# Symmetry Between True, False, and Uncertain: An Explanation

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**Abstract**

In intuitionistic fuzzy sets, there is a natural symmetry between degrees of truth and falsity. As a result, for such sets, natural similarity measures are symmetric relative to an exchange of true and false values. It has been recently shown that among such measures, the most intuitively reasonable are the ones which are also symmetric relative to an arbitrary permutation of degrees of truth, falsity, and uncertainty. This intuitive reasonableness leads to a conjecture that such permutations are not simply mathematical constructions, that these permutations also have some intuitive sense. In this paper, we show that each such permutation can indeed be represented as a composition of intuitively reasonable operations on truth values.

**Need for intuitionistic fuzzy logic: a brief reminder.** In the traditional fuzzy logic (see, e.g., [3, 4]), the degree of truth of each property $P$ on an object $x$ is characterized by a number $\mu_P(x)$ from the interval $[0,1]$. It is usually assumed that the degree to which this object $x$ has the opposite property $\neg P$ is equal to $\mu_{\neg P}(x) = 1 = -\mu_P(x)$.

In many practical situations, this assumption leads to good application results. However, in some cases, this assumption is not fully adequate.

For example, if we have no information about the property $P$, then we have no reason to prefer $P$ or $\neg P$. In this case, it makes sense to assign the same degree to both opposite statements $P(x)$ and $\neg P(x)$, i.e., to take $\mu_P(x) = \mu_{\neg P}(x)$. Since in the traditional fuzzy approach, we have $\mu_{\neg P}(x) = 1 - \mu_P(x)$, the above equality leads to $\mu_{\neg P}(x) = \mu_P(x) = 0.5$.

On the other hand, sometimes, we know a lot about $P$, and for some object $x$, we have exactly as many reasons to believe that $P$ holds as to believe that $P$ does not hold. In this case, in the traditional fuzzy logic, we also assign the same degree 0.5 to both values: $\mu_{\neg P}(x) = \mu_P(x) = 0.5$.

Thus, in this traditional approach, the same values $\mu_{\neg P}(x) = \mu_P(x) = 0.5$ may mean two drastically different situations:

- situations when we have no information about $P(x)$ at all, and
- situations when we have a large number of arguments in favor of $P(x)$ but an equally large number of arguments in favor of the opposite statement $\neg P(x)$.

To get a more adequate description of our confidence in different statements, it is desirable to be able to distinguish between these two types of situations.

**Intuitionistic fuzzy logic: a brief reminder.** To distinguish between these types of situations, K. Atanassov proposed *intuitionistic fuzzy logic*, in which for every property $P$ and for every object $x$, we have two different degrees:

- a degree $\mu_P(x)$ to which we are certain that $x$ has the property $P$, and
- a "non-membership degree" $\nu_P(x)$ to which we are certain that $x$ does not have the property $P$.

In general, $\mu_P(x)+\nu_P(x) \le 1$. By using this formalism, we can easily distinguish between the above two types of situations:

- if we have no information about $P(x)$ at all, then we take
$$\mu_P(x) = \nu_P(x) = 0;$$
- if we have a large number of arguments in favor of $P(x)$ and an equally large number of arguments in favor of the opposite statement $\neg P(x)$, then we take
$$\mu_P(x) = \nu_P(x) > 0.$$

In general, in contrast to the traditional fuzzy approach, in addition to the degree $\mu_P(x)$ with which $P$ is true and the degree $\nu_P(x)$ which which $P$ is false, we also have a *third* degree $\pi_P(x) \stackrel{\text{def}}{=} 1 - \mu_P(x) - \nu_P(x)$, the degree with which we are uncertainty about $P$.

In particular, in the case of complete ignorance, when $\mu_P(x) = \nu_P(x) = 0$, this "degree of uncertainty" attains its largest possible value 1.

**Known symmetry: between true and false.** The above formulation leads to a natural symmetry between the degrees of truth and falsity:

- in the original description,
  - $\mu_P(x)$ is the degree to which the statement $P(x)$ is true, while
  - $\nu_P(x)$ is the degree to which the statement $P(x)$ is false;

- if instead of the original statement $P(x)$, we consider its negation $\neg P(x)$, then these degrees change place:
  - $\nu_P(x)$ is the degree to which the statement $\neg P(x)$ is true, while
  - $\mu_P(x)$ is the degree to which the statement $\neg P(x)$ is false.

Not surprisingly, many formulas of intuitionsitic fuzzy logic – e.g., the formulas that describe the degree of similarity between two intuitionistic fuzzy sets $P$ and $Q$ – are invariant with respect to this true-false symmetry. In other words, the degree of similarity between the sets $(\mu_P(x), \nu_P(x))$ and $(\mu_Q(x), \nu_Q(x))$ is exactly the same as between the sets $(\nu_P(x), \mu_P(x))$ and $(\nu_Q(x), \mu_Q(x))$.

**Limitations of the known similarity measures, and the new approach.** Most traditionally proposed similarity measures for intuitionistic fuzzy sets are based on comparing the corresponding degrees of truth and degrees of falsity. However, each of the proposed measured leads, in some situations, to counter-intuitive results; see, e.g., [6, 7, 8, 9, 10, 11, 12].

These papers also show that, in order to produce similarity measures which are in a better accordance with common sense, it is necessary to also compare the degrees of uncertainty. In particular, very reasonable similarity measures arise if we treat all three degrees equally, i.e., if the expression for the similarity measure does not change if we take any permutation between the three degrees $\mu$, $\nu$, and $\pi$.

**Problem: what is the meaning of this weird symmetry?** As we have mentioned earlier, symmetry with respect to changing true (T) to false (F) makes intuitive sense. Since the weird "symmetries" (permutations) between *three* degrees T, F, and U ("unknown") lead to intuitively reasonable similarity measures, these permutations probably also have some intuitive meaning.

In this paper, we show that such permutations are not just purely mathematical tricks, each of these permutations can be represented as a composition of a small number of intuitive operations with truth values.

**The first auxiliary operation: fusion.** To come up with such a meaning, let us first consider a natural operation $\boxed{\text{f}}$ of knowledge fusion. This operation corresponds to the following typical situation: we have two sources of knowledge, and we want to combine ("fuse") knowledge from these two sources.

Our objective is to provide an intuitive explanation to a permutation of the set of three truth values T, F, U. In view of this objective, let us consider how these truth values will be fused.

If for a certain statement $S$, both fused sources consider this statement to be true, then we conclude that this statement is true, i.e., that $\text{T}\,\boxed{\text{f}}\,\text{T} = \text{T}$. If one of the fused sources claims that $S$ is true and the other source has no information about $S$, then we conclude that $S$ is true, i.e., that $\text{T}\,\boxed{\text{f}}\,\text{U} = \text{U}\,\boxed{\text{f}}\,\text{T} = \text{T}$.

Similarly, if for a certain statement $S$, both sources consider this statement to be false, then we conclude that this statement is false, i.e., that $\mathrm{F} \boxed{\mathrm{f}} \mathrm{F} = \mathrm{F}$. If one of the fused sources claims that $S$ is false and the other source has no information about $S$, then we conclude that $S$ is false, i.e., that $\mathrm{F} \boxed{\mathrm{f}} \mathrm{U} = \mathrm{U} \boxed{\mathrm{f}} \mathrm{F} = \mathrm{F}$.

If neither of the sources has any information about $S$, then in the fused knowledge base, we still have no information about it, i.e., $\mathrm{U} \boxed{\mathrm{f}} \mathrm{U} = \mathrm{U}$. Finally, if one of the sources claims that $S$ is true, and the other source claims that $S$ is false, this simply means that in the fused knowledge base, we do not have any knowledge whether $s$ is true or false, i.e., $\mathrm{T} \boxed{\mathrm{f}} \mathrm{F} = \mathrm{F} \boxed{\mathrm{f}} \mathrm{T} = \mathrm{U}$.

The resulting truth table has the following form:

| $\boxed{\mathrm{f}}$ | T | F | U |
|---|---|---|---|
| T | T | U | T |
| F | U | F | F |
| U | T | F | U |

It is worth mentioning that the relation corresponding to this fusion operation is symmetric with respect to negation: i.e., if $x \boxed{\mathrm{f}} y = z$, then

$$(\neg x) \boxed{\mathrm{f}} (\neg y) = \neg z.$$

*Comment.* This operation is a discrete version of the fusion operation used in MYCIN, the first successful expert system; see, e.g., [2, 5].

**The second auxiliary operation: equality.** Another natural operation is equality $x = y$: if two truth values are equal, the result of this operation is "true" (T), otherwise its result is false. The corresponding truth table is also easy to describe:

| = | T | F | U |
|---|---|---|---|
| T | T | F | F |
| F | F | T | F |
| U | F | F | T |

It is worth mentioning that the equality operation is invariant with respect to negation: namely, for every $x$ and $y$, the expressions $x = y$ and $(\neg x) = (\neg y)$ have the same truth value.

*Comment.* From the mathematical viewpoint, we are accustomed to view equality as a *relation*, not as an operation, but inside the computer, equality is an *operation* – just like in our description.

**Possible permutations of T, F, and U.** All possible permutations of the truth values T, F, and U are easy to enumerate:

- we have three groups of permutations, depending on whether T, F, or U is the first element after permutation, and
- within each of these three groups, there are two possible ways to place to place the remaining two truth values into two places.

Thus, we get six possible permutations, in which TFU turns into TFU, TUF, FTU, FUT, UTF, and UFT. Let us show that each of these permutations can be represented as a composition of negation and of the new operations (fusion and equality).

**TFU → TFU.** The permutation that turns TFU into itself is trivial – i.e., it does not change any truth value.

**TFU → TUF.** The permutation that transforms TFU into FUT can be represented as $x \boxed{\text{f}} (x \neq \text{U})$:

| $x$ | $x \neq \text{U}$ | $x \boxed{\text{f}} (x \neq \text{U})$ |
|---|---|---|
| T | T | T |
| F | T | U |
| U | F | F |

**TFU → FTU.** The permutation that transforms TFU into FTU is simply negation $\neg x$.

**TFU → FUT.** The permutation that transforms TFU into FUT can be represented as $\neg x \boxed{\text{f}} (x = \text{U})$:

| $x$ | $\neg x$ | $x = \text{U}$ | $\neg x \boxed{\text{f}} (x = \text{U})$ |
|---|---|---|---|
| T | F | F | F |
| F | T | F | U |
| U | U | T | T |

**TFU → UTF.** The permutation that transforms TFU into UTF can be represented as $\neg(x \boxed{\text{f}} (x = \text{U}))$:

| $x$ | $x = \text{U}$ | $x \boxed{\text{f}} (x = \text{U})$ | $\neg(x \boxed{\text{f}} (x = \text{U}))$ |
|---|---|---|---|
| T | F | U | U |
| F | F | F | T |
| U | T | T | F |

**TFU → UFT.** Finally, the permutation that transforms TFU into UFT can be represented as $x \boxed{\text{f}} (x = \text{U})$:

| $x$ | $x = \text{U}$ | $x \boxed{\text{f}} (x = \text{U})$ |
|---|---|---|
| T | F | U |
| F | F | F |
| U | T | T |

**Conclusion.** The statement is proven: every permutation can indeed be represented in terms of negation, fusion, and equality.

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