Square Root of "Not": A Major Difference Between Fuzzy and Quantum Logics

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***Abstract*—Many authors have emphasized the similarity between quantum logic and fuzzy logic. In this paper, we show that, in spite of this similarity, these logics are not identical. Specifically, we show that while quantum logic has a special "square root of not" operation which is very useful in quantum computing, fuzzy logic lacks such an operation.**

I. Similarity Between Quantum Logic and Fuzzy Logic

Both quantum logic and fuzzy logic describe uncertainty:

- quantum logic describes uncertainties of the real world (to be more precise, the uncertainty of our best theory of the world), while
- fuzzy logic described the uncertainty of our reasoning.

Due to this common origin, there is a lot of similarity between the two logics, similarities which have been emphasized in several papers on fuzzy logic; see, e.g., [6] and references therein.

II. What We Plan to Do

We plan to emphasize that, in spite of the similarity, quantum logic and fuzzy logic are not mathematically identical.

Specifically, in this paper, we show the difference on the example of one of the major features of quantum logic, a features that underlies most successful quantum computing algorithms: that in quantum logic, there is a "square root of not" operation.

III. What Is a Square Root of Not?

In precise terms, the fact that an operation $s(x)$ is a square root of "not" means that if we apply this operation twice to a truth value a, we get $\neg a$ ("not a"):

$$s(s(a)) = \neg a$$

for all a.

IV. Negation ("Not") in Classical (2-Valued) Logic

In the traditional (2-valued) logic, we have two possible truth values – "true" and "false". In the computer, "true" is usually represented as 1, and "false" as 0.

In these terms, the negation operation has a very simple form: $\neg(0) = 1$ and $\neg(1) = 0$.

V. There Is No Square Root of Not in Classical Logic

In classical logic, a unary operation $s(a)$ can be described by listing its values $s(0)$ and $s(1)$. There are two possible values of $s(0)$ and two possible values of $s(1)$, so overall, we have $2 \times 2 = 4$ possible unary operations:

- when $s(0) = s(1) = 0$, then we get a constant function whose value is "false";
- when $s(0) = s(1) = 1$, then we get a constant function whose value is "true";
- when $s(0) = 0$ and $s(1) = 1$, we get the identity function;
- finally, when $s(0) = 1$ and $s(1) = 0$, we get the negation.

In all four cases, the composition $s(s(a))$ is different from the negation:

- for the "constant false" function s, we have $s(s(a)) = s(a)$, i.e., the composition of s and s is also a constant false function;
- for the "constant true" function s, also $s(s(a)) = s(a)$, i.e., the composition of s and s is also a constant true function;;
- for the identity function s, we have $s(s(a)) = s(a)$, i.e., the composition of s and s is also the identity function;
- finally, for the negation s, the composition of s and s is the identity function.

VI. Quantum Mechanics

Since early 20th century, physicists have found out that our physical world is better described not by the classical Newtonian physics, but by the laws of quantum mechanics. The smaller the particles, the larger the deviation between the classical and quantum descriptions. So, for macro-size bodies, Newtonian mechanics provides a very accurate description. However, for molecules and atoms, it is important to take into account quantum effects.

One of the main features of quantum mechanics is the possibility of *superpositions*. Namely, each classical state s is also a quantum state – denoted by $|s\rangle$. However, in addition to this, for every n states $s_1, \ldots, s_n$, and for every n complex numbers $c_1, \ldots, c_n$ for which

$$|c_1|^2 + \ldots + |c_n|^2 = 1,$$

the following state is also possible:

$$c_1 \cdot |s_1\rangle + \ldots + c_n \cdot |s_n\rangle.$$

If, in this state, we try to measure whether we are in the state s_1, or in the state s_2, etc., then:

- we will get the state s_1 with the probability $|c_1|^2$;
- ...
- we will get the state s_n with the probability $|c_n|^2$.

The above requirement $|c_1|^2 + \ldots + |c_n|^2 = 1$ simply comes from the fact that the probabilities should add up to 1.

It is worth mentioning that if we multiply all the values c_i by the same constant $e^{\mathrm{i}\cdot\alpha}$ (with real α) whose absolute value is 1, we get the same probabilities of all the states. In quantum mechanics, mathematically different states s and $e^{\mathrm{i}\cdot\alpha}$ are therefore considered to be corresponding to the same physical state.

VII. Quantum Logic

Quantum Logic is an application of the general idea of quantum mechanics to logic. In the classical logic, there are two possible states: 0 and 1. In quantum logic, in addition to these states $|0\rangle$ and $|1\rangle$, we can have arbitrary superpositions

$$c_0 \cdot |0\rangle + c_1 \cdot |1\rangle$$

for complex values c_0 and c_1 for which $|c_0|^2 + |c_1|^2 = 1$.

These superpositions are the "truth values" of quantum logic.

VIII. Negation in Quantum Logic

For "pure" (classical) states $|0\rangle$ and $|1\rangle$, negation can be defined in a standard way:

$$\neg(|0\rangle) = |1\rangle$$

and

$$\neg(|1\rangle) = |0\rangle.$$

In general, in quantum mechanics, all operations are linear in terms of superpositions. By using this linearity, we can describe the negation of an arbitrary quantum state:

$$\neg(c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) = c_0 \cdot |1\rangle + c_1 \cdot |0\rangle.$$

IX. Square Root of Not

Let us show that in quantum mechanics, there exists an operation s whose square is, in some reasonable sense, equal to negation. Due to linearity, it is sufficient to define this operation for the basic states $|0\rangle$ and $|1\rangle$. We can define it as follows:

$$s(|0\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle;$$

$$s(|1\rangle) = \frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle.$$

Let us show that the composition of this operation with itself indeed leads to negation, i.e., that $s(s(a)) = \neg(a)$ for all a.

Due to linearity, it is sufficient to prove this fact for the basic states $|0\rangle$ and $|1\rangle$.

For $|1\rangle$, we get

$$s(s(|1\rangle) = s\left(\frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle\right).$$

Due to linearity, this means that

$$s(s(|1\rangle) = \frac{1}{\sqrt{2}} \cdot s(|0\rangle) + \frac{1}{\sqrt{2}} \cdot s(|1\rangle).$$

Substituting the above expressions for $s(|0\rangle)$ and $s(|1\rangle)$ into this formula, we conclude that

$$s(s(|1\rangle)) =$$

$$\frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle\right) + \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle\right).$$

Thus, we conclude that

$$s(s(|1\rangle)) =$$

$$\left(\frac{1}{2} \cdot |0\rangle - \frac{1}{2} \cdot |1\rangle\right) + \left(\frac{1}{2} \cdot |0\rangle + \frac{1}{2} \cdot |1\rangle\right) = |0\rangle.$$

So, for the "true" state $a = |1\rangle$, the value $s(s(a))$ is indeed equal to its negation "false".

For $|0\rangle$, we get

$$s(s(|0\rangle) = s\left(\frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle\right).$$

Due to linearity, this means that

$$s(s(|0\rangle) = \frac{1}{\sqrt{2}} \cdot s(|0\rangle) - \frac{1}{\sqrt{2}} \cdot s(|1\rangle).$$

Substituting the above expressions for $s(|0\rangle)$ and $s(|1\rangle)$ into this formula, we conclude that

$$s(s(|0\rangle)) =$$

$$\frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle - \frac{1}{\sqrt{2}} \cdot |1\rangle\right) - \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot |0\rangle + \frac{1}{\sqrt{2}} \cdot |1\rangle\right).$$

Thus, we conclude that

$$s(s(|0\rangle)) =$$

$$\left(\frac{1}{2} \cdot |0\rangle - \frac{1}{2} \cdot |1\rangle\right) - \left(\frac{1}{2} \cdot |0\rangle + \frac{1}{2} \cdot |1\rangle\right) = -|1\rangle.$$

This is literally the same state as the negation $|1\rangle$ of the original "false" state $s = |0\rangle$, but, since states s and $-s$ represent the same physical state, from the physical viewpoint, this is exactly negation.

X. Square Root of Not Is An Important Part of Quantum Algorithms

Square root of not is an important part of quantum algorithms.

For example, without using quantum effects, we need – in the worst case – at least N computational steps to search for a desired element in an unsorted list of size N. A quantum computing algorithm proposed by Grover (see, e.g., [2], [3], [8]) can find this element much faster – in $O(\sqrt{N})$ time.

An even more impressive speedup is achieved in factoring large integers. This problem is extremely important for computer security. Most security features of online communications and e-commerce use RSA encryption algorithm originally invented by R. Rivest, A. Shamir, and L. Adleman; see, e.g., [1]. At present, this algorithm provides safe communication because to decrypt RSA-encrypted messages, one needs to factor large integers, and all known factoring algorithms require an unrealistically large computation time: namely, time which grows exponentially with the length of the integer. On the other hand, there exist quantum algorithms which enable us to perform this factorization in time which only grows polynomially with this size – and which is, thus, quite practically feasible [8], [9]. Thus, quantum computers will leads to breaking most exiting encryption codes.

XI. Fuzzy Logic

Let us now consider fuzzy logic; see, e.g., [5], [7]. In fuzzy logic, in addition to the classical values 0 and 1, we also allow intermediate truth values, i.e., arbitrary real numbers from the interval $[0,1]$.

XII. Negation in Fuzzy Logic

Usually, in fuzzy logic, negation is defined as

$$\neg(a) = 1 - a.$$

In principle, there exist other negation operations, but it is known (see, e.g., [5], [7]) that they can be reduced to this standard negation by an appropriate re-scaling of the interval $[0,1]$.

XIII. There Is No Continuous Square Root of Not in Fuzzy Logic: A Statement

In fuzzy logic, usually, we only consider logical operations which are continuous functions of their inputs. This makes sense because the degree of uncertainty are only approximately know, and for operations to be meaningful, we need to require that similar values of the input degrees lead to similar values of the result of the logical operation.

Let us prove that, in contrast to quantum logic, in fuzzy logic, if we restrict ourselves to continuous operations

$$s : [0,1] \to [0,1],$$

there is no square root of negation.

XIV. Proof

We will prove the above statement by contradiction. Let us assume that there exists a continuous function

$$s : [0,1] \to [0,1]$$

for which $s(s(a)) = 1 - a$ for all a.

If $a \neq b$, then $s(a) \neq s(b)$: indeed, otherwise, if we had $s(a) = s(b)$, then we would have $s(s(a)) = s(s(b))$, hence $a = b$, and we assumed that $a \neq b$. So, s is a 1-1 function. It is known that every 1-1 function should be strictly monotonic. Thus, the function $s(a)$ is either strictly increasing or strictly decreasing.

In the first case, if the function $s(a)$ is strictly increasing, then we have $a < b$ imply $s(a) < s(b)$ and thus, $s(s(a) < s(s(b))$. Thus, the composition $s(s(a))$ is also strictly increasing – and thus cannot be equal to the strictly decreasing function $\neg(a) = 1 - a$.

In the second case, if the function $s(a)$ is strictly decreasing, then we have $a < b$ imply $s(a) > s(b)$ and thus, $s(s(a) < s(s(b))$. Thus, the composition $s(s(a))$ is strictly increasing – and thus cannot be equal to the strictly decreasing function $\neg(a) = 1 - a$.

In both cases, we get a contradiction. This contradiction shows that in fuzzy logic, there is no (continuous) square root of not.

XV. Comment: Discontinuous Square Roots of "Not" Are Possible in Fuzzy Logic

Let us show that if we do not require continuity, then a square root of not is possible in fuzzy logic. Indeed, in this case, we can consider the following piece-wise linear function $s(x)$:

- when $0 \le x < \frac{1}{4}$, we set
$$s(x) = x + \frac{1}{4};$$
- when $\frac{1}{4} \le x < \frac{1}{2}$, we set
$$s(x) = \frac{5}{4} - x;$$
- when $x = \frac{1}{2}$, we set
$$s(x) = \frac{1}{2};$$
- when $\frac{1}{2} < x \le \frac{3}{4}$, we set
$$s(x) = \frac{3}{4} - x;$$
- finally, when $\frac{3}{4} < x \le 1$, we set
$$s(x) = x - \frac{1}{4}.$$

By considering all 5 cases, we can check that $s(s(x)) = 1 - x$ for all $x \in [0,1]$. Indeed:

- When $0 \le x < \frac{1}{4}$, then for $s(x) = x + \frac{1}{4}$, we get
$$\frac{1}{4} \le s(x) < \frac{1}{2}.$$
Thus, in this case, $s(s(x)) = \frac{5}{4} - s(x)$. Since
$$s(x) = x + \frac{1}{4},$$
we get
$$s(s(x)) = \frac{5}{4} - s(x) = \frac{5}{4} - \left(x + \frac{1}{4}\right) =$$
$$\frac{5}{4} - x - \frac{1}{4} = 1 - x.$$
- When $\frac{1}{4} \le x < \frac{1}{2}$, then for $s(x) = \frac{5}{4} - x$, we get
$$\frac{3}{4} < s(x) \le 1.$$
Thus, in this case, $s(s(x)) = s(x) - \frac{1}{4}$. Since
$$s(x) = \frac{5}{4} - x,$$
we get
$$s(s(x)) = s(x) - \frac{1}{4} = \left(\frac{5}{4} - x\right) - \frac{1}{4} =$$
$$\frac{5}{4} - x - \frac{1}{4} = 1 - x.$$
- When $x = \frac{1}{2}$, then $s(x) = \frac{1}{2}$ and thus,
$$s(s(x)) = s\left(\frac{1}{2}\right) = \frac{1}{2} = 1 - x.$$
- When $\frac{1}{2} < x \le \frac{3}{4}$, then for $s(x) = \frac{3}{4} - x$, we get
$$0 \le s(x) < \frac{1}{4}.$$
Thus, in this case, $s(s(x)) = s(x) + \frac{1}{4}$. Since
$$s(x) = \frac{3}{4} - x,$$
we get
$$s(s(x)) = s(x) + \frac{1}{4} = \left(\frac{3}{4} - x\right) + \frac{1}{4} =$$
$$\frac{3}{4} - x + \frac{1}{4} = 1 - x.$$
- Finally, when $\frac{3}{4} < x \le 1$, then for $s(x) = x - \frac{1}{4}$, we get
$$\frac{1}{2} < s(x) \le \frac{3}{4}.$$
Thus, in this case, $s(s(x)) = \frac{3}{4} - s(x)$. Since
$$s(x) = x - \frac{1}{4},$$
we get
$$s(s(x)) = \frac{3}{4} - s(x) = \frac{3}{4} - \left(x - \frac{1}{4}\right) =$$
$$\frac{3}{4} - x + \frac{1}{4} = 1 - x.$$

So, this discontinuous function is indeed a square root of negation.

XVI. Conclusions and Future Work

What do the results of the paper mean for both logics?

Our main result means that in spite of the seeming similarity between the two logics, they are different. Moreover, these logics are different in an important feature (square root of "not") which is crucial for the most impressive applications of quantum logic – to the drastic computation speed-up.

This difference is not unexpected. After all, fuzzy logic is a human way of reasoning about the real-world phenomena. Most real-world phenomena are well described by classical physics, so it is not surprising that our way of reasoning about these phenomena is well-suited for classical physics, but not for the quantum physics.

Our auxiliary result means that if we add some non-classical (quantum) features into fuzzy logic, then we can emulate very intuitively unusual quantum features such as the square root of "not". Specifically, we show that in order to be able to represent a square root of "not" in fuzzy logic, it is sufficient to add discontinuity to this logic. Discontinuity is one of the original phenomena which characterized quantum phenomena – where, e.g., an atom in an excited state, instead of continuously decreasing its energy, decreases it abruptly, by emitting a quantum of energy – a photon.

It is worth mentioning that discontinuity seems to be opposite to the main idea behind fuzzy logic: that everything is a matter of degree, and that every seemingly discontinuous transition is actually continuous.

This auxiliary result raises the possibility that by combining such empirically clear quantum phenomena as discontinuity with the main intuitively clear ideas behind fuzzy logic, we can get a better explanation of very technical (and somewhat counter-intuitive) quantum phenomena such as the square root of "not".

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