When Are Two Wave Functions Distinguishable: A New Answer to Pauli's Question, with Potential Application to Quantum Cosmology

Luc Longpré and Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
{longpre,vladik}@utep.edu

Abstract

Traditional quantum mechanics (QM) predicts probabilities of different events. If we describe an elementary particle, then, experimentally, these probabilities mean that if we repeat the same measurement procedure with multiple particles in the same state, the resulting sequence of measurement results will be random w.r.t. the corresponding probability measure. In quantum cosmology, QM is used to describe the world as a whole; we have only one copy of the world, so multiple measurements are impossible. How to interpret these probabilities?

In this paper, we use the approach of the algorithmic information theory to come up with a reasonable interpretation. This interpretation is in good accordance with the arguments presented by several physicists (such as D. Finkelstein) that a wave function is not always a physically reasonable description of a quantum state.

1 Standard quantum description: brief reminder

Intended audience and a need for reminders. The main objective of this paper is to apply the notions of algorithmic information theory and Kolmogorov randomness to quantum physics. We therefore expect this paper to be of some interest both

- to specialists in Kolmogorov randomness who are interested in possible applications, and
- to specialists in quantum physics who may be interested in physical consequences.

We realize that few people are well familiar with both research areas of quantum physics and Kolmogorov randomness. So, to make this paper more readable to both audiences, we need to include brief introductions to both areas.

Readers who are well familiar with quantum physics can skip the physical introduction (which follows right after this comment), and readers who are familiar with the algorithmic information theory can skip the introduction to Kolmogorov-Martin-Löf randomness (which follows later).

Classical (pre-quantum) description. In the traditional classical (pre-quantum) description of particles, the state of each elementary particle can be characterized by its location $x \in R^3$ in the 3-D space R^3, by its momentum $p = m \cdot v$ (where m denotes the particle's mass and v its velocity), etc.

Similarly, the state of an N-particle system is characterized by describing the coordinates $x_1, \ldots, x_N$ and momenta $p_1, \ldots, p_N$ of all the particles.

Quantum description is probabilistic. In quantum physics, a particle does not have a certain location x or a certain momentum p: if we measure location of several particles prepared in the same state, we get different locations with different probabilities. Similarly, we get different values of the momentum with different probabilities.

In contrast to the classical (pre-quantum) case, a *state* of a quantum particle does not enable us to determine the exact location or the exact momentum; instead, a quantum state uniquely determine the *probabilities* of different locations and/or different momenta.

Case of a single particle. In the traditional quantum mechanics, a state is described by a complex-valued function ψ called a *wave function*.

For example, a state of a single particle is described by a complex-valued function $\psi(x)$ defined on the 3-D space ($x \in R^3$). Under this description, for every set S, the probability to find this particle in an area $S \subseteq R^3$ is equal to the integral $\int_S |\psi(x)|^2\,dx$, where $|\psi(x)|$ denotes the absolute value (magnitude) of the complex number $\psi(x)$. In other words, the function $|\psi(x)|^2$ is the probability density function of the probability function which describes the particle's location.

The total probability to get any value $x \in R^3$ should be equal to 1, so we must have $\int_{R^3} |\psi(x)|^2\,dx = 1$.

The probability to find the moment p within a certain area S is similarly equal to $\int_S |F(\psi)(\omega)|^2\,d\omega$, where $F(\psi)$ denotes the Fourier transform of the wave function $\psi(x)$.

Linear structure on the set of wave functions. One of the fundamental notions of quantum physics is the notion of superposition. In terms of wave functions, superposition of n states $\psi_1(x), \ldots, \psi_n(x)$ simply means a linear combination $\psi(x) \stackrel{\text{def}}{=} c_1 \cdot \psi_1(x) + \ldots + c_n \cdot \psi_n(x)$ of the corresponding wave functions $\psi_i(x)$, for appropriate (complex-valued) constants c_i.

Bilinear product on the set of all quantum states. From the purely mathematical viewpoint, we can consider arbitrary linear combinations. However, for such composition, the integral $\int_{R^3} |\psi(x)|^2\,dx$ may be different from 1. For example, if we multiply a wave function $\psi(x)$ by a constant $c > 1$, then for the new function $\psi'(x) = c \cdot \psi(x)$, this integral is equal to $c^2 > 1$.

From the physical viewpoint, however, this integral describes the probability to find a particle somewhere, so it must be equal to 1. So, from the physical viewpoint, the only linear combinations which make physical sense are the ones for which this integral is equal to 1.

For $\psi(x) = \sum\limits_{i=1}^{n} c_i \cdot \psi_i(x)$, the integral takes the form

$$\int_{R^3} |\psi(x)|^2\,dx = \int_{R^3} \psi(x) \cdot \psi^*(x)\,dx = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \cdot c_j^* \cdot \langle \psi_i \,|\, \psi_j \rangle,$$

where $\langle \psi_i \,|\, \psi_j \rangle \stackrel{\text{def}}{=} \int_{R^3} \psi_i(x) \cdot \psi_j^*(x)\,dx$. Thus, from the physical viewpoint, it is important to study such expressions $\langle \psi \,|\, \psi' \rangle$.

Case of multi-particle systems. Similarly, a state of an N-particle system is described by a function $\psi(x_1, \ldots, x_N)$, where $x_1, \ldots, x_N$ are coordinates of these particles. Here, the multi-dimensional probability density function $|\psi(x_1, \ldots, x_N)|^2$ describes the probabilities of different locations $x_1, \ldots, x_N$, and the fact that the overall probability should be equal to 1 means that $\int |\psi(x_1, \ldots, x_N)|^2\,dx_1 \ldots dx_n = 1$.

Similarly to the 1-dimensional case, superposition is represented by a linear combination.

Towards a general case. In both cases, the expressions $\langle \psi \,|\, \psi' \rangle$ satisfy the properties that $\langle \psi' \,|\, \psi \rangle = \langle \psi \,|\, \psi' \rangle^*$, $\langle \psi \,|\psi \rangle \geq 0$, and that

$$\langle c_1 \cdot \psi_1 + c_2 \cdot \psi_2 \,|\, \psi' \rangle = c_1 \cdot \langle \psi_1 \,|\, \psi' \rangle + c_2 \cdot \langle \psi_2 \,|\, \psi' \rangle;$$

$$\langle \psi \,|\, c_1 \cdot \psi_1' + c_2 \cdot \psi_2' \rangle = c_1^* \cdot \langle \psi \,|\, \psi_1' \rangle + c_2^* \cdot \langle \psi \,|\, \psi_2' \rangle.$$

General case: a Hilbert space description. In the general case, we need a linear structure to define composition. To take into consideration that the overall probability should be equal to 1, we need a complex-valued bilinear function $\langle \psi \,|\, \psi' \rangle$ defined on pairs of states, which satisfies the above properties. A linear space with a bilinear operation which satisfies these properties is called a *Hilbert space* H.

For example, in the case of a single particle, the corresponding Hilbert space is the space of all square integrable functions $\psi(x)$ from the space R^3 to real numbers, i.e., functions for which $\|\psi\|^2 \stackrel{\text{def}}{=} \int |\psi(x)|^2\,dx < +\infty$.

Of course, not all elements of a linear space can be actual states of a quantum system, only those elements for which $\|\psi\| \stackrel{\text{def}}{=} \langle \psi \,|\, \psi \rangle = 1$. In other words, physical states form a *unit sphere* $\{\psi : \|\psi\|^2 = 1\}$ in a Hilbert space.

Measurements: a general description. In quantum physics, a measurement is described by a self-adjoint operator $A : H \to H$, i.e., an operator for which $\langle \psi \mid A\psi' \rangle = \langle A\psi \mid \psi' \rangle$ for all states $\psi, \psi' \in H$. For example, the measurement of an x coordinate can be described by an operator $x : \psi(x) \to x \cdot \psi(x)$.

Eigenvalues λ_j of the operator A represent possible values of the measurement result. When the measurement result is λ_j, the original state ψ of the quantum system "transforms" into a new state $\psi' = c \cdot P_j(\psi)$, where P_j is a projection to the corresponding eigenspace, and c is a normalizing factor (which ensures that $\|\psi'\|^2 = 1$). The probability that the measurement will result in the j-th eigenvalue λ_j is equal to $|P_j(\psi)|^2$.

2 Pauli's question: what is known and what is still open

Pauli's original question: how uniquely can we determine the quantum state based on the measurements? For a single particle, if all we measure are coordinates, then we can only determine the absolute values $|\psi(x)|$ of the wave function $\psi(x)$. By measuring other characteristics such as momentum, we can gain more information about the wave function.

W. Pauli asked the following natural question (see, e.g., [27]): to what extent can we determine the wave function from the measurements?

First clarification: we need a generator generating systems in the same state. After we measure, e.g., coordinates of a single particle, the original state transforms into the corresponding eigenstate, i.e., into a state in which a particle is located in the corresponding spatial location with probability 1. After this measurement, the remaining information about the original state is lost.

What Pauli had in mind was a typical quantum situation when we do not only have a single particle, we also have a (potentially infinite) ensemble of particles generated in the same state ψ. For that, we need to have a generator which generates particles in the same state.

For example, we may have a laser generating photons in a given state, or a special type of radioactivity. Once we have such an ensemble, we can measure coordinates on some of these particles, momenta on some other particles, etc. So, by applying different measurements procedures A and applying each such procedure many times, we can determine the probabilities $|P_j(\psi)|^2$ of all possible eigenvalues λ_j.

Second clarification: we need a projective Hilbert space. From the above description of the measurement process, we can easily conclude that for every state ψ, for every measurement A, and for every real number α, for the state $\psi' = e^{\mathrm{i}\cdot\alpha} \cdot \psi$, the probability to get any measurement result λ_j is the same for the original state ψ.

So, from the physical viewpoint, the elements ψ and $\psi' = e^{\mathrm{i}\cdot\alpha} \cdot \psi$ of the Hilbert space H actually describe the same state. Thus, strictly speaking, a quantum state should be associated not with a *single* element $\psi \in H$, but rather with an *equivalence class* $\{e^{\mathrm{i}\cdot\alpha} \cdot \psi\}_\alpha$ of such elements.

In geometry, the collection of such equivalence classes is called a *projective space*. So, in these geometric terms, we can say that the actual space of possible states is a *projective Hilbert space* $P(H)$.

In these terms, Pauli's question is: is it possible to uniquely determine the state $\psi \in P(H)$ based on the (physically meaningful) measurements? And if we cannot determine the state uniquely, "how uniquely" can we determine this state – i.e., when can two different states $\psi \neq \psi'$ be distinguished by appropriate measurements?

Pauli's question: what is known. It is known that for particles, by performing appropriate measurements, we can uniquely determine the original state $\psi \in P(H)$; see, e.g., [9, 11, 17, 23, 24, 25].

Specifically, in addition to measuring coordinates and momenta at the initial moment of time, we can place the particle in some (physically meaningful) potential fields and re-measure coordinates and momenta after a certain time. Based on the results of these measurements, we can uniquely reconstruct $\psi \in P(H)$.

Pauli's question: open problems related to practical implementation. Before describing the main problem of interest to us, let us mention several open problems directly related to the above results.

These problems are related to the fact that the above results show that *in principle*, we can uniquely reconstruct the quantum state ψ by performing appropriate measurements. This positive answer does not yet mean that we have an easy-to-implement efficient practical procedure for such reconstruction. To design such a procedure, we must solve two types of open problems.

The first class of open problems is related to the fact that the practical implementation of the specific complex measurement procedures prescribed in the above papers may be practically difficult. It is therefore reasonable to try to restrict ourselves to easier-to-implement procedure. Once we restrict ourselves to operators A from a certain class $\mathcal{A}$, it is reasonable to ask whether measurements A from a given class $\mathcal{A}$ enable us to uniquely recover the quantum state. Many of such questions are still open.

Another class of open problems is related to the *computational complexity* of reconstructing ψ. In general, the problem of reconstructing ψ from the given measurement results is computationally difficult (to be precise, NP-hard); see, e.g., [18]. Crudely speaking, this means that in some cases, for this reconstruction, we need computation time which exceeds the lifetime of the Universe. So, from the practical viewpoint, we must make sure that in the reconstruction scheme we propose, not only we can theoretically reconstruct ψ, but that we can perform all needed computations in reasonable time.

Pauli's question: a remaining fundamental problem. In the original question, we assumed that we have a generator which generates particles prepared in *the same* state ψ, and the question was how we can determine this state ψ.

In practice, we can have an ensemble of particles prepared in *different* states $\psi_1, \psi_2, \ldots, \psi_n, \ldots$ This may happen, e.g., when we observe rare high-energy events in cosmic rays.

For example, a faraway quasar emits a series of particle in the state ψ_1; these high-energy particles travel to all sides of the Universe, but only one of these particles reaches our detector: all the others end up on different planets and maybe even in different galaxies. Next, another quasar (or another cosmic event) emits another bunch of particles, all prepared in a different state ψ_2; again, we observe only one of these states, etc.

In such situations, the original Pauli's question takes the following form: what can we determine about the original sequence of states $\Psi = (\psi_1, \ldots, \psi_n, \ldots)$ based on our measurements?

Mathematical comment. From the mathematical viewpoint, if we have two independent particles in the state $\psi_1 \in H_1$ and $\psi_2 \in H_2$, then the state of the 2-particle system can be described as a tensor product of these states: $\psi = \psi_1 \otimes \psi_2 \in H_1 \otimes H_2$.

Similarly, the state of an infinite sequences of independent particles can be described as an infinite tensor product $\psi_1 \otimes \psi_2 \otimes \ldots$

Simple observation: we cannot uniquely determine the sequence of states. As we have mentioned, if we only measure a state once, we cannot uniquely determine this state. In our new situation, we measure every state ψ_i exactly once, so we cannot uniquely determine each of these states.

Thus, in contrast to the generator case, when the state ψ can be uniquely determined, we cannot uniquely determine the sequence of states Ψ. The question is: what can we determine? when can we distinguish between the two different sequences $\Psi = (\psi_1, \psi_2, \ldots)$ and $\Psi' = (\psi'_1, \psi'_2, \ldots)$?

What would be a good answer to this question. As we have just mentioned, for the case of a generator, we can uniquely determine the quantum state $\psi \in P(H)$ from measurements.

Our path to this answer was not as direct as it may now seem. First, we mentioned that states are described by functions $\psi(x)$ from the Hilbert space H. Then, we observed that we cannot uniquely determine the corresponding function $\psi(x) \in H$ from measurements, because a different function $\psi'(x) = e^{\mathrm{i}\cdot\alpha} \cdot \psi(x)$ leads to exact same measurement results. We resolved this non-uniqueness by considering the corresponding factor-space (= the set of equivalence classes), which in that case was the projective Hilbert space $P(H)$. On this factor-space, reconstruction is unique.

Similarly, for general ensembles, we know that the sequence Ψ cannot be uniquely determined by measurements: some sequences $\Psi' \neq \Psi$ cannot be distinguished by measurements. We would like to describe the corresponding equivalence classes so that in the resulting factor-space, reconstruction will be unique.

This is, in essence, what we will do in this paper.

Comment 1. Several results presented in this paper have been previously announced in [21].

Comment 2. The fact that from the physical viewpoint, wave functions may not be the best descriptions of states, have been emphasized by many physicists; see, e.g., Finkelstein [5]. In the present paper, we translate the corresponding physical arguments into a new mathematical formalism: namely, we provide new mathematical objects for describing quantum states, objects which are, in our opinion, closer to the physical intuition.

This question has potential applications to quantum cosmology. Let us show that there is a fundamental situation in which such a factor-space can be very useful: quantum cosmology.

According to modern physics, all the physical processes in the world should be described by quantum mechanics. Usually, the state of a quantum object is described by a wave function. This description makes perfect physical sense when we analyze, e.g., the states of elementary particles. We usually have a large ensemble of similar particles in the same state ψ; thus, by measuring different quantities on different particles from this ensemble, we can uniquely reconstruct the state ψ.

From the purely mathematical viewpoint, we can apply the same formalism to the description of the Universe as a whole, and claim that the state of the Universe is described by a wave function ψ. However, our Universe is unique, so we only have a single object in this state. We have already mentioned that by performing measurements on a unique objects, we cannot determine the state. Thus, the wave function describing the state of the Universe cannot be experimentally determined and thus, has no direct physical sense.

This argument is in line with the usual physicists' claim that a wave function cannot be determined (and thus does not make physical sense) for a *unique* object – a wave function only makes physical sense if we have an *ensemble* of identical objects prepared in the same state.

Comment. In principle, we can try to go around this problem by assuming that instead of a unique Universe, we have a "multi-verse" consisting of many alternative universes. However, from the operationalistic viewpoint, this does not solve our problem:

- If our universe does not interact with other universes, then we cannot check what is going on in those universes, and so, this assumption does not have any operationalistic sense.
- On the other hand, if our Universe does interact with other universes (e.g., along the lines of A. D. Sakharov's paper [28] cited in [22]), then in reality, the whole set of these universes should be considered the true (multi-dimensional) universe, and the problem appears again.

Applications to quantum cosmology: continued. How can the above description help? The Universe consists of many parts with very weak interaction between them. Therefore, with a good accuracy, we can assume that these parts are independent and can, therefore, be described by separate wave functions $\psi_1, \ldots, \psi_n, \ldots$ From the mathematical viewpoint, the state of the Universe thus corresponds to the sequence $\Psi = (\psi_1, \ldots)$ of functions ψ_i.

If we know the corresponding factor-space, then we will be able to say that a proper physical description of the state of the universe is by an element of this factor space.

3 Auxiliary reminder: closeness in the projective Hilbert space

To describe how accurately we can reconstruct individual states, we need a metric on the set of all states. We have already mentioned that for an ensemble of objects in different states ψ_i, we cannot reconstruct each state ψ_i uniquely. So, a natural question is: how close can we reconstruct this state?

This leads to another question: what is a natural way to described closeness of states in the projective Hilbert space?

In Hilbert space, there is a natural metric but it is not sufficient. In a Hilbert space, there is a natural distance between the two elements ψ and ψ': the value $\|\psi - \psi'\|$.

However, this value is not exactly what we want. Indeed,

- as we have mentioned, the wave functions ψ and $\psi' = e^{\mathrm{i}\cdot\alpha} \cdot \psi$ describe the same physical state, but
- the Hilbert distance between these two states is non-zero: e.g., for $\alpha = \pi$, we have $\psi' = -\psi$, hence $\psi - \psi' = \psi - (-\psi) = 2\psi$ and $\|\psi - \psi'\| = 2\|\psi\| = 2$.

Natural idea. A natural way to transform the Hilbert distance into a physically adequate distance is to consider the smallest possible distance between ψ and *all* the wave functions representing ψ', i.e., define

$$d(\psi, \psi') \stackrel{\text{def}}{=} \min_{\alpha} \|\psi - e^{\mathrm{i}\cdot\alpha} \cdot \psi'\|.$$

Let us simplify the resulting formula into an easy-to-compute expression.

Derivation of the corresponding formula. To simplify the above formula, we take into account that $d^2(\psi,\psi') \stackrel{\text{def}}{=} \min\limits_{\alpha} \|\psi - e^{\mathrm{i}\cdot\alpha}\cdot\psi'\|^2$ and that $\|a\|^2 = \langle a\,|\,a\rangle$. As a result, we conclude that

$$\|\psi - e^{\mathrm{i}\cdot\alpha}\cdot\psi'\|^2 = \langle \psi - e^{\mathrm{i}\cdot\alpha}\cdot\psi'\,|\,\psi - e^{\mathrm{i}\cdot\alpha}\cdot\psi'\rangle =$$

$$\langle\psi\,|\,\psi\rangle + \langle\psi'\,|\,\psi'\rangle - e^{\mathrm{i}\cdot\alpha}\cdot\langle\psi\,|\,\psi'\rangle - e^{-\mathrm{i}\cdot\alpha}\cdot\langle\psi'\,|\,\psi\rangle.$$

Taking into account that $\|\psi\|^2 = \|\psi'\|^2 = 1$ and that $\langle\psi'\,|\,\psi\rangle = \langle\psi\,|\,\psi'\rangle^*$, we conclude that

$$\|\psi - e^{\mathrm{i}\cdot\alpha}\cdot\psi'\|^2 = 2 - e^{\mathrm{i}\cdot\alpha}\cdot\langle\psi\,|\,\psi'\rangle - (e^{\mathrm{i}\cdot\alpha}\cdot\langle\psi\,|\,\psi'\rangle)^* = 2 - 2\cdot\mathrm{Re}(e^{\mathrm{i}\cdot\alpha}\cdot\langle\psi\,|\,\psi'\rangle).$$

This difference is the smallest when the real part of the product $e^{\mathrm{i}\cdot\alpha}\cdot\langle\psi\,|\,\psi'\rangle$ takes the largest possible value. The real part of a complex number cannot exceed its magnitude, and for the appropriate phase, the real part is equal to the magnitude. Thus, the minimum in the definition of $d^2(\psi,\psi')$ is attained when the real part becomes the magnitude. So, we arrive at the following formula:

Resulting formula.

$$d^2(\psi,\psi') = 2 - 2|\langle\psi\,|\,\psi'\rangle|.$$

4 Kolmogorov randomness as a way to formalize distinguishability

Need for an algorithmic notion of randomness. From the physical viewpoint, what does it mean to be able to distinguish between two different sequences of states $\Psi = (\psi_1,\psi_2,\ldots)$ and $\Psi' = (\psi'_1,\psi'_2,\ldots)$? It means that we can select appropriate measurements A_i, $i = 1,2,\ldots$ (with eigenvalues $\lambda_{i1},\lambda_{i2},\ldots$) so that after applying A_i to the state ψ_i, we get a sequence of eigenvalues λ_{ij} which cannot occur if we apply these same operators to the states ψ'_i.

In quantum physics, we can only predict the probabilities of different values. In other words, the only prediction that we can make about the sequence of the measurement results is that this sequence is, in some reasonable sense, "random" with respect to the corresponding probability measure.

To describe this in formal terms, we need to have a formalized ("algorithmic") definition of randomness.

Mathematical comment. In terms of the tensor product state $\psi = \psi_1\otimes\psi_2\otimes\ldots$, measuring A_i in a state ψ_i is equivalent to measuring a single tensor product operator $A_1\otimes A_2\otimes\ldots$ in the state ψ.

Kolmogorov-Martin-Löf randomness: motivations. A formalized definition of randomness was provided by Kolmogorov and Martin-Löf; see, e.g., [20]. Let us explain its main ideas on the example of the simplest case when we have n independent random variables each of which take the value 0 or 1 with probability 1/2.

When we say that the actual sequence $\omega = (\omega_1\omega_2\ldots)$ is random, we mean that it should satisfy all the laws of probability: for example, the ratio of 0s in a subsequence $\omega_1\ldots\omega_n$ should tend to $1/2$, the deviation between this ratio and $1/2$ should asymptotically be described by an appropriate Gaussian distribution, etc.

In probability theory, all these probability laws are formulated as follows: for almost all sequences (i.e., with probability 1), the ratio tends to 1/2, the distribution of the deviation tends to Gaussian, etc. For each such law, there is a set of probability measure 0, and all sequences outside this set satisfy this law. In these terms, the fact that a sequence is random means that it does not belong to any of these sets of measure 0.

Of course, we cannot simply claim that the sequence ω does not belong to any set of measure 0: because ω belongs to the 1-element set $\{\omega\}$, and for the probability measure corresponding to our simple situation, every 1-element set has probability 0. The good news is that when we talk about the laws of probability, we only mean laws which can be expressed by a finite sequence of symbols in some formal (mathematical) language. There are only countably many such sequences. So, if we restrict ourselves to sets of probability 0 which are definable (in some reasonable set), then we can define a random sequence as a sequence which does not belong to any of such sets.

These are countably many definable sets of measure 0, so their union also has measure 0, and thus, almost all sequences are random in this sense.

Kolmogorov-Martin-Löf randomness: resulting definition. In a nutshell, this is the whole definition of Kolmogorov-Martin-Löf randomness. We fix some formal language, and we say that a sequence is *random* with respect to a given probability measure if it does not belong to any set of measure 0 which is definable in this language (i.e., which can be described by a formula from this formal language).

For details and results related to this definition, see, e.g., [20].

Kolmogorov-Martin-Löf randomness have been used in quantum physics. The idea of using the above notion of randomness was first proposed in [1]. Following applications have been overviewed in [20]; see also [13, 14, 15, 16].

Our new application is different from the previous ones. By itself, the notion of algorithmic randomness does not change any experimental results, it simply formalizes the intuitive notion of randomness. In accordance with this fact, the results of the previous applications of this notion to quantum mechanics were mainly foundational.

Our new result is different: we not only provide an answer to a *foundational* question of when two sequences of states can be distinguished, we also produce an exact *analytical* criteria for such distinguishability.

Now, we are ready for a formal description of our result.

5 Definitions and the main results

Definition 1. *Let $H_1, H_2, \ldots$ be a sequence of Hilbert spaces.*

- *By a* sequence of states*, we mean a sequence* ($\Psi = (\psi_1, \psi_2, \ldots)$ *for which* $\psi_i \in H_i$.
- *By a* sequence of measurements*, we mean a sequence* $A = (A_1, A_2, \ldots)$ *of self-adjoint operators* $A_i : H_i \to H_i$.
- *For each i, for each $\psi_i \in H_i$, and for each $A_i : H_i \to H_i$, we define a probability measure $\mu_i(A_i, \psi_i)$ on the set of real numbers as follows: it is located on the eigenvalues $\lambda_{i1}, \lambda_{i2}, \ldots$ of the operator A_i, and the probability of the j-th eigenvalues λ_{ij} is equal to $|P_{ij}(\psi_i)|^2$, where P_{ij} denotes the projection on the corresponding eigenspace.*
- *For each Ψ and A, we define the measure $\mu(A, \Psi)$ on the set of all sequences of real numbers as the Cartesian product of the measures μ_i.*
- *We say that a sequence of real numbers is a* possible result *of measuring A on Ψ if this sequence is random w.r.t. $\mu(A, \Psi)$; the set of all such possible results will be denoted by* $\mathrm{Poss}(A, \Psi)$.
- *We say that a sequence of measurements A* cannot distinguish *between the sequence of states Ψ and Ψ' if every possible result of measuring A on Ψ is also a possible result of measuring A on Ψ':* $\mathrm{Poss}(A, \Psi) = \mathrm{Poss}(A, \Psi')$.
- *We say that sequences Ψ and Ψ' are* distinguishable *if there exists A for which no possible result of measuring A on Ψ can occur when we measure A on Ψ':* $\mathrm{Poss}(A, \Psi) \cap \mathrm{Poss}(A, \Psi') = \emptyset$.

In other words, sequences are distinguishable if we can organize an appropriate sequence of measurements which enables us to distinguish between them. The following result shows that it *is* important to select appropriate measurements.

Proposition 1. *For every two sequences of states Ψ and Ψ', there exists a sequence of measurements A which cannot distinguish them.*

Comment. For readers' convenience, the proofs are placed in the special appendix.

Theorem. *Two sequences of states $\Psi = (\psi_1, \psi_2, \ldots)$ and $\Psi' = (\psi_1', \psi_2', \ldots)$ are distinguishable if and only if*

- *either $\psi_i \perp \psi_i'$ (i.e., $\langle \psi_i \,|\, \psi_i' \rangle = 0$) for some i,*
- *or $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi_i') = +\infty$.*

Discussion. One can argue that the case when $\psi \perp \psi_i'$ is not very physical. Indeed, in practice, states are usually generated with some accuracy, and if we slightly deviate from ψ_i or from ψ_i', we lose orthogonality. It is therefore reasonable to redefine distinguishability in such a way that it will be stable with respect to such minor deviations.

Definition 2. *Let $\Psi = (\psi_1, \psi_2, \ldots)$ and $\widetilde{\Psi} = (\widetilde{\psi}_1, \widetilde{\psi}_2, \ldots)$ be two sequences of states, and let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ be a sequence of positive real numbers. We say that Ψ and Ψ' are* ε-close *if $d(\psi_i, \psi_i') \le \varepsilon_i$ for all i.*

Definition 3. *We say that two sequences of states $\Psi = (\psi_1, \psi_2, \ldots)$ and $\Psi' = (\psi_1', \psi_2', \ldots)$ are* stably distinguishable *if the following two statements hold:*

- *Ψ and Ψ' are distinguishable, and*
- *there exists a sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ of positive real numbers with the following property: if $\widetilde{\Psi}$ is ε-close to Ψ and $\widetilde{\Psi'}$ is ε-close to Ψ', then $\widetilde{\Psi}$ and $\widetilde{\Psi'}$ are also distinguishable.*

In other words, the two sequences are stably distinguishable if they are guaranteed to be distinguishable even if we implement them with some inaccuracy.

Proposition 2. *Two sequences of states $\Psi = (\psi_1, \psi_2, \ldots)$ and $\Psi' = (\psi_1', \psi_2', \ldots)$ are stably distinguishable if and only if $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi_i') = +\infty$.*

So, the sequences are strongly distinguishable if and only if $d^2(\Psi, \Psi') = +\infty$, where

$$d^2(\Psi, \Psi') \stackrel{\text{def}}{=} \sum_{i=1}^{\infty} d^2(\psi_i, \psi_i').$$

Thus, the states cannot be distinguished if and only if $d^2(\Psi, \Psi') < +\infty$.

The relation $d^2(\Psi, \Psi') < +\infty$ is an equivalence relation, so the desired space of sequences of sets is the factor-set of the tensor product $H_1 \otimes H_2 \otimes \ldots$ over this relation. In particular, a proper description of quantum cosmology equations should produce an element of this factor space.

6 Auxiliary results and an open problem

First auxiliary result: weak distinguishability. In the above text, we defined the two sequences to be distinguishable if we can distinguish them based on an arbitrary measurement result. It may be reasonable to define *weak distinguishability* as the possibility to distinguish based on *some* measurement results. A natural question is: when can the two sequences of states be weakly distinguished?

Definition 4. *We say that sequences Ψ and Ψ' are* weakly distinguishable *if there exists A and a possible result of measuring A on Ψ which cannot occur when we measure A on Ψ' (or vice versa), i.e.,* $\mathrm{Poss}(A, \Psi) \neq \mathrm{Poss}(A, \Psi')$

Proposition 3. *Two sequences of states Ψ and Ψ' are weakly distinguishable if and only if they are different: $\Psi \neq \Psi'$.*

Second auxiliary result: going back to the case when we have a sequence of identical states. The above result is about the situation when different states ψ_i are, in general, different. However, it has an interesting consequence for the case when we have states $\psi_1 = \psi_2 = \ldots = \psi$ generated by the same state generator. Indeed, in such a generator situation, we simply assume that all the state are indeed the same. A natural question is: can we check this assumption experimentally? In other words, if in reality, states are different, will we be able to distinguish this situation from the case when they are identical?

The above theorem provides an answer to this question: what we need is to distinguish between the sequence of states $\Psi = (\psi_1, \psi_2, \ldots)$, and the ideal sequence $(\psi, \psi, \ldots)$:

Corollary 1. *A sequence of states $\Psi = (\psi, \psi_2, \ldots)$ can be distinguished from the ideal sequence $(\psi, \psi, \ldots)$ if and only if:*

- *either $\psi_i \perp \psi$ for some i*
- *or $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi) = +\infty$.*

Corollary 2. *A sequence of states $\Psi = (\psi, \psi_2, \ldots)$ can be stably distinguished from the ideal sequence $(\psi, \psi, \ldots)$ if and only if $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi) = +\infty$.*

Remaining open problem: going beyond Kolmogorov-Martin-Löf definition of randomness. In this paper, we used Kolmogorov-Martin-Löf description of randomness. It is worth mentioning that while this definition captures *most* of physicists' ideas about randomness, it does *not* fully capture *all* of them. This can be illustrated already on the simplest probability measure, when we have a fair coin which can fall heads or tails with equal probability 1/2.

In the Kolmogorov-Martin-Löf formalization, if we have an infinite sequence $\omega = (\omega_1\omega_2\ldots)$ which is random with respect to the corresponding probability measure, and we add 1,000,000 heads H...H in front of this sequence, the resulting new sequence H...H$\omega_1\omega_2\ldots$ is still random. This is clearly counter-intuitive: we do not expect a sequence which starts with a million heads to be truly random.

This fact has been realized early on. There exist modifications of the Kolmogorov-Martin-Löf definition in which this phenomenon does not happen – and which are thus in better accordance with the physicists' intuition about randomness; see, e.g., [4, 12, 20] and references therein. It is therefore desirable to extend our analysis to these modified definitions of randomness.

7 Conclusions

It is well known that if we have a generator which generates objects in the same quantum state ψ, then, by applying different measurements $A_1, A_2, \ldots$ to different objects from the resulting ensemble, we can uniquely determine this state ψ (uniquely modulo a known transformation $\psi \to e^{\mathrm{i}\cdot\alpha}\cdot\psi$). For such a generator situation, this possibility provides a positive answer to the question asked by W. Pauli: to what extent can we reconstruct a wave function from measurements.

If different objects from the ensemble are in different states $\psi_1, \psi_2, \ldots$, and we are only allowed one measurement A_i for each of these states, then, of course, we cannot reconstruct the sequence $\Psi = (\psi_1, \psi_2, \ldots)$ uniquely. For such a situation, Pauli's question takes the following form: when is it possible to select appropriate measurements which would enable us to distinguish between two given sequences $\Psi = (\psi_1, \psi_2, \ldots)$ and $\Psi' = (\psi'_1, \psi'_2, \ldots)$? One possibility is an (unstable) case when $\psi_i \perp \psi'_i$. If we only consider stable distinguishability, then the only possibility is $d^2(\Psi, \Psi') \stackrel{\text{def}}{=} \sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi'_i) = +\infty$.

Thus, in this case, a proper description of the state of this sequence of object is a sequence of wave functions factored over the relation $d^2(\Psi, \Psi') < +\infty$.

This result is of potential interest to quantum cosmology where we deal with a unique object: our Universe (so there is no generator to produce many copies of it :-).

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Appendix: Proofs

Proof of Proposition 1. To prove the proposition, let us select, for each i, an appropriate operator A_i. For those i for which $\psi_i = \psi'_i$, every A_i will do, since here clearly $\mu(A_i, \psi_i) = \mu(A_i, \psi'_i)$.

Let us now consider the values i for which $\psi_i \not\perp \psi'_i$, i.e., for which $\langle \psi_i \,|\, \psi'_i \rangle \neq 0$. In this case, let us take $c_i \stackrel{\rm def}{=} \dfrac{\langle \psi_i \,|\, \psi'_i \rangle}{|\langle \psi_i \,|\, \psi'_i \rangle|}$. By construction, this complex ratio c_i is a number of magnitude 1, i.e., $|c_i|^2 = c_i \cdot c_i^* = 1$.

To find an appropriate A_i, we take two vectors $d_i = \psi_i - c_i \cdot \psi'_i$ and $s_i = \psi_i + c_i \cdot \psi'_i$. Let us show that these two vectors are orthogonal. Indeed,

$$\langle d_i \,|\, s_i \rangle = \langle \psi_i - c_i \cdot \psi'_i \,|\, \psi_i + c_i \cdot \psi'_i \rangle =$$

$$\langle \psi_i \,|\, \psi_i \rangle - c_i \cdot \langle \psi'_i \,|\, \psi_i \rangle + c_i^* \cdot \langle \psi_i \,|\, \psi'_i \rangle - c_i \cdot c_i^* \cdot \langle \psi'_i \,|\, \psi'_i \rangle.$$

Since both ψ_i and ψ'_i are states, we have $\langle \psi_i \,|\psi_i \rangle = \langle \psi'_i \,|\, \psi'_i \rangle = 1$. Due to $c_i \cdot c_i^* = 1$, we conclude that

$$\langle d_i \,|\, s_i \rangle = -c_i \cdot \langle \psi'_i \,|\, \psi_i \rangle + c_i^* \cdot \langle \psi_i \,|\, \psi'_i \rangle.$$

Substituting the definition of c_i and using the fact that $\langle b \,|\, a \rangle = \langle a \,|\, b \rangle^*$, we conclude that $\langle d_i \,|\, s_i \rangle = 0$, i.e., that $d_i \perp s_i$.

We can therefore normalize these two orthogonal vectors into $D_i = \dfrac{d_i}{\|d_i\|}$ and $S_i = \dfrac{s_i}{\|s_i\|}$, and take an operator A_i for which D_i and S_i are eigenvectors corresponding to different eigenvalues. For D_i, the corresponding probabilities are equal to

$$|\langle \psi_i \,|\, D_i \rangle|^2 = \frac{|\langle \psi_i \,|\, d_i \rangle|^2}{\|d_i\|^2}$$

and

$$|\langle \psi'_i \,|\, D_i\rangle|^2 = \frac{|\langle \psi'_i \,|\, d_i\rangle|^2}{\|d_i\|^2}.$$

Thus, to prove that these probabilities coincide, it is sufficient to prove that

$$|\langle \psi_i \,|\, d_i\rangle|^2 = |\langle \psi'_i \,|\, d_i\rangle|^2.$$

Indeed,

$$\langle \psi_i \,|\, d_i\rangle = \langle \psi_i \,|\, \psi_i - c_i \cdot \psi'_i\rangle = \langle \psi_i \,|\, \psi_i\rangle - c_i^* \cdot \langle \psi_i \,|\psi'_i\rangle.$$

Using the definition of c_i and the fact that $\langle \psi_i \,|\, \psi_i\rangle = 1$, we thus conclude that

$$\langle \psi_i \,|\, d_i\rangle = 1 - \frac{\langle \psi_i \,|\, \psi'_i\rangle^* \cdot \langle \psi_i \,|\, \psi'_i\rangle}{|\langle \psi_i \,|\, \psi'_i\rangle|} = 1 - |\langle \psi_i \,|\, \psi'_i\rangle|.$$

Similarly,

$$\langle \psi'_i \,|\, d_i\rangle = \langle \psi'_i \,|\, \psi_i - c_i \cdot \psi'_i\rangle = \langle \psi'_i \,|\, \psi_i\rangle - c_i^* \cdot \langle \psi'_i \,|\psi'_i\rangle.$$

Using the definition of c_i and the fact that $\langle \psi'_i \,|\, \psi'_i\rangle = 1$, we thus conclude that

$$\langle \psi'_i \,|\, d_i\rangle = c_i^* \cdot |\langle \psi_i \,|\psi'_i\rangle| - c_i^* = -c_i^* \cdot (1 - |\langle \psi_i \,|\, \psi_i\rangle|).$$

Since $|c_i| = |c_i^*| = 1$, we conclude that indeed $|\langle \psi_i \,|\, d_i\rangle|^2 = |\langle \psi'_i \,|\, d_i\rangle|^2$.

For S_i, the proof is similar. The proposition is proven.

Proof of the Theorem.

1°. If for some i, we have $\psi_i \perp \psi'_i$, then we can take a projection on ψ_i as the operator A_i. Then:

- for the sequence of states which contains ψ_i, the result of i-th measurement will be 1; and
- for the sequence of states which contains ψ'_i, the result of i-th measurement will be 0.

Thus, we can easily distinguish the two sequences of states.

2°. To complete the proof, we must prove that if $\psi_i \not\perp \psi'_i$ for all i, then the existence of the sequence of measurements A for which $\mathrm{Poss}(A, \Psi) \cap \mathrm{Poss}(A, \Psi') = \emptyset$ is equivalent to $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi'_i) = +\infty$.

2.1°. Let us first prove that if $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi'_i) = +\infty$, then there exists a sequence of measurement A_i for which $\mathrm{Poss}(A, \Psi) \cap \mathrm{Poss}(A, \Psi') = \emptyset$.

Indeed, let us take, as A_i, projection on ψ_i. Then, for the sequence of states $\Psi = (\psi_1, \psi_2, \ldots)$, the only possible result of measuring A is the sequence $(1, 1, \ldots)$.

Let us show that this sequence is not possible if we measure A on the sequence of states Ψ'. Indeed, in this case, the probability of having 1 in the i-th measurement is equal to $|\langle\psi_i\,|\,\psi'_i\rangle|^2$. Thus, the overall probability of this sequence is equal to $\prod\limits_{i=1}^{\infty}|\langle\psi_i\,|\,\psi'_i\rangle|^2$, i.e., to p^2, where $p\stackrel{\rm def}{=}\prod\limits_{i=1}^{\infty}|\langle\psi_i\,|\,\psi'_i\rangle|$.

Let us prove that $p=0$; then, $p^2=0$ and thus, the sequence of measurement results $(1,1,\ldots)$ is not possible for Ψ' – since it belongs to the definable set of $\mu(A,\Psi')$-measure 0.

Indeed, due to the above formula for the distance $d(\psi,\psi')$ on the projective Hilbert space, we conclude that $|\langle\psi_i\,|\psi'_i\rangle| = 1-\dfrac{d^2(\psi_i,\psi'_i)}{2}$. So, the desired probability p takes the form

$$p=\prod_{i=1}^{\infty}\left(1-\frac{d^2(\psi_i,\psi'_i)}{2}\right).$$

Since none of the states are mutually orthogonal, none of the terms in the product are 0s. It is known that in this situation, the product p converges to a non-zero value if and only if the corresponding sum

$$\sum_{i=1}^{\infty}\frac{d^2(\psi_i,\psi'_i)}{2}=\frac{1}{2}\cdot\sum_{i=1}^{\infty}d^2(\psi_i,\psi'_i)$$

converges. Since this sum diverges, we have $p=0$.

2.2°. Let us now prove that if there exists a sequence of measurement A_i and a sequence of real numbers which is possible for both sequences of states Ψ and Ψ', then $\sum\limits_{i=1}^{\infty}d^2(\psi_i,\psi'_i)=+\infty$.

We will illustrate the proof of this statement on the example of binary measurements, when each operator A_i only has two eigenvalues; this proof can be easily extended to the general case.

2.2.1°. In general, the non-existence of a common random sequence means that there exists a definable set which has measure 0 in the sense of the first measure and whose complement has measure 0 in the sense of the second measure. In measure theory and probability theory, probability measures with such a property (without the word "definable") are called *(mutually) singular*. So, the absence of a common random vector means that the product measures $\mu(A,\Psi)$ and $\mu(A,\Psi')$ are mutually singular.

Mutual singularity of two probability measures, with densities $\rho(\omega)$ and $\rho'(\omega)$ w.r.t. some measure ρ_0, can be checked by computing their Hellinger distance (see, e.g., [6, 7, 8, 10, 26]):

$$d_H(\rho,\rho')\stackrel{\rm def}{=}\sqrt{\int(\sqrt{\rho}-\sqrt{\rho'})^2\,d\rho_0}.$$

It is worth mentioning that this distance does not depend on the choice of ρ_0.

Specifically, the measure ρ and ρ' are mutually singular if and only if $d_H(\rho, \rho') = \sqrt{2}$, i.e., if and only if $d^2(\rho, \rho') = 2$. By definition of the Hellinger distance, $d^2(\rho, \rho') = \int \rho \, d\rho_0 + \int \rho' \, d\rho_0 - 2 \cdot H(\rho, \rho')$, where

$$H(\rho, \rho') \stackrel{\text{def}}{=} \int \sqrt{\rho} \cdot \sqrt{\rho'} \, d\rho_0$$

is called *Hellinger affinity.* Since ρ and ρ' are density functions, we have $\int \rho \, d\rho_0 = \int \rho' \, d\rho_0 = 1$ hence $d_H^2(\rho, \rho') = 2 - 2 \cdot H(\rho, \rho')$. So, $d^2(\rho, \rho') = 2$ if and only if $H(\rho, \rho') = 0$.

Thus, the two measures ρ and ρ' are mutually singular if and only if $H(\rho, \rho') = 0$.

2.2.2°. Following [32], let us describe the explicit expression for $H(\rho, \rho')$ for the case when we have product measures.

To be more precise, we consider the case when both measures ρ and ρ' are located on infinite binary sequences $\omega = (\omega_1 \omega_2 \ldots)$:

- For each i, the probability of $\omega_i = 1$ is equal, correspondingly, to p_i (for ρ) and to p_i' (for ρ').
- The probability of $\omega_i = 0$ is equal to $1 - p_i$ (for ρ) and to $1 - p_i'$ (for ρ').

Different elements ω_i are assumed to be independent.

In this case, as a measure ρ_0, it is reasonable to take the standard probability measure on the set of all infinite binary sequences, i.e., a probability measure in which each bit ω_i appears with probability $1/2$, and different bits ω_i and ω_j $(i \neq j)$ are statistically independent. Then, for each i, the i-th probability density $\rho_i(\omega_i)$ is equal:

- to $\rho_i(1) = \dfrac{p_i}{1/2} = 2p_i$ when $\omega_i = 1$ and
- to $\rho_i(0) = \dfrac{1 - p_i}{1/2} = 2(1 - p_i)$ for $\omega_i = 0$.

Both cases can be described by a single expression

$$\rho_i(\omega_i) = 2 \cdot [(1 - \omega_i) + (2\omega_i - 1) \cdot p_i].$$

Similarly, for the measure ρ', the i-th probability density takes the form

$$\rho_i'(\omega_i) = 2 \cdot [(1 - \omega_i) + (2\omega_i - 1) \cdot p_i'].$$

Since the bits ω_i are independent, the overall probability density ρ is equal to the product of these densities: $\rho(\omega) = \prod_{i=1}^{\infty} \rho_i(\omega_i)$ and $\rho'(\omega) = \prod_{i=1}^{\infty} \rho_i'(\omega_i)$. Thus, $H(\rho, \rho')$ is the expected value of the corresponding product:

$$H(\rho, \rho') = E\left[\sqrt{\rho(\omega)} \cdot \sqrt{\rho'(\omega)}\right] = E\left[\prod_{i=1}^{\infty} \sqrt{\rho_i(\omega_i) \cdot \rho_i'(\omega_i)}\right].$$

Since the values ω_i are independent, the expected value of this product is equal to the product of the corresponding expected values:

$$H(\rho, \rho') = E\left[\sqrt{\rho(\omega)} \cdot \sqrt{\rho'(\omega)}\right] = \prod_{i=1}^{\infty} E_i,$$

where we denoted $E_i \stackrel{\text{def}}{=} E\left[\sqrt{\rho_i(\omega_i) \cdot \rho'_i(\omega_i)}\right]$.

The i-th expected value E_i means that we combine the values corresponding to $\omega_i = 1$ and $\omega_i = 0$ with probabilities $1/2$: for every function f, we have $E[f(\omega_i)] = (1/2) \cdot f(1) + (1/2) \cdot f(0)$. In particular, in our case, we have

$$E_i = (1/2) \cdot \sqrt{2p_i} \cdot \sqrt{2p'_i} + (1/2) \cdot \sqrt{2(1-p_i)} \cdot \sqrt{2(1-p'_i)},$$

hence

$$E_i = \sqrt{p_i} \cdot \sqrt{p'_i} + \sqrt{1-p_i} \cdot \sqrt{1-p'_i}.$$

2.2.3°. It is known that, in general, an infinite product $\prod_{i=1}^{\infty}(1-a_i)$ of positive numbers $1 - a_i > 0$ is equal to 0 if and only if the sum $\sum_{i=1}^{\infty} a_i$ is infinite. To apply this fact to our case, we must take $a_i = 1 - E_i$. Then, $H(\rho, \rho') = \prod_{i=1}^{\infty} E_i = 0$ if and only if $\sum_{i=1}^{\infty}(1-E_i) = +\infty$. So, if the measures ρ and ρ' are mutually singular and $H(\rho, \rho') = 0$, we have $\sum_{i=1}^{\infty}(1-E_i) = +\infty$.

The expression $1 - E_i$ can be reformulated as

$$1 - E_i = \frac{1}{2} \cdot (2 - 2E_i) =$$

$$\frac{1}{2} \cdot \left[p_i + (1-p_i) + p'_i + (1-p'_i) - 2 \cdot \sqrt{p_i} \cdot \sqrt{p'_i} - 2 \cdot \sqrt{1-p_i} \cdot \sqrt{1-p'_i}\right] =$$

$$\frac{1}{2} \cdot \left[\left(\sqrt{p_i} - \sqrt{p'_i}\right)^2 + \left(\sqrt{1-p_i} - \sqrt{1-p'_i}\right)^2\right].$$

So, if the measures ρ and ρ' are mutually signular, then

$$\sum_{i=1}^{\infty}\left[\left(\sqrt{p_i} - \sqrt{p'_i}\right)^2 + \left(\sqrt{1-p_i} - \sqrt{1-p'_i}\right)^2\right] = +\infty.$$

2.2.4°. In terms of Kolmogorov-Martin-Löf randomness, this problem was analyzed by Vovk in [30] (see also Problem 4.24 in [19]). In particular, Vovk proved that for the product measures corresponding to the probabilities $(p_1, p_2, \ldots)$

and $(p'_1, p'_2, \ldots)$, the non-existence of a common possible (= random) sequence is indeed equivalent to

$$\sum_{i=1}^{\infty} \left[\left(\sqrt{p_i} - \sqrt{p'_i} \right)^2 + \left(\sqrt{1-p_i} - \sqrt{1-p'_i} \right)^2 \right] = +\infty.$$

For the product measures $\mu(A, \Psi)$ and $\mu(A, \Psi')$, we have $p_i = \|P_{i1}(\psi_i)\|^2$, $p'_i = \|P_{i1}(\psi'_i)\|^2$ $1 - p_i = \|P_{i2}(\psi_i)\|^2$, and $1 - p'_i = \|P_{i2}(\psi'_i)\|^2$ for the corresponding projection operators P_{i1} and P_{i2}. Therefore, $\sqrt{p_i} - \sqrt{p'_i} = \|P_{i1}(\psi_i)\| - \|P_{i1}(\psi'_i)\|$, $\sqrt{1-p_i} - \sqrt{1-p'_i} = \|P_{i2}(\psi_i)\| - \|P_{i2}(\psi'_i)\|$, and the above condition takes the form

$$\sum_{i=1}^{\infty} \left[(\|P_{i1}(\psi_i)\| - \|P_{i1}(\psi'_i)\|)^2 + (\|P_{i2}(\psi_i)\| - \|P_{i2}(\psi'_i)\|)^2 \right] = +\infty.$$

2.2.5°. By definition of a distance $d(\psi_i, \psi'_i)$ in the projective Hilbert space, we have $\psi_i = e^{\mathrm{i}\cdot\alpha} \cdot \psi_i + \delta_i$ for some vector δ_i for which $\|\delta_i\| = d(\psi_i, \psi'_i)$. Since each projection P_{ij} is a linear operator, we have $P_{ij}(\psi_i) = e^{\mathrm{i}\cdot\alpha} \cdot P_{ij}(\psi'_i) + P_{ij}(\delta_i)$. Thus, the triangle inequality leads to

$$\big| \|P_{ij}(\psi_i)\| - \|e^{\mathrm{i}\cdot\alpha} \cdot P_{ij}(\psi'_i)\| \big| \le \|P_{ij}(\delta_i)\|$$

and hence, to

$$(\|P_{ij}(\psi_i)\| - \|e^{\mathrm{i}\cdot\alpha} \cdot P_{ij}(\psi'_i)\|)^2 \le \|P_{ij}(\delta_i)\|^2.$$

The length of a vector in a Hilbert space does not change when we multiply this vector by $e^{\mathrm{i}\cdot\alpha}$, so we have

$$(\|P_{ij}(\psi_i)\| - \|P_{ij}(\psi'_i)\|)^2 \le \|P_{ij}(\delta_i)\|^2.$$

The length of a projection is always smaller than or equal that the length of a vector, so $\|P_{ij}(\delta_i)\| \le \|\delta_i\| = d(\psi_i, \psi'_i)$ and $\|P_{ij}(\delta_i)\|^2 \le d^2(\psi_i, \psi'_i)$. Hence, for every j, we have $(\|P_{ij}(\psi_i)\| - \|P_{ij}(\psi'_i)\|)^2 \le d^2(\psi_i, \psi'_i)$. Therefore,

$$(\|P_{i1}(\psi_i)\| - \|P_{i1}(\psi'_i)\|)^2 + (\|P_{i2}(\psi_i)\| - \|P_{i2}(\psi'_i)\|)^2 \le 2 \cdot d^2(\psi_i, \psi'_i).$$

Thus, the condition

$$\sum_{i=1}^{\infty} \left[(\|P_{i1}(\psi_i)\| - \|P_{i1}(\psi'_i)\|)^2 + (\|P_{i2}(\psi_i)\| - \|P_{i2}(\psi'_i)\|)^2 \right] = +\infty$$

implies that $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi'_i) = +\infty$. For the case of binary measurements, the theorem is proven.

Similar arguments apply in the general case as well.

Proof of Proposition 2. Once can easily check that the condition $\psi_i \perp \psi'_i$ is not stable, while the condition $\sum\limits_{i=1}^{\infty} d^2(\psi_i, \psi'_i) = +\infty$ is preserved if we slightly modify the states ψ_i and ψ'_i.

Proof of Proposition 3. If the sequences Ψ and Ψ' are weakly distinguishable, then, of course, they are different. Let us prove that if they are different, then they are weakly distinguishable.

Indeed, if $\Psi \neq \Psi'$, this means that for some i, we have $\psi_i \neq \psi'_i$. By using orthonomalization, we can find a linear combination ψ of ψ_i and ψ'_i which is orthogonal to ψ_i. Since the states ψ_i and ψ'_i are different, this linear combination is not orthogonal to ψ'_i. Let us now take, as A_i, a projection on ψ. Since $|\langle\psi'_i\,|\,\psi\rangle| > 0$, there exists a random sequence for which the result of i-measurement is 1 and which is random with respect to $\mu(A, \Psi')$ – i.e., which is a possible measurement result when measuring A at Ψ'.

On the other hand, since $\psi \perp \psi_i$, this sequence cannot appear when we measure A on Ψ. The statement is proven.