Special Relativity-Type Space-Times Naturally Lead to Quasi-Pseudometrics

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Abstract

The standard 4-dimensional space-time of special relativity is based on the 3-dimensional Euclidean metric. In 1967, H. Busemann showed that similar space-time models can be based on an arbitrary metric space. In this paper, we search for the broadest possible generalization of a metric under which Einstein's construction leads to a physically reasonable space-time model. It turns out that this broadest possible generalization is related to the known notion of a quasi-pseudometric.

1 Space-Time of General Relativity and Its Natural Generalization

Space-time of special relativity. Before Einstein, it was usually assumed that in principle, we can have arbitrarily fast physical processes. This assumption led to the following simple description of causality between events. Each event (t, x) can be described by its time t and its location x. So, if an event (s, y) corresponds to a later moment of time $s > t$, then, in principle (irrespective of how far the corresponding spatial points x and y are from each other), the event (t, x) can causally influence the event (s, y); we will denote causal relation by $(t, x) \preceq (s, y)$.

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In other words, in pre-Einstein Newtonian physics, the causality relation $\preceq$ can be described as follows: $(t,x) \preceq (s,y)$ if and only if either $t < s$, or $t = s$ and $x = y$.

In his 1905 Special Relativity Theory, Einstein postulated that the velocities of all physical processes are limited by the speed of light c. As a result of this limitation, for an event (t,x) to be able to influence the event (s,y), we must have not only $t \le s$, we must also make sure that during the time $s-t$ the influence can indeed cover the distance between the spatial points x and y, i.e., that $s - t \ge \frac{d(x,y)}{c}$.

This condition can be described in an even simpler form if we change the units for measuring space and/or time in such a way that in the new units, the speed of light is equal to 1. For example, as theoretical physicists often do, we can use "light seconds" to measure distance, or use the time 1 m/c (during which the light covers 1 meter) as a new unit of time. In such units, since $c = 1$, the causality relation of special relativity takes the simplified form: $s - t \ge d(x,y)$.

Let us describe this relation in precise terms.

Before we give an exact definition, we should mention that at the time of the Special Relativity, for Einstein, $d(x,y)$ meant the standard Euclidean distance. However, later Einstein himself started considering curved (non-Euclidean) spaces and curved space-times – which eventually led to his General Relativity Theory. With this in mind, let us present Einstein's causality in its most general form.

In the following text, $\mathbb{R}$ will denote the set of all real numbers, and $\mathbb{R}_0^+$ will denote the set of all non-negative real numbers.

Definition 1 *Let X be a set, and let $d : X \times X \to \mathbb{R}$ be a function. By a* causality relation*, we mean the following relation $\preceq$ between points of the Cartesian product $\mathbb{R} \times X$:*

$$(t,x) \preceq (s,y) \leftrightarrow s - t \ge d(x,y). \qquad (1)$$

Possible generalizations. Einstein considered this definition for $X = \mathbb{R}^3$ and Euclidean metric d. In 1967, Busemann analyzed the case when X is a general metric space with a metric d [1]; see also [9, 13].

A natural question is: what are the conditions on the functions d under which the above causality relation $\preceq$ is physically reasonable, e.g., is a (partial) pre-order (= reflexive and transitive relation)?

Main result. Let us provide the full characterization of such functions d.

Proposition 1 *For a set X and a function $d : X \times X \to \mathbb{R}$, the following two statements are equivalent to each other:*

- *the causality relation (1) is a pre-order;*
- *the function d satisfies the following two conditions:*

(a) $d(x,x)=0$ *for all* x*;*

(b) $d(x,z)\le d(x,y)+d(y,z)$ *for all* x*,* y*, and* z*.*

Comments. These conditions are similar to the conditions that define a metric, but with two differences:

- first, in contrast to metric, $d(x,y)=0$ does not necessarily imply $x=y$;
- second, unlike metric, the function $d(x,y)$ does not have to be symmetric.

If we add symmetry, then the above conditions would automatically imply that $d(x,y)\ge 0$ for all x and y: indeed, we would have $0=d(x,x)\le d(x,y)+d(y,x)=2d(x,y)$. Without symmetry, we can only conclude that $d(x,y)+d(y,x)\ge 0$, but the function d is not necessarily non-negative.

Non-negative functions d which satisfy the properties (a) and (b) are well-known: they are called *quasi-pseudometrics*; see, e.g., [3, 5, 11, 12, 14]. These functions are used as a natural asymmetric generalization of metrics in optimization problems, when we must describe, e.g., a cost (or time) $d(x,y)$ of going from x to y. For example, if x is downhill from y, the cost $d(x,y)$ of going from x to y is different from the cost $d(y,x)$ from going from y to x.

We can formulate the following Corollary to the above Proposition:

Corollary 1 *The causality relation generated by a non-negative function* d *is a pre-order if and only if* d *is a quasi-pseudometric.*

This corollary reveals the importance of quasi-pseudometric spaces in space-time geometry.

Comments. We use quasi-pseudometrics on the proper space. This use generalizes the use of a standard metric on a 3-dimensional proper space to describe the *causality structure* of a 4-dimensional space-time of Special Relativity. It is worth mentioning that in [4, 6, 7], another use of quasi-pseudometrics in space-time geometry has been proposed: to describe the *topology* of the space-time as a whole.

In several recent investigations belonging to the computational theory of (generalized) metric spaces the *poset of formal balls* plays an important role (see e.g. [8] and [10]). We recall here that a *formal ball* of a metric space (X,d) is a pair (r,x) with $r\in \mathbb{R}_0^+$ and $x\in X$, and that the set of all formal balls of X is partially ordered by $(r,x)\sqsubseteq(s,y)\Leftrightarrow d(x,y)\le r-s$. Obviously these concepts are closely related to our definition of a causality relation given above.

Proof of Proposition 1.

1°. Let us first show that if $\preceq$ is a pre-order, then d satisfies the conditions (a) and (b).

We first prove the triangle inequality (b). Let us take any points x, y, and z, and prove that $d(x,z)\le d(x,y)+d(y,z)$. For that, we take $t=d(x,y)$ and

$s = d(y,z)$. Then, by the formula (1), we conclude that $(0,x) \preceq (t,y)$, and $(t,y) \preceq (t+s,z)$. Due to transitivity, we have $(0,x) \preceq (t+s,z)$, i.e., due to the formula (1), $d(x,z) \le t+s$. By definition of t and s, this leads to the desired inequality $d(x,z) \le d(x,y) + d(y,z)$.

Let us now prove the condition (a). By using the triangle inequality for $x = y = z$, we conclude that $d(x,x) \le d(x,x) + d(x,x)$, hence $d(x,x) \ge 0$. Now, the reflexivity condition means that $(t,x) \preceq (t,x)$ for every t and every x, i.e., in view of the formula (1), that $t - t = 0 \ge d(x,x)$. Since $d(x,x) \ge 0$ and $d(x,x) \le 0$, we conclude that $d(x,x) = 0$.

2°. Conversely, let us now show that if d satisfies the conditions (a) and (b), then (1) is a pre-order.

Indeed, for every t and x, we have $d(x,x) = 0$, hence $t - t \ge d(x,x)$ and $(t,x) \preceq (t,x)$.

Let us now prove transitivity. We assume that $(t,x) \preceq (s,y) \preceq (q,z)$. By definition (1), this means that $s - t \ge d(x,y)$ and $q - s \ge d(y,z)$. By adding these two inequalities, we conclude that $q - t \ge d(x,y) + d(y,z)$. Since d satisfies the triangle inequality $d(x,y) + d(y,z) \ge d(x,z)$, hence $q - t \ge d(x,z)$ and $(t,x) \preceq (q,z)$.

The proposition is proven.

What if we require causality to be an order, not just a pre-order? If we require the causality relation (1) to be a (partial) *order*, i.e., to satisfy the additional property that $e \preceq e'$ and $e' \preceq e$ imply $e = e'$, then we get the following result:

Proposition 2 *For a set X and a function $d : X \times X \to \mathbb{R}$, the following two statements are equivalent to each other:*

- *the causality relation (1) is an order;*
- *the function d satisfies the following three conditions:*
  - (a) $d(x,x) = 0$ *for all x;*
  - (b) $d(x,z) \le d(x,y) + d(y,z)$ *for all x, y, and z;*
  - (c) *if $d(x,y) + d(y,x) = 0$, then $x = y$.*

Comment. In theoretical computer science, non-negative functions d which satisfy these properties are usually called *quasi-metrics*. It is worth mentioning that in mathematics, a "quasi-metric" is often assumed also to satisfy a condition stronger than (c), namely that $d(x,y) = 0$ implies that $x = y$.

Proof. Let us first show that the order requirement leads to the condition (c).

Indeed, if $d(x,y) + d(y,x) = 0$, then we get $(0,x) \preceq (d(x,y),y)$ and $(d(x,y),y) \preceq (0,x)$, hence $(0,x) = (d(x,y),y)$ and, therefore, $x = y$.

Conversely, let us assume that the condition (c) is satisfied; we then prove that the relation $\preceq$ is indeed an order. Indeed, let $(t,x)\preceq(s,y)$ and $(s,y)\preceq(t,x)$. Let us prove that then $t=s$ and $x=y$.

By definition (1), this means that $s-t\geq d(x,y)$ and $t-s\geq d(y,x)$. By adding these two inequalities, we conclude that $0\geq d(x,y)+d(y,x)$. On the other hand, from the triangle inequality (b), we conclude that $0=d(x,x)\leq d(x,y)+d(y,x)$, hence $d(x,y)+d(y,x)=0$. Due to property (c), we thus get $x=y$. Hence $d(x,y)=d(y,x)\geq 0$ and $t=s$. The proposition is proven.

2 Symmetries Naturally Lead to the Corresponding Space-Times

In the previous section, we described a class of space-times that we obtained as a result of a rather mathematically sounding generalization. Let us show that this class of space-times has a direct physical meaning. Indeed, in a space-time model $\mathbb{R}\times X$ with the causal relation (1), a temporal shift $T_{t_0}:(t,x)\to(t+t_0,x)$ preserves causality. Such shifts form a 1-parametric symmetry group, in the sense that T_0 is an identity map, and $T_t\circ T_s=T_{t+s}$ for all t and s, where $\circ$ denotes the composition of the two maps.

We will show that, conversely, under reasonable assumptions, every space-time that allows a 1-parametric group of "temporal shifts" can be represented in a form (1).

Definition 2 *Let $(E,\preceq)$ be a pre-ordered set. The set E is called a* space-time*, its elements are called* events*, and the relation $\preceq$ is called* causality relation *(or simply* causality*, for short).*

- *We say that a map $T:E\to E$ is* causality-preserving *if for every two events e and e', $e\preceq e'$ if and only if $T(e)\preceq T(e')$.*
- *We say that a map $T:E\to E$ is a* positive temporal shift *if $e\preceq T(e)$ and $T(e)\not\preceq e$ for all events e.*
- *We say that a map $T:E\to E$ is a* negative temporal shift *if $T(e)\preceq e$ and $e\not\preceq T(e)$ for all events e.*

Let us assume that for every real number $t\in\mathbb{R}$, there is a map $T_t:E\to E$.

- *We say that the maps T_t form a* 1-parametric group *if*
  - *T_0 is an identity map, i.e., $T_0(e)=e$ for all $e\in E$, and*
  - *for every t and s, $T_t\circ T_s=T_{t+s}$.*
- *We say that the space-time E is* closed *under T_t when the following property holds for every two events e and e':*
  - *if $t_n\to t$ and for every n, $e\preceq T_{t_n}(e')$, then $e\preceq T_t(e')$.*

- *We say that the space-time E is* connected *under the group T_t if for every $e, e' \in E$, there exists a t for which $e \preceq T_t(e')$.*

Proposition 3 *Let $(E, \preceq)$ be a pre-ordered set, and let T_t be a 1-parametric group of causality-preserving transformations such that:*

- *for every $t > 0$, the transformation T_t is a positive temporal shift, and*
- *the space-time E is closed and connected under T_t.*

Then, there exist a set X and a function $d : X \times X \to \mathbb{R}$ that satisfies the conditions (a)–(b) such that the pre-ordered set $(E, \preceq)$ is isomorphic to the Cartesian product $\mathbb{R} \times X$ with the order (1).

Comment. For ordered sets, a similar proposition holds, with a function d satisfying the additional condition (c).

Proof.

1°. Let us first define the set X.

Since the maps T_t form a group, the relation $e \sim e' \stackrel{\text{def}}{=} \exists t\, (T_t(e) = e')$ is an equivalence relation (meaning that e and e' belong to the same orbit of this group). Indeed:

- The property $e \sim e$ holds for $t = 0$.
- If $e \sim e'$, then $T_t(e) = e'$ for some t, hence $T_{-t}(e') = e$ hence $e' \sim e$.
- Finally, if $e \sim e'$ and $e' \sim e''$, this means that $e' = T_t(e)$ and $T_s(e') = e''$ for some t and s, hence $T_s(T_t(e)) = T_{t+s}(e) = e''$ and $e \sim e''$.

As the set X, we will take the factor-set $E/\sim$, i.e., the set of all equivalence classes with respect to the equivalence relation $\sim$.

2°. Let us now define the function d. To do that, in each equivalence class $x \in E/\sim$, we select an element; we will denote an element corresponding to the class x by $\bar{x}$. Now, we can define $d(x, y)$ as follows:

$$d(x, y) \stackrel{\text{def}}{=} \inf\{t : \bar{x} \preceq T_t(\bar{y})\}.$$

3°. Let us prove that the value $d(x, y)$ is finite for all x and y, and that $d(x, y)$ indeed satisfies conditions (a)-(b).

3.1°. We first prove that for every x and y, the above-defined value $d(x, y)$ is finite.

Indeed, since E is connected, there exists a number t for which $\bar{x} \preceq T_t(\bar{y})$, hence the set $\{t : \bar{x} \preceq T_t(\bar{y})\}$ is non-empty, and $d(x, y) < +\infty$.

On the other hand, due to the same connectedness, there exists a real number s for which $\bar{y} \preceq T_s(\bar{x})$. Since T_{-s} is a causality-preserving map, we conclude that $T_{-s}(\bar{y}) \preceq T_{-s}(T_s(\bar{x})) = \bar{x}$. So, if $t < -s$, we cannot have $\bar{x} \preceq T_t(\bar{y})$ because otherwise we would have $T_{-s}(\bar{y}) \preceq \bar{x} \preceq T_t(\bar{y})$ and $T_{-s}(\bar{y}) \preceq T_t(\bar{y})$. Thus, for $e \stackrel{\rm def}{=} T_t(\bar{y})$, we would have $T_{-s-t}(e) \preceq e$ with $(-s) - t > 0$ – which contradicts to our assumption that transformations T_τ with $\tau > 0$ are positive temporal shifts. Thus, the set $\{t : \bar{x} \preceq T_t(\bar{y})\}$ cannot contain any values smaller than $-s$; so for its infimum $d(x, y)$, we get $d(x, y) \geq -s > -\infty$.

Thus, the value $d(x, y)$ is always finite.

3.2°. Let us now prove that $d(x, x) = 0$ for all x.

Indeed, for $t = 0$, we have $\bar{x} \preceq T_0(\bar{x}) = \bar{x}$, hence the set $\{t : \bar{x} \preceq T_t(\bar{x})\}$ contains 0.

On the other hand, for every negative t, i.e., for every $t = -s$ for some $s > 0$, the map T_s is a positive temporal shift, hence $T_s(\bar{x}) \not\preceq \bar{x}$. Since $T_t = T_{-s}$ is a causality-preserving transformation, we conclude that $\bar{x} = T_t(T_s(\bar{x})) \not\preceq T_t(\bar{x})$. Thus, the set $\{t : \bar{x} \preceq T_t(\bar{x})\}$ cannot contain any negative numbers.

Since the set $\{t : \bar{x} \preceq T_t(\bar{x})\}$ contains only non-negative numbers and contains 0, 0 is clearly its infimum, so $d(x, x) = 0$.

3.3°. Let us now prove that $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, and z.

Indeed, by definition of an infimum, for every $\varepsilon > 0$, there exists a number $t \leq d(x, y) + \varepsilon$ for which $\bar{x} \preceq T_t(\bar{y})$. Similarly, there exists a number $s \leq d(y, z)+\varepsilon$ for which $\bar{y} \preceq T_s(\bar{z})$. Since T_t is a causality-preserving transformation, we conclude that $T_t(\bar{y}) \preceq T_t(T_s(\bar{z})) = T_{t+s}(\bar{z})$ and, by transitivity of pre-order, that $\bar{x} \preceq T_{t+s}(\bar{z})$. Hence, the infimum $d(x, z)$ of the set $\{u : \bar{x} \preceq T_u(\bar{z})\}$ cannot exceed $t+s$: $d(x, z) \leq t+s$. Since $t \leq d(x, y)+\varepsilon$ and $s \leq d(y, z)+\varepsilon$, we conclude that $d(x, z) \leq d(x, y) + d(y, z) + 2\varepsilon$. This is true for every $\varepsilon > 0$, hence in the limit $\varepsilon \to 0$, we get the desired triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$.

4°. We have proven that X indeed satisfies conditions (a) and (b). Let us now prove that E is indeed isomorphic to the Cartesian product $\mathbb{R} \times X$ with the pre-order (1).

Specifically, we will prove that the map $(t, x) \to T_t(\bar{x})$ is the desired isometry.

4.1°. First, we will prove that this map is an injection, i.e., that different pairs $(t, x) \neq (s, y)$ get mapped into different events $T_t(\bar{x}) \neq T_s(\bar{y})$.

By definition of the relation $\sim$, for each t, the event $T_t(\bar{x})$ belongs to the same equivalence class as $\bar{x}$, i.e., to the equivalence class x. So, if $x \neq y$, this means that x and y are different equivalence classes, and thus, $T_t(\bar{x}) \in x$ cannot be equal to $T_s(\bar{y}) \in y$.

To complete the proof of injectivity, it is therefore sufficient to consider the case when $x = y$ and $t \neq s$. Without losing generality, we can assume that $t < s$. In this case, $T_s(\bar{x}) = T_{s-t}(T_t(\bar{x}))$; since T_{s-t} is a positive temporal shift, we conclude that $T_s(\bar{x}) \not\preceq T_t(\bar{x})$, in particular, that $T_s(\bar{x}) \neq T_t(\bar{x})$. Injectivity is proven.

4.2°. Let us now prove that the map $(t, x) \to T_t(\bar{x})$ is surjective, i.e., that for every $e \in E$, there exist t and x for which $e = T_t(\bar{x})$.

Indeed, let x be the equivalence class that contains the event e. Since $\bar{x}$ denotes the selected event from this equivalence class, we have $e \sim \bar{x}$. By definition of the relation $\sim$, this means that $e = T_t(\bar{x})$ for some real number t. Surjectivity is proven.

4.3°. To complete the proof, we must show that the original causality relation on the space-time E coincides with the relation (1), i.e., that $T_x(\bar{x}) \preceq T_s(\bar{y})$ if and only if $s - t \geq d(x, y)$.

4.3.1°. Let us first prove that if $T_t(\bar{x}) \preceq T_s(\bar{y})$, then $s - t \geq d(x, y)$.

Indeed, let $T_t(\bar{x}) \preceq T_s(\bar{y})$. Since T_{-t} is a causality-preserving transformation, we conclude that $T_{-t}(T_t(\bar{x})) \preceq T_{-t}(T_s(\bar{y}))$, i.e., that $\bar{x} \preceq T_{s-t}(\bar{y})$. By definition of $d(x, y)$ as the infimum of the set $\{u : \bar{x} \preceq T_u(\bar{y})\}$, this means that $d(x, y) \leq s - t$.

4.3.2°. Let us now prove that if $s - t \geq d(x, y)$, then $T_t(\bar{x}) \preceq T_s(\bar{y})$.

Indeed, by definition of $d(x, y)$ as the infimum, for every n, there exists a value τ_n such that $d(x, y) \leq \tau_n \leq d(x, y) + 1/n$ and $\bar{x} \preceq T_{\tau_n}(\bar{y})$. Since T_t is a causality-preserving transformation, we conclude that $T_t(\bar{x}) \preceq T_{t+\tau_n}(\bar{y})$. When $n \to \infty$, we have $\tau_n \to d(x, y)$ hence $t + \tau_n \to t + d(x, y)$. Due to the closedness of the space-time E, we thus conclude that $T_t(\bar{x}) \preceq T_{t+d(x,y)}(\bar{y})$.

If $s - t = d(x, y)$, then $t + d(x, y) = s$ and we have the desired causality relation. If $s - t > d(x, y)$, i.e., if $\Delta \stackrel{\text{def}}{=} s - t - d(x, y) > 0$, then T_Δ is a positive temporal shift hence $T_{t+d(x,y)}(\bar{y}) \preceq T_\Delta(T_{t+d(x,y)}(\bar{y})) = T_{\Delta+t+d(x,y)}(\bar{y}) = T_s(\bar{y})$. By transitivity, we thus get $T_t(\bar{x}) \preceq T_s(\bar{y})$.

The isomorphism is proven, hence the proposition is proven.

Comments.

- In the above proof, the definition of the function $d(x, y)$ depended on the selection of an element $\bar{x}$ in each corresponding class x. If we select a different element $\bar{x}'$ in each class, then, in general, we end up with a different function $d'(x, y)$. How are these functions related?

  Since for every class x, elements $\bar{x}$ and $\bar{x}'$ belong to the same equivalence class, there exists a value $t(x)$ depending on x for which $T_{t(x)}(\bar{x}) = \bar{x}'$. Similarly to the above proof, we can conclude that this value $t(x)$ is uniquely determined. From $T_{t(x)}(\bar{x}) = \bar{x}'$, $T_{t(y)}(\bar{y}) = \bar{y}'$, and the fact that the maps T_t form a group of causality-preserving maps, we can conclude that $\bar{x} \preceq T_t(\bar{y})$ is equivalent to

  $$\bar{x}' = T_{t(x)}(\bar{x}) \preceq T_{t(x)}(T_t(\bar{y})) = T_{t(x)+t}(T_{-t(y)}(\bar{y}')) = T_{t+t(x)-t(y)}(\bar{y}').$$

  Thus, from the definition of d, we now conclude that

  $$d'(x, y) = d(x, y) + t(x) - t(y).$$

  Vice versa, for every function $t(x)$, we can select new elements $T_{t(x)}(\bar{x}) = \bar{x}'$ in each class, and for this selection, the resulting function d' will have the above form.

If one of these functions d is symmetric, that is $d(x,y) = d(y,x)$, then $d'(x,y) = d(x,y) + t(x) - t(y)$ is only symmetric for $t(x) = \text{const}$; in this case, $d'(x,y) = d(x,y)$. Thus, in contrast to the general case, where a function d is not uniquely determined, a symmetric function d is determined uniquely.

- If we do not assume that the space-time is connected, then we get similar results but with a function $d(x,y)$ that can attain infinite values.

3 From Causality to Kinematic Metric

Einstein did not just provide the description of the causality relation $\preceq$. For the case when $(t,x) \preceq (s,y)$, he also explained how we can quantify the amount of proper time that it takes for an inertial particle starting at the spatial point x at moment t to reach the point y at moment $s > t$. The corresponding value $\tau((t,x),(s,y))$ – sometimes called *kinematic metric* – is described by the well-known formula:

$$\tau((t,x),(s,y)) = \sqrt{(s-t)^2 - d^2(x,y)}. \tag{2}$$

This proper time satisfies the following "anti-triangle" inequality:

$$\text{if } e \preceq e' \preceq e'', \text{ then } \tau(e,e'') \geq \tau(e,e') + \tau(e',e''). \tag{3}$$

This inequality makes perfect physical sense. Indeed, it is known that, according to special relativity, the time slows down when we travel with a large speed; the closer this speed to the speed of light, the slower the time. Thus, we can reach e'' from e in almost 0 proper time if we travel with a speed close to the speed of light. The longest time is when we do not travel at all, i.e., if we keep an inertial motion without any accelerations and decelerations. In other words, if we follow a single inertial path, the resulting proper time $\tau(e,e'')$ is longer than (or equal to) the time $\tau(e,e') + \tau(e',e'')$ needed for a two-segment path $e \to e' \to e''$.

Busemann has shown [1] that this anti-triangle inequality holds for an arbitrary metric $d(x,y)$; moreover, it holds for a more general formula

$$\tau((t,x),(s,y)) = \sqrt[\alpha]{(s-t)^\alpha - d^\alpha(x,y)}, \tag{4}$$

where $\alpha \geq 1$ is a fixed real number. It is natural to ask whether this inequality holds for (non-negative) quasi-pseudometrics as well. The answer is "yes":

Proposition 4 *For every quasi-pseudometric d and for every $\alpha \geq 1$, the kinematic metric (4) satisfies the anti-triangle inequality (3).*

This result provides one more argument that quasi-pseudometrics are important in the analysis of space-time models.

Proof. This proof is similar to the proof given in [1]. Let $(t_1,x_1)\preceq(t_2,x_2)\preceq(t_3,x_3)$. Let us denote $d_1\stackrel{\rm def}{=}d(x_1,x_2)$, $d_2\stackrel{\rm def}{=}d(x_2,x_3)$, $d\stackrel{\rm def}{=}d(x_1,x_3)$, $\tau_1\stackrel{\rm def}{=}\sqrt[\alpha]{(t_2-t_1)^\alpha-d_1^\alpha}$, $\tau_2\stackrel{\rm def}{=}\sqrt[\alpha]{(t_3-t_2)^\alpha-d_2^\alpha}$, and $\tau\stackrel{\rm def}{=}\sqrt[\alpha]{(t_3-t_1)^\alpha-d^\alpha}$. From these definitions, we conclude that $t_2-t_1=\sqrt[\alpha]{\tau_1^\alpha+d_1^\alpha}$ and $t_3-t_2=\sqrt[\alpha]{\tau_2^\alpha+d_2^\alpha}$.

By adding these two equalities, we get

$$t_3-t_1=\sqrt[\alpha]{\tau_1^\alpha+d_1^\alpha}+\sqrt[\alpha]{\tau_2^\alpha+d_2^\alpha}. \qquad (5)$$

We know that for every $\alpha\ge 1$, the l^p-expression $\|(\tau,d)\|\stackrel{\rm def}{=}\sqrt[\alpha]{\tau^\alpha+d^\alpha}$ is a norm, i.e., $\|(\tau_1+\tau_2,d_1+d_2)\|\le\|(\tau_1,d_1)\|+\|(\tau_2,d_2)\|$, or, equivalently,

$$\sqrt[\alpha]{(\tau_1+\tau_2)^\alpha+(d_1+d_2)^\alpha}\le\sqrt[\alpha]{\tau_1^\alpha+d_1^\alpha}+\sqrt[\alpha]{\tau_2^\alpha+d_2^\alpha}. \qquad (6)$$

Combining (5) and (6), we conclude that

$$t_3-t_1\ge\sqrt[\alpha]{(\tau_1+\tau_2)^\alpha+(d_1+d_2)^\alpha}. \qquad (7)$$

Raising both sides of this inequality to the power α, we get

$$(t_3-t_1)^\alpha\ge(\tau_1+\tau_2)^\alpha+(d_1+d_2)^\alpha.$$

By moving $(d_1+d_2)^\alpha$ to the left-hand side, we get

$$(t_3-t_1)^\alpha-(d_1+d_2)^\alpha\ge(\tau_1+\tau_2)^\alpha. \qquad (8)$$

Due to the triangle inequality, $d\stackrel{\rm def}{=}d(x_1,x_3)\le d(x_1,x_2)+d(x_2,x_3)=d_1+d_2$, hence $d^\alpha\le(d_1+d_2)^\alpha$. So, from the equation (8), we conclude that

$$(t_3-t_1)^\alpha-d^\alpha\ge(\tau_1+\tau_2)^\alpha.$$

By taking the α-th root of both sides, we conclude that

$$\sqrt[\alpha]{(t_3-t_1)^\alpha-d^\alpha}\ge\tau_1+\tau_2,$$

i.e., that $\tau\ge\tau_1+\tau_2$ – the desired anti-triangle inequality.

The proposition is proven.

This result can be further generalized. Namely, H. Busemann, in [1], generalizes the construction (4) from the case when the first component of the Cartesian product is the real line to the more general case when this first component is an arbitrary space-time. Let us describe this construction in detail.

Let $(E,\preceq)$ be a pre-ordered set, and let $\tau(e,e')$ be a function that is defined for all pairs (e,e') for which $e\preceq e'$, and that satisfies the property (3). Let (X,d) be a metric space. Then, on the Cartesian product $E\times X$, we can define a pre-order relation as follows:

$$(e,x)\preceq(e',x')\leftrightarrow e\preceq e'\,\&\,(x=x'\vee(x\ne x'\,\&\,\tau(e,e')\ge d(x,x'))). \qquad (9)$$

For this relation, for every $\alpha \geq 1$, the expression

$$\tau_\alpha((e,x),(e',x')) = \sqrt[\alpha]{\tau^\alpha(e,e') - d^\alpha(x,x')} \tag{10}$$

also satisfies the anti-triangle inequality (3).

Let us prove that this result can be extended to the case when d is a quasi-pseudometric.

Proposition 5 *Let $(E, \preceq)$ be a pre-ordered set, and let $\tau(e,e')$ be a non-negative function that is defined for all pairs (e,e') for which $e \preceq e'$, and that satisfies the property (3). Let (X,d) be a quasi-pseudometric space. Then, on the Cartesian product $E \times X$, the formula (9) defines a pre-order, and the expression (10) satisfies the anti-triangle inequality (3).*

Proof.

1°. Let us first prove that the formula (9) indeed defines a pre-order.

Indeed, $(e,x) \preceq (e,x)$ is true. Let us now prove transitivity. Let $(e,x) \preceq (e',x')$ and $(e',x') \preceq (e'',x'')$; this means, in particular, that $e \preceq e'$ and $e' \preceq e''$. Since $\preceq$ is a pre-order, we conclude that $e \preceq e''$. We need to prove that $(e,x) \preceq (e'',x'')$. Let us consider all possible situations.

1.1°. We first consider the case when $x = x'$ and $x' = x''$.

In this case, $x = x''$, so, by the definition (9), we get $(e,x) \preceq (e'',x'')$.

1.2°. Let us now consider the case when $x = x'$ and $x' \neq x''$.

In this case, $x = x' \neq x''$. From $(e',x') \preceq (e'',x'')$ and $x' \neq x''$, we conclude that

$$\tau(e',e'') \geq d(x',x''). \tag{11}$$

Since $e \preceq e' \preceq e''$, from the anti-triangle inequality we get $\tau(e,e'') \geq \tau(e,e') + \tau(e',e'')$. Since τ is a non-negative function, we conclude that $\tau(e,e'') \geq \tau(e',e'')$. Using (11), we now get $\tau(e,e'') \geq d(x',x'') = d(x,x'')$ and $x \neq x''$, i.e., $(e,x) \preceq (e'',x'')$.

1.3°. Let us consider the case when $x \neq x'$ and $x' = x''$.

In this case, $x \neq x' = x''$, so $x \neq x''$. From $(e,x) \preceq (e',x')$ and $x \neq x'$, we conclude that

$$\tau(e,e') \geq d(x,x'). \tag{12}$$

Since $e \preceq e' \preceq e''$, from the anti-triangle inequality, we get $\tau(e,e'') \geq \tau(e,e') + \tau(e',e'')$. Since τ is a non-negative function, we conclude that $\tau(e,e'') \geq \tau(e,e')$. Using (12), we now get $\tau(e,e'') \geq d(x,x') = d(x,x'')$ and $x \neq x''$, i.e., $(e,x) \preceq (e'',x'')$.

1.4°. Finally, let us consider the case when $x \neq x'$ and $x' \neq x''$.

Since $x \neq x'$, the assumption $(e,x) \preceq (e',x')$ implies

$$\tau(e,e') \geq d(x,x'). \tag{13}$$

Similarly, since $x' \neq x''$, the assumption $(e', x') \preceq (e'', x'')$ implies

$$\tau(e', e'') \geq d(x', x''). \tag{14}$$

In this case, either $x = x''$ – in which case $e \preceq e''$ implies $(e, x) \preceq (e'', x'')$, or $x \neq x''$. But then from $e \preceq e' \preceq e''$ and the anti-triangle inequality, we conclude that $\tau(e, e'') \geq \tau(e, e') + \tau(e', e'')$. From (13) and (14), we can now get $\tau(e, e'') \geq d(x, x') + d(x', x'')$. By applying the triangle inequality $d(x, x') + d(x', x'') \geq d(x, x'')$, we get the desired inequality $\tau(e, e'') \geq d(x, x'')$.

So, the formula (9) indeed defines a pre-order.

2°. Let us now prove that the expression (10) satisfies the anti-triangle inequality, i.e., that if $(e, x) \preceq (e', x')$ and $(e', x') \preceq (e'', x'')$, then

$$\tau_\alpha((e, x), (e', x')) \geq \tau_\alpha((e, x), (e', x')) + \tau_\alpha((e', x'), (e'', x'')). \tag{15}$$

According to (9), if $(e, x) \preceq (e', x')$, then either $x = x'$ or $\tau(e, e') \geq d(x, x')$. In the case $x = x'$, we have $d(x, x') = 0$ and, since τ is a non-negative function, we also have $\tau(e, e') \geq d(x, x')$; so, the latter inequality follows from $(e, x) \preceq (e', x')$.

Similarly as above, let us denote $d_1 \stackrel{\text{def}}{=} d(x, x')$, $d_2 \stackrel{\text{def}}{=} d(x', x'')$, $d \stackrel{\text{def}}{=} d(x, x'')$, $\tau_1 \stackrel{\text{def}}{=} \sqrt[\alpha]{\tau^\alpha(e, e') - d_1^\alpha}$, $\tau_2 \stackrel{\text{def}}{=} \sqrt[\alpha]{\tau^\alpha(e', e'') - d_2^\alpha}$, and $\tau \stackrel{\text{def}}{=} \sqrt[\alpha]{\tau^\alpha(e, e'') - d^\alpha}$. From these definitions, we conclude that $\tau(e, e') = \sqrt[\alpha]{\tau_1^\alpha + d_1^\alpha}$ and $\tau(e', e'') = \sqrt[\alpha]{\tau_2^\alpha + d_2^\alpha}$.

By adding these two equalities, we get

$$\tau(e, e') + \tau(e', e'') = \sqrt[\alpha]{\tau_1^\alpha + d_1^\alpha} + \sqrt[\alpha]{\tau_2^\alpha + d_2^\alpha}. \tag{16}$$

We know that for every $\alpha \geq 1$, the l^p-expression $\|(\tau, d)\| \stackrel{\text{def}}{=} \sqrt[\alpha]{\tau^\alpha + d^\alpha}$ is a norm, i.e., $\|(\tau_1 + \tau_2, d_1 + d_2)\| \leq \|(\tau_1, d_1)\| + \|(\tau_2, d_2)\|$, or, equivalently,

$$\sqrt[\alpha]{(\tau_1 + \tau_2)^\alpha + (d_1 + d_2)^\alpha} \leq \sqrt[\alpha]{\tau_1^\alpha + d_1^\alpha} + \sqrt[\alpha]{\tau_2^\alpha + d_2^\alpha}. \tag{17}$$

Combining (16) and (17), we conclude that

$$\tau(e, e') + \tau(e', e'') \geq \sqrt[\alpha]{(\tau_1 + \tau_2)^\alpha + (d_1 + d_2)^\alpha}. \tag{18}$$

From the anti-triangle inequality $\tau(e, e'') \geq \tau(e, e') + \tau(e', e'')$ for τ, we deduce that

$$\tau(e, e'') \geq \sqrt[\alpha]{(\tau_1 + \tau_2)^\alpha + (d_1 + d_2)^\alpha}. \tag{19}$$

Raising both sides of this inequality to the power α, we get

$$\tau^\alpha(e, e'') \geq (\tau_1 + \tau_2)^\alpha + (d_1 + d_2)^\alpha.$$

By moving $(d_1 + d_2)^\alpha$ to the left-hand side, we get

$$\tau^\alpha(e, e'') - (d_1 + d_2)^\alpha \geq (\tau_1 + \tau_2)^\alpha. \tag{20}$$

Due to the triangle inequality, $d \stackrel{\text{def}}{=} d(x, x'') \le d(x, x') + d(x', x'') = d_1 + d_2$, hence $d^\alpha \le (d_1 + d_2)^\alpha$. So, from the equation (20), we deduce that

$$\tau^\alpha(e, e'') - d^\alpha \ge (\tau_1 + \tau_2)^\alpha.$$

By taking the α-th root of both sides, we conclude that

$$\sqrt[\alpha]{\tau^\alpha(e, e'') - d^\alpha} \ge \tau_1 + \tau_2,$$

i.e., that $\tau \ge \tau_1 + \tau_2$ – the desired anti-triangle inequality.

The proposition is proven.

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