“Weird” Fuzzy Notations: An Algebraic Interpretation

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Abstract

Traditionally, fuzzy logic used non-standard notations like

$$m_1/x_1 + \ldots + m_n/x_n$$

for a function that attains the value m_1 at x_1, …, and the value m_n at x_n. In this paper, we provide an algebraic explanation for these notations.

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Formulation of the problem. In fuzzy logic, traditionally, researchers and practitioners used non-standard notations to describe functions; see, e.g., [1]. In these notations, an expression of the type

$$m_1/x_1 + m_2/x_2 + \ldots + m_n/x_n$$

indicates a function that is defined on the set $\{x_1, x_2, \ldots, x_n\}$ and that takes:

- the value m_1 for $x = x_1$,
- the value m_2 for $x = x_2$,
- …, and
- the value m_n for $x = x_n$.

To a mathematician, these non-standard notations are very confusing.

In this paper, we provide an algebraic justification for these “weird” notations, justification that will helpfully make them somewhat less confusing.

Main idea: application of a function to a value as a "multiplication" operation. In mathematics, the division operation a/b is usually understood as the inverse to a "multiplication" operation ab. Thus, to provide a reasonable interpretation for the fuzzy "division" operation, we must find the appropriate "multiplication" operation.

In the context in which the above notations are used, we have a universal set U, the set T of possible values, and we have *partial* functions defined on this set, i.e., functions from the set U (or from its proper subset) to the set T. The only operation that we have is the operation of applying a function f to the value $x \in U$.

It is therefore reasonable to use this application operation as the multiplication operation.

Comment. This usage is in full agreement with the usual notations, in which the result of applying a function f to the value x is denoted either by $f(x)$, or simply by fx. This simplified notation is exactly the notation for a multiplication operation.

Resulting division operations: discussion. For this multiplication operation, what is the resulting division operation? For commutative multiplication operations, a division operation corresponding to a multiplication operation is defined as follows: $a/b = c$ if and only if $a = bc$. For non-commutative multiplication operations (and the operation fx is clearly non-commutative, since xf does not even make sense), we can distinguish between left and right divisions:

- in the left division, $a/b = c$ if and only if $a = bc$; and
- in the right division, $a/b = c$ if and only if $a = cb$.

In our case, when $a = bc$, then b is a function, c is an element of the universal set U, and a is the element of the set T. Thus, the corresponding left division operation would correspond to dividing an element $a \in T$ by a function. The only case that leads to dividing an element $a \in T$ by a value $x \in U$ is the right division.

Since the condition $m = fx$ means that $f(x) = m$, the right division means the following: $f = m/x$ if and only if $f(x) = m$. This interpretation cannot be taken literally, since there are many different functions for which $f(x) = m$, and they cannot be all equal to the same object m/x.

However, in the class of all the functions for which $m = fx$, there exists the *smallest* one (in terms of inclusion): a function which is defined only at a single point x and whose value is equal to m. It is therefore reasonable to define this smallest element as the desired "ratio" m/x.

Comment. This definition is in line with the way fuzzy implication $a \to b$ is sometimes defined (see, e.g., [1]): as the smallest possible degree c for which $c \,\&\, a = b$, where & is the fuzzy "and" operation (t-norm).

Relation to function composition as multiplication. In addition to applying a function to an object, we can also consider composition of functions. A composition is also sometimes denoted simply by fg (e.g., $\log\sin(x)$ is a usual notation for $\log(\sin(x))$), so it is also natural to view it as a multiplication operation.

This multiplication operation is in line with the above definition of division: e.g., if $f = m/x$, and $g = n/m$, then formally, $gf = (n/m)(m/x) = n/x$. And indeed, here:

- $f = m/x$ means that $f(x) = m$ and f is undefined for all other x;
- $g = n/m$ means that $g(m) = n$;
- hence $g(f(x)) = g(m) = n$ (and $g(f(y))$ is undefined for all $y \neq x$), which is exactly what $gf = n/x$ means.

Meaning of the sum. In our interpretation, each expression like m_i/x_i means a partial function which are defined at only one point x_i and has the value m_i at this point. Since in mathematics, a function f is defined as a set of (ordered) pairs $(x, f(x))$, the notation m_i/x_i means a set consisting of a single ordered pair: $m_i/x_i = \{(x_i, m_i)\}$.

A natural "addition" operation for sets is union. It is not a standard notation for the union, but it is not as non-standard as the notations for fuzzy sets:

- a few decades ago, union was indeed routinely denoted by $+$, and
- even now, in many engineering applications, addition is used as a symbol for set union (and for the corresponding logical "or" operation).

Moreover, while the union is not any more routinely described by the plus sign $+$, the minus sign $-$, a typical sign of an operation which is inverse to $+$, is still routinely used to describe the difference between the two sets.

Also, in Boolean algebra, $+$ is often used to describe the "exclusive or" operation, which, for our one-point functions m_i/x_i, is equivalent to the union.

Conclusion. So, we will interpret the sum

$$m_1/x_1 + \ldots + m_n/x_n$$

as the union of the partial functions $\{(x_1, m_1)\}$, ..., $\{(x_n, m_n)\}$, i.e., as the set of pairs

$$\{(x_1, m_1), \ldots, (x_n, m_n)\},$$

which is a function that maps x_1 into m_1, ..., and maps x_n into m_n – exactly the meaning that we are trying to interpret.

Now, this seemingly weird expression has a reasonable algebraic explanation.

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References

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