Set-Valued Extensions of Fuzzy Logic: Classification Theorems

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***Abstract*—Experts are often not 100% confident in their statements. In traditional fuzzy logic, the expert's degree of confidence in each of his or her statements is described by a number from the interval $[0,1]$. However, due to similar uncertainty, an expert often cannot describe his or her degree by a *single* number. It is therefore reasonable to describe this degree by, e.g., a *set* of numbers. In this paper, we show that under reasonable conditions, the class of such sets coincides either with the class of all 1-point sets (i.e., with the traditional fuzzy set set of all numbers), or with the class of all subintervals of the interval $[0,1]$, or with the class of all closed subsets of the interval $[0,1]$. Thus, if we want to go beyond standard fuzzy logic and still avoid sets of arbitrary complexity, we have to use intervals. These classification results shows the importance of interval-valued fuzzy logics.**

I. Formulation of the Problem

A. Fuzzy Logic: Brief Reminder

In classical (2-valued) logic, every statement is either true or false. Such a 2-valued logic is often not adequate in describing expert knowledge, because experts are usually not fully confident about their statements.

To formally describe this uncertainty in human reasoning, L. A. Zadeh introduced the notion of *fuzzy logic*; see, e.g., [2], [4]. In fuzzy logic, a person's degree of confidence is described by a number from the interval $[0,1]$, so that absolute confidence in a statement corresponds to 1, absolute confidence in its negation corresponds to 0, and intermediate values correspond to intermediate degrees of confidence.

In fuzzy logic, once we know the degree of confidence a in a statement A and the degree of confidence b in a statement B, we usually estimate the degree of confidence in composite statements $A \wedge B$ and $A \vee B$ as, correspondingly,

$$a \wedge b \stackrel{\text{def}}{=} \min(a,b)$$

and

$$a \vee b \stackrel{\text{def}}{=} \max(a,b).$$

B. Mappings Which Preserve Standard Fuzzy Logic Operations

One can easily check that if a bijection (1-1 onto mapping)

$$\varphi : [0,1] \to [0,1]$$

is monotonic, then it preserves both operations $\wedge$ and $\vee$ in the sense that

$$\varphi(a) \wedge \varphi(b) = \varphi(a \wedge b)$$

and

$$\varphi(a) \vee \varphi(b) = \varphi(a \vee b).$$

Vice versa, if a bijection φ preserves the operations $\wedge$ and $\vee$, then it is monotonic.

In mathematical terms, a strictly monotonic continuous function from $[0,1]$ to $[0,1]$ for which $\varphi(0) = 0$ and $\varphi(1) = 1$ is called an *automorphism* of the structure $([0,1], \wedge, \vee)$. The set of all automorphisms is called the *automorphism group* of the structure $([0,1], \wedge, \vee)$.

C. From Single-Valued Fuzzy Logic to Interval-Valued and Set-Valued Ones

As we have mentioned earlier, experts are usually not fully confident about their statements. In traditional fuzzy logic, the expert's degree of confidence in each of his or her statements is described by a number from the interval $[0,1]$. However, due to similar uncertainty, an expert often cannot describe his or her degree by a single number.

It is therefore reasonable to describe this degree by, e.g., a *set* of possible values.

There is a natural extension of operations $\wedge$ and $\vee$ to such sets. Indeed, a set A means that all values $a \in A$ are possible, B means that all the values $b \in B$ are possible; so the set $A \wedge B$ of possible values of $a \wedge b$ is formed by all the values $a \wedge b$ where $a \in A$ and $b \in B$:

$$A \wedge B \stackrel{\text{def}}{=} \{a \wedge b : a \in A, b \in B\}. \qquad (1)$$

Similarly,

$$A \vee B \stackrel{\text{def}}{=} \{a \vee b : a \in A, b \in B\}. \qquad (2)$$

In many applications, researchers have been successfully using *intervals* of possible values; see, e.g., [2], [3], [4]; however, it is possible to consider more general sets as well [5]. A natural question is: which sets should we consider?

D. We Want an Extension

Since we are talking about *extensions* of the traditional fuzzy logic, it is reasonable to require that the desired class of sets $\mathcal{S}$ contain all one-element sets (corresponding to traditional fuzzy values).

E. We Want Invariance

It is also reasonable to assume that the class $\mathcal{S}$ is invariant under automorphisms of the traditional fuzzy logic.

In precise terms, if S is a possible set (i.e., if $S \in \mathcal{S}$), and $\varphi(x)$ is a strictly increasing continuous function with $\varphi(0) = 0$ and $\varphi(1) = 1$, then the image $\varphi(S) = \{\varphi(s) : s \in S\}$ should also be a possible set – i.e., we should have $\varphi(S) \in \mathcal{S}$.

F. We Want Closure under $\wedge$ and $\vee$

Another reasonable requirement is that the class $\mathcal{S}$ be closed under naturally defined operations $\wedge$ and $\vee$.

G. It Is Sufficient to Consider Closed Sets

There is one more property that is natural to assume. If, according to a set $S \in \mathcal{S}$, the values $s_1, s_2, \ldots, s_k, \ldots$ are all possible (i.e., $s_k \in S$), and the sequence s_k converges to a certain number s, then no matter how accurately we compute s, we will always find a number s_k that is indistinguishable from s (and possible). Therefore, it is natural to assume that this limit value s is also possible.

In other words, it is natural to assume that every set $S \in \mathcal{S}$ contains all its limit points, i.e., that it is a *closed* set.

H. It Is Sufficient to Consider Closed Classes of Sets

A similar requirement can be formulated for different sets $S \in \mathcal{S}$.

Indeed, on the class of all bounded closed sets, there is a natural metric – Hausdorff distance $d_H(S, S')$. This distance is defined as the smallest $\varepsilon > 0$ for which S is contained in the ε-neighborhood of S' and S' is contained in the ε-neighborhood of S. In more precise terms, the Hausdorff distance is the smallest number ε for which

$$\forall s \in S \, \exists s' \in S' \, (d(s, s') \leq \varepsilon)$$

and

$$\forall s' \in S' \, \exists s \in S \, (d(s, s') \leq \varepsilon),$$

where $d(s, s') = |s - s'|$ is the standard distance between the points on the real line.

Informally, it means that if $d_H(S, S') \leq \varepsilon$, and we only know the values $s \in S$ and $s' \in S'$ with accuracy ε, then we cannot distinguish between the sets S and S'.

So, if the sets $S_1, S_2, \ldots, S_k, \ldots$ are all possible (i.e., $S_i \in \mathcal{S}$), and the sequence of sets S_k converges to a certain set S (i.e., $d_H(S_k, S) \to 0$), then no matter how accurately we compute the values, we will always find a set S_k that is indistinguishable from the set S (and possible). Therefore, it is natural to assume that this limit set S is also possible.

In other words, it is natural to assume that the class $\mathcal{S}$ contains all its limit points, i.e., that it is a *closed* class under the Hausdorff metric.

We are now ready to formulate the main classification result.

II. MAIN RESULT

Definition 1. *A class $\mathcal{S}$ of closed non-empty subsets of the interval $[0, 1]$ is called a* set-valued extension of fuzzy logic *if it satisfies the following conditions:*

(i) *the class $\mathcal{S}$ contains all 1-element sets $\{s\}$, $s \in [0, 1]$;*
(ii) *the class $\mathcal{S}$ is closed under "and" and "or" operations (1) and (2);*
(iii) *the class $\mathcal{S}$ is closed under arbitrary automorphisms of $([0, 1], \wedge, \vee)$, i.e., if $S \in \mathcal{S}$ and $\varphi(x)$ is a strictly increasing function for which $\varphi(0) = 0$ and $\varphi(1) = 1$, then $\varphi(S) \in \mathcal{S}$; and*
(iv) *the class $\mathcal{S}$ is closed under Hausdorff metric.*

Theorem 1. *Every set-valued extension of fuzzy logic coincides with one of the following three classes:*

- *the class P of all one-point sets $\{s\}$;*
- *the class I of all subintervals $[\underline{s}, \overline{s}] \subseteq [0, 1]$ of the interval $[0, 1]$;*
- *the class C of all closed subsets S of the interval $[0, 1]$.*

Comments.

- This result shows that under reasonable conditions, every set-valued extension of fuzzy logic coincides either with the traditional fuzzy logic, or with interval-valued fuzzy logic, or with the class of all closed subsets of the interval $[0, 1]$. So, if want to go beyond traditional single-valued fuzzy sets and do not want to consider arbitrarily complex closed sets, we must use intervals. This classification result shows the importance of interval-valued fuzzy logics.
- Our proofs are similar in style to the proof from set-valued analysis; see, e.g., [1].
- For reader's convenience, all the proofs are placed in special Appendices.

III. AUXILIARY RESULTS

A. First Auxiliary Result: No Need to Require Single-Valued Fuzzy Values

Since single-valued fuzzy values are probably un-realistic, it may be not necessary to require that one-point sets belong to the class $\mathcal{S}$. It turns out that for our classification, it is not necessary to require one-point sets, it is sufficient to require that there is at least one set $S \in \mathcal{S}$ which corresponds to "pure uncertainty", i.e., does not contain 0 and does not contain 1.

Theorem 2. *Every class $\mathcal{S}$ of closed non-empty subsets of the interval $[0, 1]$ which satisfies the condition*

(i$'$) *the class $\mathcal{S}$ contains a set S for which $0 \notin S$ and $1 \notin S$,*

and conditions (ii)-(iv), coincides with one of the three classes P, I, and C described in Theorem 1.

B. A General Classification Result

A natural question is: What happens if the opposite to (i$'$) is true, i.e., if every set $S \in \mathcal{S}$ contains either 0 or 1? In this case, several other classes are possible:

Theorem 3. *Every class $\mathcal{S}$ of closed non-empty subsets of the interval $[0,1]$ which satisfies the condition*

(i$''$) *every set $S \in \mathcal{S}$ contains either 0 or 1,*

and conditions (ii)-(iv), is a union of one or more of the following classes:

- *the class consisting of a single set $\{0\}$;*
- *the class consisting of a single set $\{1\}$;*
- *the class consisting of a single interval $[0,1]$;*
- *the class I_0 of all subintervals $S \subseteq [0,1]$ which contain 0, i.e., the class of all subintervals of the type $[0,\overline{s}]$;*
- *the class I_1 of all subintervals $S \subseteq [0,1]$ which contain 1, i.e., the class of all subintervals of the type $[\underline{s},1]$;*
- *the class I_{01} of all sets $S \subseteq [0,1]$ of the type $[0,\underline{s}] \cup [\overline{s},1]$;*
- *the class C_0 of all closed subsets $S \subseteq [0,1]$ which contain 0;*
- *the class C_1 of all closed subsets $S \subseteq [0,1]$ which contain 1;*
- *the class C_{01} of all closed subsets $S \subseteq [0,1]$ which contain both 0 and 1.*

Comment. Here, one of the cases is when we have a 3-valued logic (true = 1, false = 0, and unknown = $[0,1)$) or its sublogic (including the case of classical logic $\mathcal{S} = \{\{0\},\{1\}\}$). In all other cases, we have either intervals or arbitrarily complex closed set. So, here too, if we do not want arbitrarily complex sets, we must restrict ourselves to intervals.

C. From Set-Valued to Type-2 Fuzzy Logic

Set-values fuzzy logics are a particular case of general type-2 fuzzy sets, in which a degree of confidence is itself a fuzzy set. In particular, instead of intervals, it is reasonable to consider *fuzzy numbers* as fuzzy sets, i.e., membership functions which increase to 1 and then decrease back to 0. An important case is *strictly monotonic* fuzzy numbers (e.g., triangular ones) in which the membership function continuously strictly increase to 1 and then continuously strictly decreases back to 0.

It is worth mentioning that every two such functions can be transformed into each other by an appropriate automorphism $\varphi : [0,1] \to [0,1]$, and that very other fuzzy number can be represented as a limit of strictly monotonic ones. Thus, if a class $\mathcal{S}$ of fuzzy sets contains *at least one* strictly increasing fuzzy number and that it is invariant under automorphisms and closed (in the sense of the appropriately defined Hausdorff metric), that $\mathcal{S}$ should contain *all* fuzzy numbers.

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REFERENCES

[1] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, 2006.
[2] G. Klir and B. Yuan, *Fuzzy sets and fuzzy logic: theory and applications.* Prentice Hall, Upper Saddle River, New Jersey, 1995.
[3] J. M. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper Saddle River, New Jersey, 2001.
[4] H. T. Nguyen and E. A. Walker, *First Course in Fuzzy Logic*, CRC Press, Boca Raton, Florida, 2006.
[5] C. Walker and E. Walker, "Sets with type-2 operations", *Proceedings of the 2006 NAFIPS International Conference*, Montreal, Canada, June 3–6, 2006.

APPENDIX A: PROOF OF THEOREM 1

$1°$. If the class $\mathcal{S}$ consists only of one-point sets, then we clearly have the first case of the theorem.

It is therefore sufficient to consider only classes $\mathcal{S}$ which contain at least one set which has two or more points.

$2°$. Let us first consider the case when all the sets from the class $\mathcal{S}$ are intervals. We will prove that in this case, the class $\mathcal{S}$ contains the interval $[0,1]$.

Indeed, in this case, the class $\mathcal{S}$ contains at least one non-degenerate interval $[\underline{s},\overline{s}]$, with $\underline{s} < \overline{s}$. Let us denote the midpoint of this interval by $s_0 = \dfrac{\underline{s}+\overline{s}}{2}$.

$2.1°$. Let us prove that then $\mathcal{S}$ contains the interval $[0,\overline{s}]$.

This is automatically true if $\underline{s} = 0$. If $\underline{s} > 0$, then, for every $\varepsilon \in (0,\underline{s})$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(\underline{s}) = \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, $\varphi_\varepsilon(\overline{s}) = \overline{s}$, and $\varphi_\varepsilon(1) = 1$. Then, for every ε, we get $\varphi_\varepsilon([\underline{s},\overline{s}]) = [\varepsilon,\overline{s}] \in \mathcal{S}$. In the limit $\varepsilon \to 0$, we get $[\varepsilon,\overline{s}] \to [0,\overline{s}]$. So, from the fact that the class $\mathcal{S}$ is closed, we conclude that $[0,\overline{s}] \in \mathcal{S}$.

$2.2°$. Let us now prove that then $\mathcal{S}$ contains the interval $[0,1]$.

This is automatically true if $\overline{s} = 1$. If $\overline{s} < 1$, then, for every $\varepsilon \in (0, 1-\overline{s})$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(\overline{s}/2) = \overline{s}/2$, $\varphi_\varepsilon(\overline{s}) = 1-\varepsilon$, and $\varphi_\varepsilon(1) = 1$. Then, for every ε, we get $\varphi_\varepsilon([0,\overline{s}]) = [0, 1-\varepsilon] \in \mathcal{S}$. In the limit $\varepsilon \to 0$, we get $[0,1-\varepsilon] \to [0,1]$. So, from the fact that the class $\mathcal{S}$ is closed, we conclude that $[0,1] \in \mathcal{S}$.

$2.3°$. To complete the proof for this case, we need to show that the class $\mathcal{S}$ contains an arbitrary intervals $[a,b]$ with $a \le b$.

Indeed, we have already proven that $[0,1] \in \mathcal{S}$, and by definition, the class $\mathcal{S}$ contains all 1-point sets, in particular, it contains $\{a\}$ and $\{b\}$. Since the class $\mathcal{S}$ is closed under $\wedge$ and $\vee$, we thus conclude that $[0,1] \vee \{a\} \in \mathcal{S}$. One can easily check that

$$[0,1] \vee \{a\} = \{\max(s,a) : s \in [0,1]\} = [a,1].$$

Similarly, we conclude that $[a,1] \wedge \{b\} = [a,b] \in \mathcal{S}$.

So, the class $\mathcal{S}$ contains all intervals. Since we are in the case when all its elements are intervals, the class $\mathcal{S}$ is thus the class of all subintervals of the interval $[0,1]$.

3°. Let us now consider the remaining case when the class $\mathcal{S}$ contains a closed set S which is not an interval.

3.1°. Let us prove that in this case, the class $\mathcal{S}$ contains the set $\{0, 1\}$.

Let $s^- \stackrel{\text{def}}{=} \inf S$ and $s^+ \stackrel{\text{def}}{=} \sup S$; then, by definition of $\inf S$ and $\sup S$, we have $S \subseteq [s^-, s^+]$. Since the set S is not an interval, we must have $S \neq [s^-, s^+]$, i.e., there must exist a point $s_0 \in [s^-, s^+]$ which is not contained in the set S.

Since the set S is closed, it contains it limit points s^- and s^+; thus, the point $s_0 \notin S$ must be strictly between s^- and s^+: $s^- < s_0 < s^+$. From this, we conclude that $0 < s_0 < 1$.

A complement to a closed set is open. So, since $s \notin S$, there exists a whole open interval $(\underline{s}, \overline{s})$ containing the point s which has no common point with S. Let us denote $t^- \stackrel{\text{def}}{=} \dfrac{\underline{s} + s_0}{2}$ and $t^+ \stackrel{\text{def}}{=} \dfrac{s_0 + \overline{s}}{2}$. Then, $0 < t^- < s_0 < t^+ < 1$, and $S \cap [t^-, t^+] = \emptyset$, i.e., $S \subseteq [0, t^-] \cup [t^+, 1]$, with $s^- \in S \cap [0, t^-]$ and $s^+ \in S \cap [t^+, 1]$.

For every $\varepsilon \in (0, \min(s_0, 1 - s_0))$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(t^-) = \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, $\varphi_\varepsilon(t^+) = 1 - \varepsilon$, and $\varphi_\varepsilon(1) = 1$.

Then, from $0 \leq s^- < t^-$ and $s^- \in S$, we conclude that $\varphi_\varepsilon(s^-) \leq \varepsilon$, i.e., that the set $\varphi_\varepsilon(S)$ contains a point from the interval $[0, \varepsilon]$. Similarly, from $t^+ < s^+ \leq 1$ and $s^+ \in S$, we conclude that $\varphi_\varepsilon(s^-) \geq 1 - \varepsilon$, i.e., that the set $\varphi_\varepsilon(S)$ contains a point from the interval $[1 - \varepsilon, 1]$.

Here, $\varphi_\varepsilon([0, t^-]) = [0, \varepsilon]$ and $\varphi_\varepsilon([t^+, 1]) = [1 - \varepsilon, 1]$. So, from $S \subseteq [0, t^-] \cup [t^+, 1]$, it follows that

$$\varphi_\varepsilon(S) \subseteq [0, \varepsilon] \cup [1 - \varepsilon, 1].$$

In the limit $\varepsilon \to 0$, we conclude that the sequence of sets $\varphi_\varepsilon(S)$ tends to the set $\{0, 1\}$. Thus, the class $\mathcal{S}$ indeed contains the set $\{0, 1\}$.

3.2°. Let us now prove that the class $\mathcal{S}$ contains an arbitrary finite set $\{p_1, p_2, \ldots, p_n\}$ with $0 < p_2 < \ldots < p_n$.

Indeed, from $\{0, 1\} \in \mathcal{S}$ and $\{p_n\} \in \mathcal{S}$, we conclude that $\{0, 1\} \wedge \{p_n\} = \{0, p_n\} \in \mathcal{S}$.

Let us now prove by induction over k, that $\{0, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\} \in \mathcal{S}$. Indeed, we have shown it for $k = 0$. If we have this inclusion for k, then

$$\{0, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\} \vee \{0, p_{n-k-1}\} \in \mathcal{S}.$$

Here, $0 \vee p_{n-k-1} = p_{n-k-1}$, and for every other element p_i, we have $p_i \vee p_{n-k-1} = p_i$, hence we have

$$\{0, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\} \vee \{0, p_{n-k-1}\} =$$
$$\{0, p_{n-k-1}, p_{n-k}, p_{(n-k)+1}, \ldots, p_n\} \in \mathcal{S}.$$

The statement is proven.

For $k = n - 1$, we get the conclusion $\{0, p_1, p_2, \ldots, p_n\} \in \mathcal{S}$. From this, we conclude that $\{0, p_1, p_2, \ldots, p_n\} \vee \{p_1\} \in \mathcal{S}$, and one can easily check that

$$\{0, p_1, p_2, \ldots, p_n\} \vee \{p_1\} = \{p_1, p_2, \ldots, p_n\}.$$

Thus, an arbitrary finite set indeed belongs to the class $\mathcal{S}$.

3.3°. Let us now prove that the class $\mathcal{S}$ contains an arbitrary closed set $S \subseteq [0, 1]$.

Indeed, for every ε, we can consider a finite approximation S_ε to the set S, by taking the set of all the grid points $k \cdot \varepsilon$ (with integer k) for which $[k \cdot \varepsilon, (k+1) \cdot \varepsilon] \cap S \neq \emptyset$. One can easily check that in the limit $\varepsilon \to 0$, we have $S_\varepsilon \to S$. Thus, from the fact that the class $\mathcal{S}$ contains all finite sets S_ε, we conclude that the class $\mathcal{S}$ must also contain their limit S.

The theorem is proven.

Appendix B: Proof of Theorem 2

1°. Let us first prove that the class S contains a one-point set $\{s_0\}$ for some $s_0 \in (0, 1)$.

Let us pick one of the classes $S \in \mathcal{S}$ which does not contain 0 or 1. If this class is already a one-point set, we are done, so it is sufficient to consider the case when this set is not a one-point set.

We already know that for $s^- = \inf S$ and $s^+ = \sup S$, we have $s^- \in S$, $s^+ \in S$, and $S \subseteq [s^-, s^+]$. Since $0 \notin S$ and $1 \notin S$, we thus conclude that $s^- \neq 0$ (i.e., $s^- > 0$) and $s^+ \neq 1$ (i.e., $s^+ < 1$). So, $S \subseteq [s^-, s^+]$ for some s^- and s^+ for which $0 < s^- \leq s^+ < 1$.

In the case $s^- = s^+$, the set S would be a one-point set. Since we assumed that S is not a one-point set, we have $s^- < s^+$. Let us denote the midpoint of the interval $[s^-, s^+]$ by s_0. Here, $0 < s_0 < 1$.

For every $\varepsilon \in (0, \min(s_0, 1 - s_0))$, let us construct a strictly increasing piece-wise linear function $\varphi_\varepsilon(x)$ for which $\varphi_\varepsilon(0) = 0$, $\varphi_\varepsilon(s^-) = s_0 - \varepsilon$, $\varphi_\varepsilon(s_0) = s_0$, $\varphi_\varepsilon(s^+) = s_0 + \varepsilon$, and $\varphi_\varepsilon(1) = 1$.

Then, from the fact that $S \in \mathcal{S}$ and $S \subseteq [s^-, s^+]$, we conclude that $\varphi_\varepsilon(S) \in \mathcal{S}$ and

$$\varphi_\varepsilon(S) \subseteq \varphi([s^-, s^+]) = [s_0 - \varepsilon, s_0 + \varepsilon].$$

In the limit $\varepsilon \to 0$, these sets tend to a one-point set $\{s_0\}$. Thus, the class $\mathcal{S}$ indeed contains a one-point set $\{s_0\}$.

2°. Let us now prove that the class S contains all one-point sets $\{s\}$ for which $s \in (0, 1)$.

Indeed, for every $s \in (0, 1)$, we can construct a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s_0) = s$, and $\varphi(1) = 1$. For this function $\varphi(x)$, we have $\varphi(\{s_0\}) = \{s\}$, so indeed $\{s\} \in \mathcal{S}$.

3°. Next, let prove that the class S contains all one-point sets $\{s\}$.

After Part 2 of this proof, the only missing one-point sets are $\{0\}$ and $\{1\}$. The first set can be represented as a limit of sets $\{1/n\} \in \mathcal{S}$ and is, thus, also an element of the class $\mathcal{S}$. The second set $\{1\}$, in its turn, is the limit of sets $\{1 - 1/n\} \in \mathcal{S}$, so we also have $\{1\} \in \mathcal{S}$.

Thus, the condition (i) is satisfied, and the result follows from Theorem 1.

Appendix C: Proof of Theorem 3

1°. One can easily check that an arbitrary union of the above classes indeed satisfies conditions (i′) and (ii)-(iv).

2°. Let us first consider the case when we have a set $S \in \mathcal{S}$ that contains both 0 and 1 but is different from the interval $[0, 1]$. Since the set $S \in \mathcal{S}$ is different from $[0, 1]$ and contain 0 and 1, it must have "holes", i.e., non-empty complement $-S$.

2.1°. If every such set $S \in \mathcal{S}$ has only one hole, i.e., if its complement is a connected interval, then all such sets have the form $[0, a] \cup [b, 1]$. By applying appropriate $\varphi(x)$, we can show that with each set of this type, every other set of this type also belongs to $\mathcal{S}$ – and in the limit when $a_n \to b$, we conclude that the entire interval $[0, 1]$ also belongs to $\mathcal{S}$. So, in this case, we have the class of all sets of the type $[0, \underline{s}] \cup [\overline{s}, 1]$.

2.2°. The only remaining situation is when there is a set $S \in \mathcal{S}$ that contains 0, 1, and at least two holes, i.e., for which $S \subseteq [0, s_1] \cup [s_2, s_3] \cup [s_4, 1]$ for some values $0 \le s_1 < s_2 \le s_2 < s_4 \le 1$ for which $0 \in S$, $S \cup [s_2, s_3] \neq \emptyset$, and $1 \in S$.

In this case, by using appropriate functions $\varphi_\varepsilon(x)$, we "compress" the interval $[0, s_1]$ into a single point 0, the interval $[s_4, 1]$ into a single point 1, and the interval $[s_2, s_3]$ into a single midpoint $s_0 \in (0, 1)$. Thus, we conclude that a 3-point set $S_0 \stackrel{\text{def}}{=} \{0, s_0, 1\}$ belongs to the class $\mathcal{S}$.

For an arbitrary value $s \in (0, 1)$, by using a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s_0) = s$, and $\varphi(1) = 1$, we can now conclude that $\varphi(S_0) = \{0, s, 1\} \in \mathcal{S}$.

For $s = 0$ and $s = 1$, we can take a limit and thus conclude that $\{0, s, 1\} \in \mathcal{S}$ for all values $s \in [0, 1]$.

It is easy to check that for every two sets A and A',

$$(\{0, 1\} \cup A) \vee (\{0, 1\} \cup A') = \{0, 1\} \cup (A \cup A'). \quad (3)$$

Indeed, by definition of the set "or" operation, every element of $S \vee S'$ has the form $s \vee s' = \max(s, s')$ for some $s \in S$ and $s' \in S'$ and is, thus, equal either to $s \in S$ or to $s' \in S'$. Thus, every element of the set $S \vee S'$ belongs to the union $S \cup S'$. On the other hand, every element $s \in S$ can be represented as $s \vee 0$, and every element $s' \in S'$ as $0 \vee s'$ – hence every element of the union indeed belongs to $A \vee A'$.

We start with sets $\{0, s, 1\}$ which correspond to 1-element sets $A = \{s\}$. An arbitrary finite set can be represented as a union of its one-element subsets. Thus, due to the equality (3), we can conclude that $\mathcal{S}$ contains sets $\{0, 1\} \cup A$ for an arbitrary finite A – i.e., that $\mathcal{S}$ contains an arbitrary finite set which contains 0 and 1.

Since every closed set can be represented as a limit of finite sets, in the limit, we conclude that $\mathcal{S}$ contains an arbitrary closed set which contains 0 and 1, i.e., $C_{01} \subseteq \mathcal{S}$.

3°. If we have a set $S_0 \neq [0, 1]$ that contains 0, does not contain 1, and is different from $\{0\}$, then we also have two possibilities: either all such sets are intervals, or one of them is not an interval.

3.1°. Let us first consider a situation in which all such sets $S \in \mathcal{S}$ are intervals. Since $0 \in S$, they can only be intervals of type $[0, s]$. In particular, the interval S_0 (whose existence we have just assumed) is different from $\{0\}$ and does not contain 1, so it has the form $[0, s_0]$ for some $s_0 \in (0, 1)$.

By using an appropriate $\varphi(x)$, we conclude that every interval of the type $[0, s]$ with $s \in (0, 1)$ also belongs to $\mathcal{S}$. By taking a limit, we deduce that $\mathcal{S}$ contains all intervals $[0, s]$, i.e., that $I_0 \subseteq \mathcal{S}$.

3.2°. Let us now consider a situation in which there exist a non-interval set $S \in \mathcal{S}$ which contains 0 but does not contain 1.

As we have shown earlier, $S \subseteq [s^-, s^+]$, with $s^- = \inf S \in S$ and $s^+ = \sup S \in S$. Since $1 \notin S$, we have $s^+ < 1$; since $S \neq \{0\}$, we have $0 < s^+$. Due to the fact the set S is not an interval, it must have a hole, i.e., we have $S \subseteq [0, s_1] \cup [s_2, s^+]$ for some valued $0 \le s_1 < s_2 \le s^+ < 1$ for which $0 \in S$ and $s^+ \in S$.

By using appropriate functions $\varphi_\varepsilon(x)$, we "compress" the interval $[0, s_1]$ into a single point 0 and the interval $[s_2, s^+]$ into a single point $s^+ \in (0, 1)$. Thus, we conclude that a 2-point set $S_0 \stackrel{\text{def}}{=} \{0, s^+\}$ belongs to the class $\mathcal{S}$.

For an arbitrary value $s \in (0, 1)$, by using a strictly increasing piece-wise linear function $\varphi(x)$ for which $\varphi(0) = 0$, $\varphi(s^+) = s$, and $\varphi(1) = 1$, we can now conclude that $\varphi(S_0) = \{0, s\} \in \mathcal{S}$.

For $s = 0$ and $s = 1$, we can take a limit and thus conclude that $\{0, s\} \in \mathcal{S}$ for all values $s \in [0, 1]$.

It is easy to check that for every two sets A and A',

$$(\{0\} \cup A) \vee (\{0\} \cup A') = \{0\} \cup (A \cup A'). \quad (4)$$

We start with sets $\{0, s\}$ which correspond to 1-element sets $A = \{s\}$. An arbitrary finite set can be represented as a union of its one-element subsets. Thus, due to the equality (4), we can conclude that $\mathcal{S}$ contains sets $\{0\} \cup A$ for an arbitrary finite A – i.e., that $\mathcal{S}$ contains an arbitrary finite set which contains 0. In the limit, we conclude that $\mathcal{S}$ contains an arbitrary closed set which contains 0, i.e., that $C_0 \subseteq \mathcal{S}$.

4°. If we have a set $S_0 \neq [0, 1]$ that contains 1, does not contain 0, and is different from $\{1\}$, then we can use a similar argument to conclude that either $I_1 \subseteq \mathcal{S}$ or $C_1 \subseteq \mathcal{S}$. The only difference is that instead of (4), we must use a dual formula

$$(\{1\} \cup A) \wedge (\{1\} \cup A') = \{1\} \cup (A \cup A'). \quad (5)$$

The theorem is proven.