relation between our approach and the groupoid approach.

**Conclusion.** The main objective of this paper was to formulate a general notion of physical invariance in category terms. Surprisingly, producing such a definition turns out to be more complex than we originally thought.

Our definition captures features of invariance that we believe to be crucial in the physical context. Since the notion of invariance is extremely important in working science, we want to present this definition to interested readers for critical analysis. We canvassed two intermediate definitions before arriving at one that we believe to be final. It may be that our definition is indeed final; in this case, the readers' critical analysis is necessary to confirm this fact.

It could also be that our definition is not really final, it is just one more step towards the ideal definition. In other words, it is possible that we missed some subtle features of invariance, that would require us to produce a more adequate albeit more complex definition. It is likely also to be the case that the effort to characterize physical invariance will depend, in part on the state of our physics. As such, we remain open-minded with respect to our definition.

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