In Some Curved Spaces, We Can Solve NP-Hard Problems in Polynomial Time: Towards Matiyasevich's Dream

Vladik Kreinovich[1] and Maurice Margenstern[2]
[1]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
vladik@utep.edu
[2]Université Paul Verlaine – Metz
UFR MIM, LITA, EA 3097
Ile du Saulcy 57045 Metz
Cédex France margens@univ-metz.fr

Abstract

In late 1970s and early 1980s, Yuri Matiyasevich actively used his knowledge of engineering and physical phenomena to come up with parallelized schemes for solving NP-hard problems in polynomial time. In this paper, we describe one such scheme in which we use parallel computation in curved spaces.

1 Introduction and Formulation of the Problem

Many practical problems are NP-hard. It is well known that many important practical problems are NP-hard; see, e.g., [7, 9, 22]. Under the usual hypothesis that P$\neq$NP, NP-hardness has the following intuitive meaning: every algorithm which solves all the instances of the corresponding problem requires, for some instances, non-realistic hyper-polynomial (probably exponential) time on a Turing machine (and thus, on most known computational devices).

How can we solve NP-hard problems? Matiyasevich's dream. The difficulty with NP-hard practical problems is that their solution on the existing computational devices requires an un-realistically large time.

Meanwhile, computers are getting faster and faster. The main reason for this speed-up is that computer designers are constantly incorporating new engineering and physical ideas. Is is therefore reasonable to look for engineering and physical ideas which would enable us to solve NP-hard problems in feasible

(polynomial) time. After the main theory of NP-hardness was developed in the early 1970s, several researchers have tried to find such ideas. One of the first pioneers in this research direction was Yuri Matiyasevich.

In late 1970s and early 1980s, Matiyasevich actively used his knowledge of engineering and physical phenomena to come up with parallelized schemes for solving NP-hard problems in polynomial time. He presented several related ideas in his talks and papers; see, e.g., [33, 34]. Some of the technical and physical ideas on which he has worked at that time (based on super-conductivity, non-traditional chemical reactions, and other unusual physical and engineering phenomena) did not yet lead to promising breakthroughs; however, others ideas can be viewed as early predecessors of such successful techniques as DNA computing; see, e.g., [1, 2].

Overall, his actively pursued dream of solving NP-hard problems in reasonable time has encouraged many researchers to come up with other ideas which led to practically useful algorithms. For example, Matiyasevich's pursuit of such a speedup was one of the main inspirations behind psychology-motivated semi-heuristic fast algorithms for solving NP-hard problems which were developed by S. Maslov in the early 1980s and described in the above-cited book [33]; see also [20].

In this paper, we will follow up on a different idea: the use of parallel computation in curved spaces to speed up the solution of NP-hard problems.

Parallelism is not a panacea. When running an algorithm takes too much time on a single computer, a natural idea is to run several computers in parallel. At first glance, it may seem possible that adding very many processors can enable us to run a previously non-feasible algorithm really fast. However, there are geometric arguments (and we will present them right now) that although we can drastically decrease the computation time by using parallel processors, we cannot make a non-feasible algorithm feasible; see, e.g., [8, 35].

Indeed, according to modern physics, the fastest speed of any process is the speed of light c. So, if a parallel computer finishes its computations in time t and it contains a processor which is located at a distance $r > c \cdot t$ from the user, then this processor will not influence the result of the algorithm: even when traveling with a speed of light, the signals from this processor will reach the user only *after* the time $r/c > t$ (i.e., after the computations are over). Therefore, all processors that participate in this computation are located at a distance $r \le c \cdot t$ from the user, i.e., inside a sphere of radius $r = c \cdot t$ with a center at the location L of the user.

On each technological level, there are lower bounds d_0 and V_0, correspondingly, on the size and on volume of a processor. So, within a sphere of radius r, we can fit no more than $N \stackrel{\text{def}}{=} V(r)/V_0$ processors, where $V(r)$ denotes the volume of a sphere of radius r with the center at the point L. In the Euclidean space, $V(r) = (4/3) \cdot \pi \cdot r^3$, so, with $r = c \cdot t$, we can fit no more than $C \cdot t^3$ processors, where $C \stackrel{\text{def}}{=} (4/3) \cdot \pi \cdot c^3$. We can simulate this parallel computation on a sequential machine moment-after-moment: first, we simulate what each of

N processors does in the first moment of time, then what they all do in the second moment of time, etc.

To simulate each moment of time of a parallel machine, we thus need N moments of time on a sequential machine. So, the algorithm that took time t on a parallel machine with N processors, can be simulated on a sequential machine in time $T = N \cdot t$. Since $N \le C \cdot t^3$, we thus have $T \le C \cdot t^4$, hence $t \ge C^{1/4} \cdot T^{1/4}$.

So, if a problem requires exponential time T on a sequential computer, it will still require exponential time t (smaller bit still exponential) on a parallel computer. Similarly, if a problem can be resolved in polynomial time t on a parallel computer, then it can also be solved in a polynomial time T (larger but still polynomial) on a sequential computer.

In short, no matter how many processors we have, parallelism does not automatically lead to a possibility of solving NP-hard problems in polynomial time.

Space-time curvature can help to speed up parallel computations: what is known. According to modern physics, real space-time in not Euclidean, it is curved. Because of this curvature, the dependence $V(r)$ of the volume of the sphere on its radius r is, in general, different from the Euclidean formula $V(r) \sim r^3$.

In particular, in a hyperbolic (Lobachevsky) space, historically the first non-Euclidean space, the volume $V(r)$ grows exponentially with the radius r; see, e.g., [4]. A similar exponential growth happens for other more realistic physical spaces; see, e.g., [35]. In our papers, we have shown that this exponential growth can be actually used to speed up parallel computations. Namely, we can fit an exponential number of processors within the given distance from the user and thus, check exponential number of cases in feasible time. For example, to solve a propositional satisfiability problem with n Boolean variables $x_1, \ldots, x_n$ (the classical NP-hard problem), we can have 2^n processors each of which is assigned an n-dimensional Boolean vector; these processors, in parallel, check whether a given propositional formula F holds for the corresponding vectors.

In [35], the resulting speed up was presented as a theoretical possibility, based on the fact that, e.g., in a hyperbolic space, we can fit an exponentially numbers ($\sim \exp(\alpha \cdot r)$) of non-intersecting spheres of radius r_0 into a sphere of radius r. In that paper, we did not present any explicit scheme for placing the processors.

In [6, 10, 17, 23, 24, 25, 26, 27, 28, 29, 30, 31], explicit iterative geometric schemes are described that enables us to naturally fill the hyperbolic sphere of large radius r with exponentially many small spheres of radius r_0.

Remaining problem. We have mentioned that there exist algorithms which use processors in curved spaces to solve NP-hard problems in polynomial time. These algorithms are based on the possibility to place exponentially many processors within a sphere of a given radius.

However, these algorithms implicitly assume that these (exponentially many) processors are already there. In principle, we can place these processors one by one, but that placing would require exponential time. As a result, with such sequential placing, the overall time for placing and computation will still be exponential.

So, the problem is how to parallelize the placing of all these processors, so that the resulting overall time will be polynomial.

What we plan to do. In this paper, we describe such a scheme for parallel placing in polynomial time.

2 Towards Proposed Solution: Informal Description of the Main Ideas

How to describe this problem in algorithmic terms: available techniques. The original Turing machines and similar computational devices (implicitly) assume that we already have infinitely many memory cells. It does not allow for designing and placing of new cells and/or new processors.

Of course, we can always view an infinite tape of a Turing machine as only *potentially* infinite, in the sense that at any given moment of time, we only have finitely many cells on this tape, and new cells are added when necessary. However, in this view, cells are added one by one – exactly what we wanted to avoid.

There is, however, a known extension of Turing machines which was specifically designed to take into account the possibility of adding new cells and/or new processors. This extension was originally proposed by Kolmogorov and Uspensky in the 1950s [18, 19]; overviews of the resulting Kolmogorov-Uspensky algorithms can be found, e.g., in [3, 12, 16, 40]. Such algorithms are known to have several advantages over more traditional Turing machine-type definitions: e.g., according to [3]: "Leonid Levin used a universal Kolmogorov machine to construct his algorithm for NP problems that is optimal up to a multiplicative constant [21]; see also [11]. The up-to-a-multiplicative-constant form is not believed to be achievable for the multi-tape Turing machine model popular in theoretical computer science."

Kolmogorov-Uspensky algorithms served as a basis for an even more general formalization of the abstract notion of computability: the notion of an Abstract State Machine [5, 12, 13, 14, 15, 16, 37].

What needs to be modified. In principle, Kolmogorov-Uspensky algorithms can be described in two versions:

- based on graphs in Euclidean space and
- based on abstract graphs.

The Euclidean space description is clearly not what we need.

In the abstract graph approach, we count each connection between vertices as 1 computational step. For example, in this version, we can build a full graph of size 2^n in which every node is connected to every other node, with connection between every two nodes taking exactly 1 step. Of course, we can thus solve satisfiability problem in linear number of steps – but, as have seen on the example of Euclidean space the actual distance between the corresponding vertices can be exponential and so, one "step" in this sense may mean exponential time.

What we need is a definition specifically tailored towards a given curved space.

Difficulties. At first glance, it may look like a generalization of Kolmogorov-Uspensky-type algorithms to curved spaces should be reasonably straightforward. From the purely *mathematical* viewpoint, it is indeed possible to provide a generalization. However, our goal is to not simply to produce a mathematical generalization, but to provide a generalization which will be *physically* meaningful, a generalization is which 1 computational step indeed means a single moment of time (or at least constant number of moments of time).

Let us enumerate some potential difficulties related to this goal. Let us recall that what we want are processors which not only perform computations but also actually manufacture new processors. In other words, what we want are not simply processors, but rather intelligent robots.

Both difficulties are related to the fact that these robots need to communicate with each other.

First difficulty: communicating at a large distance r may require exponentially growing energy. In the Euclidean space, it is very reasonable to assume that two robots can communicate at a distance $\le c \cdot n$. Indeed, a signal sent by any source (in particular, a signal sent by a robot) spreads around. By the time this signal reaches distance r, the energy of the original signal is spread around the (surface of the) sphere of radius r, so its (average) energy density is equal to $E_0/S(r)$, where E_0 is the original energy and $S(r)$ is the area of the sphere of radius r. (In these back-of-the-envelope computations, we assume that the transmission is ideal and ignore possible signal decay; if we take this decay into account, the difficulties only become worse.)

The signal is detectable at a distance r if its density exceeds a certain detection threshold ρ_0: $E_0/S(r) \ge \rho_0$. Thus, to be able to transmit information to a a robot at distance r, we must generate the signal with the energy $E_0 \ge S(r) \cdot \rho_0$.

In the Euclidean space, $S(r) \sim r^2$, so the required energy increases polynomially with r – this is feasible to arrange, e.g., by combining a polynomial number of energy sources. However, we are interested in spaces in which the volume $V(r)$ grows exponentially and thus, the area $S(r)$ can also grow exponentially. In such spaces, to propagate the signal through distance r, we may need an energy which grows exponentially with r – i.e., an energy which is not feasible to implement.

How to avoid this first difficulty. To avoid this difficulty, we will assume that the robots can only communicate with each other when their distance is bounded by some constant d_0.

For this to be possible, we must make sure that by combining such bounded communications, we can set up a link between arbitrarily distant points x and x'. This is true in the Euclidean space, where we can connect x and x' by a straight line segment, and on that segment, set up points $x_1 = x$, x_2, ..., at distance d_0 from other another so that $d(x, x') = d(x_1, x_2) + \ldots + d(x_{m-1}, x_m)$ and $d(x_i, x_{i+1}) \le d_0$. A similar construction works in the hyperbolic space as well.

In our analysis, we will thus restrict ourselves to "d_0-connected" metric spaces, in which such a connecting sequence of d_0-close points exists for all x and x'. (A precise definition will be given in the next section.)

Second difficulty: number of communications can grow exponentially. Even when we solve the difficulty related to the weakness of signals, we still face the second difficulty: that there is a very large number of robots and thus, a very large number of communications.

Namely, as we have mentioned, the main idea behind the desired speed up is that all exponentially many processors perform computations in parallel, and then they send their results to the user. In the Euclidean space, we have at most polynomially many robots within a zone of polynomially growing radius, and thus, we have at most polynomially many messages. We can process them sequentially or we can place polynomially many receptors at the user's location.

In a curved space, we may have exponentially many robots and thus, exponentially many signals coming to the user. If we process these signals one by one, we will need exponential time. If we place exponentially many receptors at the user's location, we will need exponentially large volume there – and we will still face a problem of sending signals from these receptors to the user.

How to avoid the second difficulty. To avoid this difficulty, in our algorithm, we will not send signals directly back to the user, we will try to collect them in stages, tree-like: several robots send their messages to a boss-robot, these boss-robots send to bosses of next level, etc., so that at each moment of time, each robot received $\le M_0$ message for some fixed constant M_0.

We will thus restrict ourselves to spaces in which every robot can have no more than M_0 neighboring computing robots. Let us describe these properties in precise terms.

3 Proposed Solution: Definitions and the Main Result

First, we need a formal definition of appropriate metric spaces

For computational purposes, it is sufficient to consider discrete spaces. Physicists usually consider *continuous* spaces as models of physical reality. However, we are interested not in abstracts points in space, but rather in points at which we can place computational devices. At a given technological level, every computational device has a certain linear size $\varepsilon > 0$. This means that if we have already placed a device at some point x, then we cannot place any other device at a distance $\le \varepsilon$ from x.

Thus, from the viewpoint of placing computational devices, instead of considering all possible points, it is sufficient to consider only points whose distance from each other exceeds ε – e.g., points on a grid of size $\ge \varepsilon$. Thus, we arrive at the following definition.

Definition 1. *Let $\varepsilon > 0$ be a real number. We say that a metric space (X, d) is* ε-discrete *if for every two points $x \neq x'$, we have $d(x, x') \ge \varepsilon$.*

Comment. In the following text, we will only consider ε-discrete metric spaces.

For such spaces, it is natural to define a volume of a set simply as its number of elements.

Definition 2. *Let (X, d) be a ε-discrete metric space, and $S \subseteq X$ be a subset of this space. By a* volume $V(S)$ *of this set S, we mean the number of elements in the set S.*

Comment. Our objective is to describe physical models in which we can perform exponentially many operations in polynomial time. For this to be possible, we must make sure that the volume of a ball in this space grows exponentially with radius.

Definition 3. *We say that an ε-discrete metric space (X, d)* has exponential growth *if for some point x_0 and for some positive real numbers R_0, A, and k, for every $R \ge R_0$, the volume $V(B_R(x_0))$ of a ball $B_R(x_0) = \{x : d(x, x_0) \le R\}$ of radius R satisfies the inequality $V(B_R(x_0)) \ge A \cdot \exp(k \cdot R)$.*

Comment. In this definition, we assumed that the exponential growth property occurs for one points x_0. It is worth mentioning that that once this property holds for one point x_0, it is true for every other point $x \in X$:

Proposition 1. *Let (X, d) have exponential growth, then let x be a point in this space. Then, for every point $x' \in X$, there exist positive real numbers R_0', A', and k', such that for every $R \ge R_0'$, the volume $V(B_R(x'))$ satisfies the inequality $V(B_R(x')) \ge A' \cdot \exp(k' \cdot R)$.*

Proof of Proposition 1. According to the triangle inequality, for every point x, we have $d(x, x') \le d(x, x_0) + d(x_0, x')$. Thus, for every radius $r > 0$, we have $B_r(x_0) \subseteq B_{r+d(x',x_0)}(x')$. In particular, for every $R \ge d(x', x_0)$, we thus have $B_R(x') \supseteq B_{R-d(x',x_0)}(x')$. Since the volume is simply defined as the number of points in a set, we have therefore have $V(B_R(x')) \supseteq V(B_{R-d(x',x_0)}(x'))$.

If $R - d(x', x_0) \ge R_0$, i.e., if $R \ge R' \stackrel{\text{def}}{=} R_0 + d(x, x_0)$, then

$$V(B_R(x')) \supseteq V(B_{R-d(x',x_0)}(x')) \ge A \cdot \exp(k \cdot (R - d(x', x_0)) = A' \cdot \exp(-k \cdot R),$$

where $A' \stackrel{\text{def}}{=} A \cdot \exp(-k \cdot d(x', x_0))$. So, the desired property indeed holds for the above A', R_0', and $k' = k$. The proposition is proven.

Comment. To implement fast computations, it is not enough to know that the metric space has exponential growth. Indeed, we may have a metric space consisting of exponentially many points in which every two points have the same distance $d \ge \varepsilon$. In this space, an arbitrary permutation preserves the distance. Since every two points can be swapped by an appropriate permutation, every two points in this space are equivalent to each other. So here, there are exponentially many points, but it is not clear how to distribute the task between all these points since they are all equivalent to each other. What helps for hyperbolic space (and for similar spaces described in [35]) is that in this space, neighborhoods of small radius are small, so in principle, we can cover them in parallel without losing exponential speed-up. Let us formalize the notion of such spaces.

Definition 4. *Let $d_0 > 0$ be a real number. We say that two points x and x' in a metric space (X, d) are* d_0-close *(or* d_0-neighbors*) if $d(x, x') \le d_0$.*

Comment. In the following text, in some cases when the value d_0 will be clear from the contents, we will simplify the text by simply writing "neighbors" and "close".

Definition 5. *Let $d_0 > 0$ be a real number. We say that a metric space (X, d) is* C-connected *if for every two points $x, x' \in X$, there is a sequence $x_1 = x, x_2, \ldots, x_{m-1}, x_m = x'$ such that for every i, x_i and x_{i+1} are d_0-neighbors and*

$$d(x, x') = d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{m-1}, x_m).$$

Definition 6. *Let $d_0 > 0$ be a given real number and $M_0 > 0$ be a given integer. We say that a metric space (X, d) has* M_0-bounded d_0-neighborhoods *if every point $x \in X$ has no more than M_0 d_0-neighbors.*

Definition 7. *Let $\varepsilon > 0$ and $d_0 > 0$ be real numbers, and $M_0 >$ be an integer. By an* potentially (ε, d_0, M_0)-computation enhancing space *(or simply* potentially computation-enhancing space*, for short), we mean an ε-discrete d_0-connected metric space with exponential growth and M_0-bounded d_0-neighborhoods.*

In the following text, we assume that a computation-enhancing space (X, d) is given, and that we also are given a point x_0 in this space. This point will be called the *user's location*.

The corresponding "robotic computer" will consist of "computational robots" located in different points of the potentially computation-enhancing space X. Let us start our definition of a robotic computer by defining the notion of a computing robot.

Definition 8. *By a* computing robot*, we mean a regular computer (e.g., a Turing machine or RAM, or a Java virtual machine) with a finite memory (sequence of bits) in which there are three additional commands:* build*,* send forward *(with memory location as a parameter), and* send backward *(with memory location as a parameter). The memory is divided into five parts:*

- *a* hardwired *part which contains a program;*
- *a* working *memory part which can be used for computations in the usual way;*
- *a* tree ID *memory part; it will be used for storing a special "tree ID" of a processor;*
- *a* regular ID *memory part; it will be used for storing a special "regular ID" of a processor;*
- *a* received *memory part; this received part is divided into M_0 subparts.*

We assume that the tree ID memory consists of a single bit (called built*) followed by* blocks *of bit size $1 + B$, where $B \stackrel{\text{def}}{=} \lceil \log_2(M_0) \rceil$.*

Comment. In a potentially computation-enhancing space (X, d), every location has $\le M_0$ (d_0-)neighbors, so a B-bit long part of the block is sufficient to differentiate between all the neighbors of a given location.

In the following definitions, we will explain how the new commands work. As a result of the build command, the newly built robots will have a tree structure; in terms of this structure:

- *send forward* means send to direct descendants, and
- *send backward* means receive from direct descendants.

In the following text, we will construct "polynomial-time" computing robots, i.e., robots in which the computation time of the corresponding Turing machine grows no more than polynomially with n, and in which the size of the memory also grows no more than polynomially.

Definition 9. *By a* state *of a computing robot, we mean the values of all the bits from its memory. We say that in a given state, a given block of an ID tree memory part is* unused *if it contains only zeros and* used *otherwise.*

Definition 10.

- *By a* robotic computer*, we mean a mapping in which there is a computing robot assigned to (some or all) points from a potentially computation-enhancing space X.*
- *By a* state *of the robotic computer, we mean the mapping which assigns to every point, the state of the computing robot assigned to this point.*
- *In the* initial state*, there is only one computational robot located at the user's location x_0; its tree ID (i.e., the contents of its tree ID memory) consists of all zeros.*

Comment. The fact in the initial state, the robot's tree ID consists of all zeros means that the value of the "built" bit is 0 (false), and that all the blocks of the tree ID memory are unused.

To complete the description of a robotic computer, we must describe how the new commands work.

Definition 10. *As a result of a "build" command, each robot for which the value of the "built" bit is 0, builds robots in all d_0-neighboring points. These robots are identical to the original robot, with the only exception of an ID number:*

- *All the used blocks of the tree ID of the old robot are copied into the ID tree memory part of the new robot.*
- *In the first previously un-used block of the tree ID memory of each new robot, we place 1 in the first bit (to indicate that the new block is used), and assign different values to the next B bits (so that different robots built by the same robot get different tree ID numbers).*

After the new computing robots are built, the original robot changes the value of its "built" bit to 1.

Comments.

- Every robot has $\leq M_0$ neighbors, so it may have to build $\leq M_0$ new robots. Due to our choice of B as $B = \lceil \log_2(M_0) \rceil$, B bits are sufficient to distinguish between all these robots.
- If a new point is d_0-close to two or more original robot locations, then several robots want to build a new robot there. In this case, we do not care how they resolve this problem, we simply assume that one (and exactly one) of these robots is built: e.g., a parent robot with the smallest tree number may have preference. (It is worth mentioning that there is a relation between this problem and the problem of deleting duplicates in geospatial databases; see, e.g., [38].)

Example. Let us assume that $M_0 = 3$, i.e., that each point has no more than 3 neighbors. In this case, $B = \lceil \log_2(3) \rceil = 2$. So, a tree ID consists of a built bit and blocks of bit size $1 + B = 3$.

Initially, we have only one robot at the user's location x_0 with a tree ID consisting of all zeros: 0 000 000.

When the *build* command is issued, this robot has 0 value of the *built* variable, so it start building new robots. Let us assume that the point x_0 has exactly two neighbors. For each of these neighbors, the first unused block is the first one; its first 0 is replaced by 1, and it consequent 2 bits are replaced by, e.g., 00 and 11. Do, the two new robots have tree IDs 0 100 000 and 0 111 000. After the initial robot finishes constructing these robots, it changes the value of its *built* bit to 1 ("true"). So, now, its tree ID is 1 000 000.

If the *build* command is issued again, the first robot no longer does anything, because its built variable now has the value 1 ("true"). However, the two new robots have a 0-valued *built* variable, so they both react to this command.

For example, the robot 0 100 000 creates 3 new robots, with tree IDs making the second block used: e.g., 0 100 101, 0 100 110, and 0 100 111. The robot 0 111 000 creates 2 new robots, with tree IDs making the second block used: e.g., 0 111 100 and 0 100 110.

Definition 11. *Robots created by a given robot are called its* direct descendants.

Comment. A robot can easily find its direct parent – by deleting the last used block in its tree ID (and ignoring the built bit). Thus, a robot can easily check whether another robot is its direct descendant.

Definition 12. *As a result of the* send forward *command, each robot sends the values of the memory cells specified by this command. This information is received by all its direct descendants and stored in the first part of their received memory.*

Definition 13. *As a result of the* send backward *command, each robot sends the values of the memory cells specified by this command. This information is received by the direct parent of this robot. The information received by a robot from different direct descendants is stored in the different subparts of this robot's received memory.*

Clarifying comment. We are not specifying in which exactly part this information is placed, this can be decided by an implementation. For example, each descendant of a given robot can be a number, and then information sent by the i-th descendant is placed in the i-th subpart of the received memory.

Comment. The above definition describes a SIMD (Single Instruction Multiple Data) type parallel computer in which all the processors, in effect, perform the same set of instructions. In principle, it is possible to extend our definition to the case of MIMD (Multiple Instructions Multiple Data). We did not do this here, since our objective is to describe, in the simplest possible terms, how to solve NP-problems in polynomial time. This can be done already with SIMD-devices only; the description is already somewhat complex, so the extension to MIMD would have made it even more complex and much more difficult to read.

How can we count computation time? In the traditional analysis of the algorithms' computation time, we usually simply count the overall number of elementary computation steps. Of course, this is not a perfect description of the actual computation time, because, in general, different elementary computation steps require somewhat different computation time. For example, multiplication requires slightly longer time than addition, etc. However, these differences are usually minor (less than an order of magnitude), so a simple count of elementary operations provides a reasonable estimation for the actual computation time. (Although, of course, when we need to select the fastest of two reasonable algorithms, counting can be misleading: to make a realistic comparison, we must assign different weights to different operations.)

The situation with computation on a robotic computer is drastically different. Indeed, these computations consists of three types of steps:

- regular computation steps,
- steps in which we build new robots, and
- steps in which the robots send and receive information.

Here, sending a message takes orders of magnitude more time than performing a regular computation step, and building a new robot takes even more time: hours or days instead of nanoseconds.

So, to make a reasonable count of the overall computation time, we cannot simply add the number N_r of the regular computational operations, the number N_b of building operations, and the number N_s of send commands, and take $N \stackrel{\text{def}}{=} N_r + N_b + N_s$ as the computation time. We must define two real numbers

$C_b > 0$ ("building time") and $C_s > 0$ ("sending time"), and then estimate the overall computation time as $t = N_r + C_b \cdot N_b + C_s \cdot N_s$.

Intuitively, the times C_b and C_s are proportional only to the number of robots in the neighborhood (the number of robots to build, or the number of robots to send signals to), so the time needed for each of these longer operations does not change with n; from the viewpoint of dependence on n, it is simply a constant.

The problem is that we do not know these constants. Fortunately, all we are interested in is whether the resulting computation time is polynomial (in terms of th length of the input) or not. Since C_b and C_s are constants, the sum $t = N_r + C_b \cdot N_b + C_s \cdot N_s$ is polynomial if and only if all three components N_r, N_b, and N_s depend polynomially on n – i.e., if and only if the simple sum $N = N_r + N_b + N_s$ polynomially depends on n.

In other words, while the actual computation time depends on the values of the (unknown) coefficients C_b and C_s, the division of algorithms into polynomial-time and not polynomial-time ones does not depend on the actual values of these coefficients. So, we can simply define polynomial time as the case when the sum N is bounded by the polynomial of n.

We are now ready for the main result.

Theorem 1. *For every potentially computation-enhancing space, there exists a robotic computer which solves the propositional satisfiability problem in polynomial time.*

Comment. In our definition, we called a space potentially computation-enhancing if in this space, volume grows exponentially with time – thus providing a potential possibility to fit exponentially many processors within a finite radius and thus, drastically parallelize NP-complete problems into polynomial overall time.

The above theorem shows that this is not only a *potential* possibility: there is an *actual* algorithmic way to fill such as pace with exponentially many computers and thus, indeed enhance computations.

4 Proof of the Main Result

General structure of the algorithm. The desired algorithms consists of the following five stages:

- first, we build a hierarchical network (tree) of 2^n robots;
- second, we assign, to each robot, a number from 0 to $2^n - 1$ (i.e., equivalently, an n-dimensional Boolean vector) in such a way that the numbers assigned to this robot and to the robots eventually generated by the this robot and its descendants form an interval in which this robot's number comes first;

- third, we distribute the original propositional formula to all the robots;
- at the forth stage, each robot checks whether the formula F holds at this robot's vector x,
- at the fifth stage, this information is sent backwards along the robot tree until all the information is gathered at the original point – with an indication of whether the original propositional formula is solvable or not.

Building stage. Let us first describe the building stage. Initially, we have only one robot in the user's location x_0. One build command creates robots in all the points of distance $\le d_0$ from x_0; another build command creates robots at all the points in locations which are $\le d_0$ far away from these new locations, etc.

Since the space is d_0-connected, every point x is connected to x_0 by a chain in which two next points are d_0-connected; thus, if we issue one build command after another, eventually, we will cover every point in the space X. How many build commands do we need? For every point x at a distance R, we have a chain of d_0-close points for which $R = d(x, x_0) = d(x_1, x_2) + \ldots + d(x_{m-1}, x_m)$. If in these chain, two points x_i and x_{i+1} are identical, we can delete the second one. Thus, without losing generality, we can assume that all the points x_i in the chain are different: $x_i \ne x_{i+1}$. Since the space is ε-discrete, $x_i \ne x_{i+1}$ implies that $d(x_i, x_{i+1}) \ge \varepsilon$, hence $R = d(x_1, x_2) + \ldots + d(x_{m-1}, x_m) \ge (m-1) \cdot \varepsilon$. Thus, $m \le 1 + R/\varepsilon$; this means that every point within the radius R can be covered by $1 + \lfloor R/\varepsilon \rfloor$ iterations of the build command.

What radius R should we choose? The larger the radius, the more robots we will be able to build within this radius. We want to have at least 2^n computing robots. Since we are in a potentially computer-enhancing space, for sufficient large R, the number of points at a distance $\le R$ from the user's location x_0 is at least $V \ge A \cdot \exp(k \cdot R)$. So, to have $\ge 2^n$ robots, we must have (for sufficiently large n) $\exp(k \cdot R) \ge 2^n / A$, i.e., $k \cdot R \ge n \cdot \ln(2) - \ln(A)$. Thus, it is sufficient to have $R = (\ln(2)/k) \cdot n - (\ln(A)/k)$.

We have already shown that the number of build commands to cover all the points within this radius grows linearly with R as $1 + R/\varepsilon$. Thus, the required number of build commands grows linearly with n, as

$$\frac{\ln(2)}{k \cdot \varepsilon} \cdot n + \left(1 - \frac{\ln(A)}{k \cdot \varepsilon}\right).$$

So, the depth of the resulting tree grows linearly with n. (We need to make sure that the ID tree memory part contains sufficiently many bits to cover that many blocks.)

Second stage: assigning regular IDs to the robots. On the second stage, we assign a number ID to each robot. This is done in two sub-stages. The purpose of the first sub-stage is that each robot computes the number of its descendants (including itself and its indirect descendants). At the end of this

sub-stage, the initial robot will have the total number of robots in the whole tree.

The first sub-stage consists of several conditional "send backward" commands:

- the first command is only applicable to the robots of the last generation, with the largest number of used blocks (there robots do not have any direct descendants);
- the next command is applicable only to the robots of the next to last generation,
- etc.,
- finally, the last command is only applicable to the robots of the first generation, with exactly one used block in their tree IDs (i.e., robots built directly by the initial robot).

At each command, each robot sends its tree ID number and its number of descendants to its parent robot. The "leaf" robots (with no descendants) send their tree ID number and the value 1. On every other step, a robot reads all the values send by its direct descendants (read from the corresponding parts of the receive part of its memory), adds them together, adds 1, and thus gets the number to send back (to its parent robot).

Each addition of two numbers of size $\le 2^n$ requires n binary steps; thus, addition to $\le M_0$ such numbers requires linear time. Overall, we need $\mathrm{const} \cdot n$ commands each of which requires linear time – to the total of $O(n^2)$ time.

Now, we can distributed regular ID numbers by going forward, from each robot to its direct descendants.This is done by performing a sequence of conditional send forward commands.

- the first command is only applicable to the initial robot;
- the second command is only applicable to the robots of the first generation (directly built by the initial robot);
- etc.
- the last command is applicable only to the robots of the next to last generation.

The initial robot have received values $N_1 + \ldots + N_m$; due to our choice of R, we have $1 + N_1 + \ldots + N_m \ge 2^n$. So, this first robot gets the ID number 0, and the interval $[0, 2^n - 1]$. The first direct descendant gets the interval $[1, N_1]$, the second gets the interval $[N_1 + 1, N_1 + N_2]$, the third gets the interval $[N_1 + N_2 + 1, N_1 + N_2 + N_3]$, etc.

Computing each of these sums requires a constant number of additions; the time of each addition is linear in the number of bits $\le n$ in each of these sums $\le 2^n - 1$, so computation of these intervals requires linear time. All these $\le M_0$

intervals and the tree ID number of the direct descendants are then sent to all direct descendant.

From this information, each direct descendant selects the interval $[\underline{N}, \overline{N}]$ corresponding to its tree ID. Then, this robot assigns the smallest value $\underline{N}$ to itself, and divides the remaining interval $[\underline{N}+1, \overline{N}]$ between its own direct descendants: if they sent the values $n_1 \ldots, n_s$, then they are assigned the intervals $[\underline{N}+1, \underline{N}+n_1]$, $[\underline{N}+n_1+1, \underline{N}+n_1+n_2]$, etc.

At the end of these procedure, each robot is assigned an interval of ID values of its descendants, and this robot's own regular ID number is, as we desired, the smallest value from this interval.

Comment. We may have $> 2^n$ robots, in which case value $> 2^n$ are not assigned at all, so the last direct ancestor gets the interval $[\ldots, 2^n]$. This means, in effect, that we will not use all the robots in our computations, only 2^n of them.

Third stage: distributing the propositional formula. On the third stage, the original propositional formula is distributed to all the robots. This is done by applying several "send forward" commands:

- first, from the initial robot to its direct descendants,
- then to the second generation, etc.

We need as many iteration as the hight of the tree – i.e., linearly many.

Fourth stage: checking $F(x)$ for every x. On the fourth step no one is sending anything: every robot checks whether $F(x)$ holds for the propositional vector x which is equal to the (binary representation of) its regular ID number.

For example, the initial robot has a regular ID number 0, so it will check whether F holds for the propositional vector $00\ldots0$ consisting of all zeroes (i.e., if all the propositional variables are false). Similarly, the last (childless) robot gets a number $2^n - 1$ whose binary representation is $11\ldots1$, so it will check whether F holds for the propositional vector $00\ldots0$ consisting of all ones (i.e., if all the propositional variables are true).

Final (fifth) stage: combining the checking results into a single answer to the propositional satisfiability question. Before describing the fifth stage in detail, let us make the following comment. In the original problem, we do not ask for what x the formula is satisfied, we just asked whether it is satisfied or not. So, in this case, it is sufficient to simply keep the information "yes" (meaning that there is a satisfying vector). In the following text, we show how to modify this problem if we *are* actually interested in finding the vector x for which $F(x)$ holds.

At the beginning of the fifth stage, each robot only has information whether $F(x)$ holds for the vector x corresponding to this robot. There are two possibilities here:

- The first possibility is that the answer to the question of checking $F(x)$ for this robot's x is "yes". In this case, the original problem is solved – the formula is satisfiable.

- The second possibility is that $F(x)$ is false. In this case, we keep the information "no" meaning that $F(x)$ is false for this x.

As we collect this information, at each robot location, we have two similar possibilities:

- The first possibility is that for one of the values x from the corresponding interval $[\underline{x}, \overline{x}]$, we have $F(x)$. In this case, we keep the value "yes". It means that the satisfiability problem is already solved; we only need to make sure that this information is passed back and not lost.

- The second possibility is that $F(x)$ is false for all the vectors x from the interval $[\underline{x}, \overline{x}]$ assigned to the node. In this case, we keep the answer "no".

After each robot collects the information from its direct descendants, it has to merge the corresponding $\leq M_0$ pieces of information with the information coming from testing $F(x)$ for its own vector x.

- If one of the resulting $\leq M_0 + 1$ pieces of information is "yes", then this "yes" (meaning that there exists a satisfying vector) is the result of the information merger.

- If all of the merged information is "no", this means that $F(x)$ is false for all the vectors x from its interval, so the merged value is "no".

(In other words, this merger is simply an "or" operation.)

Since we combine $\leq M_0 + 1$ bits, this merger requires a constant number of steps $\leq M_0 + 1 = O(1)$.

By the time we get to the original robot – the root of the robot tree – the resulting value "yes" or "no" provide the answer to the original instance of the propositional satisfiability problem.

So, the above-described robotic computer indeed solves the propositional satisfiability problem.

Checking that the resulting algorithm requires polynomial time. To complete the proof, it is now sufficient to show that it requires polynomial time. Indeed:

- Building new computers requires time propositional to the depth of the tree – which, for a potentially computation-enhancing space, grows linearly with the size n.

- Send the information forward and backward to generate regular IDs requires, as we have mentioned, quadratic time $O(n^2)$.

- Then, checking whether $F(x)$ holds for every x is done on all robots in parallel; it requires polynomial time.
- Finally, merging also requires times which is proportional to the depth of the tree, i.e., linear time.

Thus, the overall time is indeed polynomial.

The theorem is proven.

5 Auxiliary Results

5.1 First auxiliary result: producing satisfying vector instead of checking its existence

We formulated the above algorithm for the *deciding* version of the propositional satisfiability problem, where the objective is simply to check whether a given propositional formula F is satisfiable or not.

One can easily modify this algorithm so that it will serve a similar problem in which we actually want to *produce* the propositional vector x for which $F(x)$ holds. In this case, instead of reporting the single-bit piece of information "yes" to the boss-robot, a robot should pass a vector x for which $F(x)$ holds; if two or more such vectors are passed to a boss-robot, this robot selects one of them (e.g., the smallest).

5.2 General problems from the class NP

The known fact that propositional satisfiability is NP-hard means that every problem from the class NP can be reduced, by a polynomial-time reduction, to propositional satisfiability. Thus, we arrive at the following corollary:

Corollary. *For every potentially computation-enhancing space and for every problem from a class NP, an appropriate robotic computer can solve it in polynomial time.*

Proof of the Corollary. Indeed, we can:

- first use the central processor to perform the reduction to satisfiability (this requires polynomial time), and then
- solve the resulting instance of the satisfiability problem in polynomial time (as described in the proof of Theorem 1).

The overall time is thus polynomial. The Corollary is proven.

Comment. In the above description, we make a simplifying assumption: namely, we assume that each computing robot can store (and process) strings proportional to the size n of the problem, but we still consider the size of the processor to be fixed (independent of n).

In reality, the linear size ε of the processor should grow with n (e.g., linearly). As a result, when we implement this scheme for a different n, we will need robots separated by n times larger distances. $\varepsilon' \approx n \cdot \varepsilon$. In this case, each communication step requires n times longer time.

Since the product of n and a polynomial of n is still a polynomial of n, this increase keeps the overall time polynomial, so this simplifying assumption does not change the main result.

5.3 Problems from the class PSPACE

Up to now, we have strengthened and generalized the result that in the hyperbolic space, we can solve NP-complete problems in polynomial time. However, in the hyperbolic space, an even stronger result is known. Namely, it is known that by using processors appropriately placed in a hyperbolic space, we can solve, in polynomial time, not only problems from the class NP, but also problems from the more general class PSPACE (class of all the problem which can be solved within polynomial space).

It turns out that a similar extension can be proven for the case of a general potentially computation-enhancing space.

Theorem 2. *For every potentially computation-enhancing space and for every problem from a class PSPACE, an appropriate robotic computer can solve it in polynomial time.*

Proof of Theorem 2: Main idea. It is known (see, e.g., [36]), that a problem known as QSAT (it will be described later in this proof) is PSPACE-complete. This means that every other problem from the class PSPACE can be reduced to QSAT by a polynomial-time reduction. Thus, to prove that every PSPACE problem can be solved in polynomial time on a robotic computer, it is sufficient to prove that every problem from the class QSAT can be solved in polynomial time on a robotic computer.

The QSAT problem is an extension of the propositional satisfiability problem SAT, an extension which allows quantifiers over Boolean variables. To be more precise, in SAT, the objective is to find a positional vector $x = (x_1, \ldots, x_n)$ for which a propositional formula $F(x_1, \ldots, x_n)$ becomes true – or, in the deciding version, to check whether such a propositional vector exists, i.e., whether $\exists x\, F(x)$ holds. In QSAT, we must check whether a more general formula

$$\exists x \forall y \ldots \exists z\, F(x, y, \ldots, z)$$

holds.

QSAT is the "limit" of the corresponding formulas from the polynomial hierarchy, which start with SAT-related formulas $\exists x\, F(x)$ and their duals $\forall x\, F(x)$ and continue with more complex formulas $\forall x \exists y\, F(x,y)$ (and $\exists x \forall y\, F(x,y)$), etc., until we reach QSAT.

Let us show, on the example of these next-step formulas $\forall x \exists y\, F(x,y)$, how we can modify the above proof so that it will be applicable to these formulas. The same idea works for all further extensions all the way to QSAT.

In the above proof of Theorem 1, we ordered propositional vectors x in lexicographic order and assigned to each processor an interval $[\underline{x}, \overline{x}]$ of vectors, i.e., all the vectors between two the given vectors $\underline{x}$ and $\overline{x}$ ("between" in the sense of the lexicographic order). For the formula $\forall x \exists y\, F(x,y)$, we also sort the processors in lexicographic order by the order of a "long vector" xy (a concatenation $x_1, \ldots, x_n, y_1, \ldots, y_m$ of the vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$) and assign, to each processor, all the vectors (x, y) from some interval (interval in the sense of this order).

The lexicographic ordering has the property that if the vectors $x0\ldots0$ and $x1\ldots1$ belong to an interval, then all the vectors xy belong to the same interval. Thus, in general, the interval in this order can be described as follows:

- for the smallest possible x from this interval, we have a subinterval of possible values of y;
- for some (maybe empty) interval $[\underline{x}, \overline{x}]$ of possible values of x, we have all the values xy with $x \in [\underline{x}, \overline{x}]$ and arbitrary y;
- finally, for the last x, we may have a subinterval of possible values of y.

(In the degenerate case, we may have no intermediate interval, or no starting interval, or no ending interval.)

By the time we gathered information from this xy-interval at a robotic computer, we have already checked whether $F(x,y)$ holds for all (x,y) within this interval, and have already collected this information.

For the values x from the intermediate x-interval $[\underline{x}, \overline{x}]$, we thus already know whether there exists y for which $F(x,y)$. If for one of these x, such y exists, then the original problem (of checking whether $\exists x \forall y\, F(x,y)$ is true) is solved: we just need to keep this information and make sure that it is not lost during the transition to the boss-robots.

If no y was found for all these x, then we have an interval $[\underline{x}, \overline{x}]$ of values for which $\neg\forall y F(x,y)$. So, we simply keep the endpoints of this interval.

Similarly, at each of the two borderline values x, if there is a y (among those for which xy was checked) for which $\neg F(x,y)$, then the resulting x can be simply added to the intervals of xs for which $\neg\forall y F(x,y)$. Otherwise, for each of these two borderline values x, we keep an interval $[\underline{y}, \overline{y}]$ for which $F(x,y)$ holds for all y within this interval.

As a result, at each robot location, we keep one of the two following pieces of information:

- first possibility is the information that the problem is already solved;

- second possibility is that we keep
  - the interval $[\underline{x}, \overline{x}]$ of values x for which $\neg\forall y F(x, y)$, and
  - for two borderline values x, interval of values $[\underline{y}, \overline{y}]$ for which $F(x, y)$ holds for all y within these intervals.

At each robot point, we combine several ($\leq M_0+1$) such pieces of information into one. If one of these pieces contains the solution, then we just pass this information on without bothering with other piece of information. If none of these pieces contain a solution, then first, for some borderline values x, we may need to combine the y-intervals.

- It may be possible that we simply get a larger interval of possible values of y.
- As a result of the combination, we may conclude that $F(x, y)$ holds for all possible y – in which case we have a solution.

Since we combine $\leq M_0 + 1$ pieces of information, there are $\leq M_0$ borderline values to combine, so this merger can done in constant time.

This is the only modification that we need in the above proof to convert from SAT to $\exists x \forall y\, F(x, y)$.

For more complex formulas such as $\exists x \forall y \exists z\, F(x, y, z)$, an interval in the lexicographic order includes:

- an interval of possible values of x for which we have checked for all y and z,
- two borderline values (x, y) for which we have checked for intervals of z, etc.

The theorem is proven.

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