Selecting the Best Location for a Meteorological Tower: A Case Study of Multi-Objective Constraint Optimization

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1 Formulation of the Problem

Case study. In meteorological and environmental studies, it is important to get additional data from remote locations; see, e.g., [5].

One way of collecting this additional data is to place several different sensors on a single tower. It is therefore important to select the best location of a sophisticated multi-sensor meteorological tower.

In this selection, we have several criteria to satisfy [4].

- For example, the station should not be located too close to a road, so that the gas flux generated by the cars do not influence our measurements of atmospheric fluxes; in other words, the distance x_1 to the road should be larger than a certain threshold t_1: $x_1 > t_1$, or $y_1 \stackrel{\text{def}}{=} x_1 - t_1 > 0$.
- Also, the inclination x_2 at the should be smaller than a corresponding threshold t_2, because otherwise, the flux will be mostly determined by this inclination and will not be reflective of the atmospheric processes: $x_2 < t_2$, or $y_2 \stackrel{\text{def}}{=} t_2 - x_2 > 0$.

General case. In general, we have several such differences $y_1, \ldots, y_n$ all of which have to be non-negative. For each of the differences y_i, the larger its value, the better.

2 Weighted Average: A Natural Idea and Its Limitations

Problem: reminder. We want to select the best location based on the values of the differences $y_1, \ldots, y_n$. For each of the differences y_i, the larger its value, the better.

Multi-criteria optimization. Our problem is a typical setting for *multi-criteria optimization*; see, e.g., [2, 10, 11].

Weighted average. A most widely used approach to multi-criteria optimization is *weighted average*, where we assign weights $w_1, \ldots, w_n > 0$ to different criteria y_i and select an alternative for which the weighted average $w_1 \cdot y_1 + \ldots + w_n \cdot y_n$ attains the largest possible value.

This approach has been used in many practical problems ranging from selecting the lunar landing sites for the Apollo missions [1] to selecting landfill sites [3].

Additional requirement. In our problem, we have an additional requirement – that all the values y_i must be positive. Thus, we must only compare solutions with $y_i > 0$ when selecting an alternative with the largest possible value of the weighted average.

Limitations of the weighted average approach. In general, the weighted average approach often leads to reasonable solutions of the multi-criteria optimization problem. However, as we will show, in the presence of the additional positivity requirement, the weighted average approach is not fully satisfactory.

A practical multi-criteria optimization must take into account that measurements are not absolutely accurate. Indeed, the values y_i come from measurements, and measurements are never absolutely accurate. The results $\widetilde{y}_i$ of the measurements are close to the actual (unknown) values y_i of the measured quantities, but they are not exactly equal to these values. If

- we measure the values y_i with higher and higher accuracy and,
- based on the resulting measurement results $\widetilde{y}_i$, we conclude that the alternative $y = (y_1, \ldots, y_n)$ is better than some other alternative $y' = (y'_1, \ldots, y'_n)$,

then we expect that the actual alternative y is indeed either better than y' or at least of the same quality as y'. Otherwise, if we do not make this assumption, we will not be able to make any meaningful conclusions based on real-life (approximate) measurements.

The above natural requirement is not always satisfied for weighted average. Let us show that for the weighted average, this "continuity" requirement is not satisfied even in the simplest case when we have only two criteria y_1 and y_2. Indeed, let $w_1 > 0$ and $w_2 > 0$ be the weights corresponding to these two criteria. Then, the resulting strict preference relation $\succ$ has the following properties:

- if $y_1 > 0$, $y_2 > 0$, $y'_1 > 0$, and $y'_2 > 0$, and $w_1 \cdot y'_1 + w_2 \cdot y'_2 > w_1 \cdot y_1 + w_2 \cdot y_2$, then
$$y' = (y'_1, y'_2) \succ y = (y_1, y_2);$$
- if $y_1 > 0$, $y_2 > 0$, and at least one of the values y'_1 and y'_2 is non-positive, then
$$y = (y_1, y_2) \succ y' = (y'_1, y'_2).$$

Let us consider, for every $\varepsilon > 0$, the tuple $y'(\varepsilon) \stackrel{\text{def}}{=} \left(\varepsilon, 1 + \dfrac{w_1}{w_2}\right)$, with $y'_1(\varepsilon) = \varepsilon$ and $y'_2(\varepsilon) = 1 + \dfrac{w_1}{w_2}$, and also the comparison tuple $y = (1, 1)$. In this case, for every $\varepsilon > 0$, we have

$$w_1 \cdot y'_1(\varepsilon) + w_2 \cdot y'_2(\varepsilon) = w_1 \cdot \varepsilon + w_2 + w_2 \cdot \frac{w_1}{w_2} = w_1 \cdot (1 + \varepsilon) + w_2$$

and

$$w_1 \cdot y_1 + w_2 \cdot y_2 = w_1 + w_2,$$

hence $y'(\varepsilon) \succ y$. However, in the limit $\varepsilon \to 0$, we have $y'(0) = \left(0, 1 + \dfrac{w_1}{w_2}\right)$, with $y'_1(0) = 0$ and thus, $y'(0) \prec y$.

3 Towards a More Adequate Approach to Multi-Criterion Optimization

What we want: a precise description. We want to be able to compare different alternatives.

Each alternative is characterized by a tuple of n values $y = (y_1, \ldots, y_n)$, and only alternatives for which all the values y_i are positive are allowed. Thus, from the mathematical viewpoint, the set of all alternatives is the set $(R^+)^n$ of all the tuples of positive numbers.

For each two alternatives y and y', we want to tell whether y is better than y' (we will denote it by $y \succ y'$ or $y' \prec y$), or y' is better than y ($y' \succ y$), or y and y' are equally good ($y' \sim y$). These relations must satisfy natural properties. For example, if y is better than y' and y' is better than y'', then y is better than y''. In other words, the relation $\succ$ must be transitive. Similarly, the relation $\sim$ must be transitive, symmetric, and reflexive ($y \sim y$), i.e., in mathematical terms, an *equivalence relation*.

So, we want to define a pair of relations $\succ$ and $\sim$ such that $\succ$ is transitive, $\sim$ is transitive, $\sim$ is an equivalence relation, and for every y and y', one and only one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.

It is also reasonable to require that if each criterion is better, then the alternative is better as well, i.e., that if $y_i > y'_i$ for all i, then $y \succ y'$.

Comment. Pairs of relations of the above type can be alternatively characterized by a *pre-ordering* relation

$$y' \succeq y \Leftrightarrow (y' \succ y \vee y' \sim y).$$

This relation must be transitive and – in our case – total (i.e., for every y and y', we have $y \succeq y' \vee y' \succeq y$. Once we know the pre-ordering relation $\succeq$, we can reconstruct $\succ$ and $\sim$ as follows:

$$y' \succ y \Leftrightarrow (y' \succeq y \,\&\, y \not\succeq y');$$

$$y' \sim y \Leftrightarrow (y' \succeq y \,\&\, y \succeq y').$$

Scale invariance: motivation. The quantities y_i describe completely different physical notions, measured in completely different units. In our meteorological case, some of these values are wind velocities measured in meters per second, or in kilometers per hour, or miles per hour. Other values are elevations described in meters, kilometers, or feet, etc. Each of these quantities can be described in many different units. A priori, we do not know which units match each other, so it is reasonable to assume that the units used for measuring different quantities may not be exactly matched.

It is therefore reasonable to require that the relations $\succ$ and $\sim$ between the two alternatives $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ do not change if we simply change the units in which we measure each of the corresponding n quantities.

Comment. The importance of such invariance is well known in measurements theory, starting with the pioneering work on S. S. Stevens on [12]; see also the classical books [8] and [6] (especially Chapter 22), where this invariance is also called *meaningfulness*.

Scale invariance: towards a precise description. When we replace a unit in which we measure a certain quantity q by a new measuring unit which is $\lambda > 0$ times smaller, then the numerical values of this quantity increase by a factor of λ: $q \to \lambda \cdot q$. For example, 1 cm is $\lambda = 100$ times smaller than 1 m, so the length $q = 2$ m, when measured in cm, becomes $\lambda \cdot q = 2 \cdot 100 = 200$ cm.

Let λ_i denote the ratio of the old to the new units corresponding to the i-th quantity. Then, the quantity that had the value y_i in the old units will be described by a numerical value $\lambda_i \cdot y_i$ in the new units. Therefore, scale-invariance means that for all $y, y' \in (R^+)^n$ and for all $\lambda_i > 0$, we have

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Rightarrow (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \succ (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n)$$

and

$$y' = (y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n) \Rightarrow (\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \sim (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n).$$

Comment. In general, in measurements, in addition to changing the unit, we can also change the starting point. However, for the differences y_i, the starting point is fixed by the fact that 0 corresponds to the threshold value. So, in our case, only changing a measuring unit (= scaling) makes sense.

Continuity. As we have mentioned in the previous section, we also want to require that the relations $\succ$ and $\sim$ are *continuous* in the following sense: if $y'(\varepsilon) \succeq y(\varepsilon)$ for every ε, then in the limit, when $y'(\varepsilon) \to y'(0)$ and $y(\varepsilon) \to y(0)$ (in the sense of normal convergence in R^n), we should have $y'(0) \succeq y(0)$.

Let us now describe our requirements in precise terms.

Definition 1. *By a* total pre-ordering relation *on a set Y, we mean a pair of a transitive relation $\succ$ and an equivalence relation $\sim$ for which, for every $y, y' \in Y$, one and only one of the following relations hold: $y \succ y'$, $y' \succ y$, or $y \sim y'$.*

Comment. We will denote $y \succeq y' \stackrel{\text{def}}{=} (y \succ y' \vee y \sim y')$.

Definition 2. *We say that a total pre-ordering is* non-trivial *if there exist y and y' for which $y' \succ y$.*

Comment. This definition excludes the trivial pre-ordering in which every two tuples are equivalent to each other.

Definition 3. *We say that a total pre-ordering relation on the set $(R^+)^n$ is:*

- monotonic *if $y'_i > y_i$ for all i implies $y' \succ y$;*
- scale-invariant *if for all $\lambda_i > 0$:*
  - *$(y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n)$ implies $(\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \succ (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n)$, and*
  - *$(y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n)$ implies $(\lambda_1 \cdot y'_1, \ldots, \lambda_n \cdot y'_n) \sim (\lambda_1 \cdot y_1, \ldots, \lambda_n \cdot y_n)$.*
- continuous *if whenever we have a sequence $y^{(k)}$ of tuples for which $y^{(k)} \succeq y'$ for some tuple y', and the sequence $y^{(k)}$ tends to a limit y, then $y \succeq y'$.*

Theorem. *Every non-trivial monotonic scale-invariant continuous total pre-ordering relation on $(R^+)^n$ has the following form:*

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} > \prod_{i=1}^{n} y_i^{\alpha_i};$$

$$y' = (y'_1, \ldots, y'_n) \sim y = (y_1, \ldots, y_n) \Leftrightarrow \prod_{i=1}^{n} (y'_i)^{\alpha_i} = \prod_{i=1}^{n} y_i^{\alpha_i},$$

for some constants $\alpha_i > 0$.

Comment. In other words, for every non-trivial monotonic scale-invariant continuous total pre-ordering relation on $(R^+)^n$, there exist values $\alpha_1 > 0, \ldots, \alpha_n > 0$ for which the above equivalence hold. Vice versa, for each set of values $\alpha_1 > 0, \ldots, \alpha_n > 0$, the above formulas define a monotonic scale-invariant continuous pre-ordering relation on $(R^+)^n$.

It is worth mentioning that the resulting relation coincides with the asymmetric version [9] of the bargaining solution proposed by the Nobelist John Nash in 1953 [7].

Proof.

1°. Due to scale-invariance, for every $y_1, \ldots, y_n, y'_1, \ldots, y'_n$, we can take $\lambda_i = \dfrac{1}{y_i}$ and conclude that

$$(y'_1, \ldots, y'_n) \sim (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \sim (1, \ldots, 1).$$

Thus, to describe the equivalence relation $\sim$, it is sufficient to describe the set of all the vectors $z = (z_1, \ldots, z_n)$ for which $z \sim (1, \ldots, 1)$. Similarly,

$$(y'_1, \ldots, y'_n) \succ (y_1, \ldots, y_n) \Leftrightarrow \left(\frac{y'_1}{y_1}, \ldots, \frac{y'_n}{y_n}\right) \succ (1, \ldots, 1).$$

Thus, to describe the ordering relation $\succ$, it is sufficient to describe the set of all the vectors $z = (z_1, \ldots, z_n)$ for which $z \succ (1, \ldots, 1)$.

Alternatively, we can take $\lambda_i = \dfrac{1}{y'_i}$ and conclude that

$$(y'_1, \ldots, y'_n) \succ (y_1, \ldots, y_n) \Leftrightarrow (1, \ldots, 1) \succ \left(\frac{y_1}{y'_1}, \ldots, \frac{y_n}{y'_n}\right).$$

Thus, it is also sufficient to describe the set of all the vectors $z = (z_1, \ldots, z_n)$ for which $(1, \ldots, 1) \succ z$.

2°. The above equivalence involves division. To simplify the description, we can take into account that in the logarithmic space, division becomes a simple difference: $\ln\left(\dfrac{y'_i}{y_i}\right) = \ln(y'_i) - \ln(y_i)$. To use this simplification, let us consider the logarithms $Y_i \stackrel{\text{def}}{=} \ln(y_i)$ of different values. In terms of these logarithms, the original values can be reconstructed as $y_i = \exp(Y_i)$. In terms of these logarithms, we thus need to consider:

- the set $S_\sim$ of all the tuples $Z = (Z_1, \ldots, Z_n)$ for which $z = (\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1)$, and
- the set $S_\succ$ of all the tuples $Z = (Z_1, \ldots, Z_n)$ for which $z = (\exp(Z_1), \ldots, \exp(Z_n)) \succ (1, \ldots, 1)$.

We will also consider the set $S_\prec$ of all the tuples $Z = (Z_1, \ldots, Z_n)$ for which $(1, \ldots, 1) \succ z = (\exp(Z_1), \ldots, \exp(Z_n))$. Since the pre-ordering relation is total, for every tuple z, either $z \sim (1, \ldots, 1)$, or $z \succ (1, \ldots, 1)$, or $(1, \ldots, 1) \succ z$. In particular, this is true for $z = (\exp(Z_1), \ldots, \exp(Z_n))$. Thus, for every tuple Z, either $Z \in S_\sim$ or $Z \in S_\succ$ or $Z \in S_\prec$.

3°. Let us prove that the set $S_\sim$ is closed under addition, i.e., that if the tuples $Z = (Z_1, \ldots, Z_n)$ and $Z' = (Z'_1, \ldots, Z'_n)$ belong to the set $S_\sim$, then their component-wise sum

$$Z + Z' = (Z_1 + Z'_1, \ldots, Z_n + Z'_n)$$

also belongs to the set $S_\sim$.

Indeed, by definition of the set $S_\sim$, the condition $Z \in S_\sim$ means that

$$(\exp(Z_1), \ldots, \exp(Z_n)) \sim (1, \ldots, 1).$$

Using scale-invariance with $\lambda_i = \exp(Z'_i)$, we conclude that

$$(\exp(Z_1) \cdot \exp(Z'_1), \ldots, \exp(Z_n) \cdot \exp(Z'_n)) \sim (\exp(Z'_1), \ldots, \exp(Z'_n)).$$

On the other hand, the condition $Z' \in S_\sim$ means that

$$(\exp(Z'_1), \ldots, \exp(Z'_n)) \sim (1, \ldots, 1).$$

Thus, due to transitivity of the equivalence relation $\sim$, we conclude that

$$(\exp(Z_1)\cdot\exp(Z_1'),\ldots,\exp(Z_n)\cdot\exp(Z_n'))\sim(1,\ldots,1).$$

Since for every i, we have $\exp(Z_i)\cdot\exp(Z_i')=\exp(Z_i+Z_i')$, we thus conclude that

$$(\exp(Z_1+Z_1'),\ldots,\exp(Z_n+Z_n'))\sim(1,\ldots,1).$$

By definition of the set $S_\sim$, this means that the tuple $Z+Z'$ belongs to the set $S_\sim$.

4°. Similarly, we can prove that the set $S_\succ$ is closed under addition, i.e., that if the tuples $Z=(Z_1,\ldots,Z_n)$ and $Z'=(Z_1',\ldots,Z_n')$ belong to the set $S_\succ$, then their component-wise sum

$$Z+Z'=(Z_1+Z_1',\ldots,Z_n+Z_n')$$

also belongs to the set $S_\succ$.

Indeed, by definition of the set $S_\succ$, the condition $Z\in S_\succ$ means that

$$(\exp(Z_1),\ldots,\exp(Z_n))\succ(1,\ldots,1).$$

Using scale-invariance with $\lambda_i=\exp(Z_i')$, we conclude that

$$(\exp(Z_1)\cdot\exp(Z_1'),\ldots,\exp(Z_n)\cdot\exp(Z_n'))\succ(\exp(Z_1'),\ldots,\exp(Z_n')).$$

On the other hand, the condition $Z'\in S_\succ$ means that

$$(\exp(Z_1'),\ldots,\exp(Z_n'))\succ(1,\ldots,1).$$

Thus, due to transitivity of the strict preference relation $\succ$, we conclude that

$$(\exp(Z_1)\cdot\exp(Z_1'),\ldots,\exp(Z_n)\cdot\exp(Z_n'))\succ(1,\ldots,1).$$

Since for every i, we have $\exp(Z_i)\cdot\exp(Z_i')=\exp(Z_i+Z_i')$, we thus conclude that

$$(\exp(Z_1+Z_1'),\ldots,\exp(Z_n+Z_n'))\succ(1,\ldots,1).$$

By definition of the set $S_\succ$, this means that the tuple $Z+Z'$ belongs to the set $S_\succ$.

5°. A similar argument shows that the set $S_\prec$ is closed under addition, i.e., that if the tuples $Z=(Z_1,\ldots,Z_n)$ and $Z'=(Z_1',\ldots,Z_n')$ belong to the set $S_\prec$, then their component-wise sum

$$Z+Z'=(Z_1+Z_1',\ldots,Z_n+Z_n')$$

also belongs to the set $S_\prec$.

6°. Let us now prove that the set $S_\sim$ is closed under the "unary minus" operation, i.e., that if $Z=(Z_1,\ldots,Z_n)\in S_\sim$, then $-Z\stackrel{\text{def}}{=}(-Z_1,\ldots,-Z_n)$ also belongs to $S_\sim$.

Indeed, $Z\in S_\sim$ means that

$$(\exp(Z_1),\ldots,\exp(Z_n))\sim(1,\ldots,1).$$

Using scale-invariance with $\lambda_i=\exp(-Z_i)=\dfrac{1}{\exp(Z_i)}$, we conclude that

$$(1,\ldots,1)\sim(\exp(-Z_1),\ldots,\exp(-Z_n)),$$

i.e., that $-Z\in S_\sim$.

7°. Let us prove that if $Z = (Z_1, \ldots, Z_n) \in S_\succ$, then $-Z \stackrel{\text{def}}{=} (-Z_1, \ldots, -Z_n)$ belongs to $S_\prec$.

Indeed, $Z \in S_\succ$ means that

$$(\exp(Z_1), \ldots, \exp(Z_n)) \succ (1, \ldots, 1).$$

Using scale-invariance with $\lambda_i = \exp(-Z_i) = \dfrac{1}{\exp(Z_i)}$, we conclude that

$$(1, \ldots, 1) \succ (\exp(-Z_1), \ldots, \exp(-Z_n)),$$

i.e., that $-Z \in S_\prec$.

Similarly, we can show that if $Z \in S_\prec$, then $-Z \in S_\succ$.

8°. From Part 3 of this proof, it now follows that if $Z = (Z_1, \ldots, Z_n) \in S_\sim$, then $Z + Z \in S_\sim$, that $Z + (Z + Z) \in S_\sim$, etc., i.e., that for every positive integer p, the tuple

$$p \cdot Z = (p \cdot Z_1, \ldots, p \cdot Z_n)$$

also belongs to the set $S_\sim$.

By using Part 6, we can also conclude that this is true for negative integers p as well. Finally, by taking into account that the zero tuple $0 \stackrel{\text{def}}{=} (0, \ldots, 0)$ can be represented as $Z + (-Z)$, we conclude that $0 \cdot Z = 0$ also belongs to the set $S_\sim$.

Thus, if a tuple Z belongs to the set $S_\sim$, then for every integer p, the tuple $p \cdot Z$ also belongs to the set $S_\sim$.

9°. Similarly, from Parts 4 and 5 of this proof, it follows that

- if $Z = (Z_1, \ldots, Z_n) \in S_\succ$, then for every positive integer p, the tuple $p \cdot Z$ also belongs to the set $S_\succ$, and
- if $Z = (Z_1, \ldots, Z_n) \in S_\prec$, then for every positive integer p, the tuple $p \cdot Z$ also belongs to the set $S_\prec$.

10°. Let us prove that for every rational number $r = \dfrac{p}{q}$, where p is an integer and q is a positive integer, if a tuple Z belongs to the set $S_\sim$, then the tuple $r \cdot Z$ also belongs to the set $S_\sim$.

Indeed, according to Part 8, $Z \in S_\sim$ implies that $p \cdot Z \in S_\sim$.

According to Part 2, for the tuple $r \cdot Z$, we have either $r \cdot Z \in S_\sim$, or $r \cdot Z \in S_\succ$, or $r \cdot Z \in S_\prec$.

- If $r \cdot Z \in S_\succ$, then, by Part 9, we would get $p \cdot Z = q \cdot (r \cdot Z) \in S_\succ$, which contradicts our result that $p \cdot Z \in S_\sim$.
- Similarly, if $r \cdot Z \in S_\prec$, then, by Part 9, we would get $p \cdot Z = q \cdot (r \cdot Z) \in S_\prec$, which contradicts our result that $p \cdot Z \in S_\sim$.

Thus, the only remaining option is $r \cdot Z \in S_\sim$. The statement is proven.

11°. Let us now use continuity to prove that for every real number x, if a tuple Z belongs to the set $S_\sim$, then the tuple $x \cdot Z$ also belongs to the set $S_\sim$.

Indeed, a real number x can be represented as a limit of rational numbers: $r^{(k)} \to x$. According to Part 10, for every k, we have $r^{(k)} \cdot Z \in S_\sim$, i.e., the tuple $Z^{(k)} \stackrel{\text{def}}{=} (\exp(r^{(k)} \cdot Z_1), \ldots, \exp(r^{(k)} \cdot Z_n)) \sim (1, \ldots, 1)$. In particular, this means that $Z^{(k)} \succeq (1, \ldots, 1)$. In the limit, $Z^{(k)} \to (\exp(x \cdot Z_1), \ldots, \exp(x \cdot Z_n)) \succeq (1, \ldots, 1)$. By definition of the sets $S_\sim$ and $S_\succ$, this means that $x \cdot Z \in S_\sim$ or $x \cdot Z \in S_\succ$.

Similarly, for $-(x \cdot Z) = (-x) \cdot Z$, we conclude that $-x \cdot Z \in S_\sim$ or $(-x) \cdot Z \in S_\succ$. If we had $x \cdot Z \in S_\succ$, then by Part 7 we would get $(-x) \cdot Z \in S_\prec$, a contradiction. Thus, the case $x \cdot Z \in S_\succ$ is impossible, and we have $x \cdot Z \in S_\sim$. The statement is proven.

12°. According to Parts 3 and 11, the set $S_\sim$ is closed under addition and under multiplication by an arbitrary real number. Thus, if tuples $Z, \ldots, Z'$ belong to the set $S_\sim$, their arbitrary linear combination $x \cdot Z + \ldots + x' \cdot Z'$ also belongs to the set $S_\sim$. So, the set $S_\sim$ is a linear subspace of the n-dimensional space of all the tuples.

13°. The subspace $S_\sim$ cannot coincide with the entire n-dimensional space, because then the pre-ordering relation would be trivial. Thus, the dimension of this subspace must be less than or equal to $n-1$. Let us show that the dimension of this subspace is $n-1$.

Indeed, let us assume that the dimension is smaller than $n-1$. Since the pre-ordering is non-trivial, there exist tuples $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ for which $y \succ y'$ and thus, $Z = (Z_1, \ldots, Z_n) \in S_\succ$, where $Z_i = \ln\left(\frac{y_i}{y'_i}\right)$. From $Z \in S_\succ$, we conclude that $-Z \in S_\prec$.

Since the linear space $S_\sim$ is a less than $(n-1)$-dimensional subspace of an n-dimensional linear space, there is a path connecting $Z \in S_\succ$ and $-Z \in S_\prec$ which avoids $S_\sim$. In mathematical terms, this path is a continuous mapping $\gamma : [0,1] \to R^n$ for which $\gamma(0) = Z$ and $\gamma(1) = -Z$. Since this path avoids $S_\sim$, every point $\gamma(t)$ on this path belongs either to $S_\succ$ or to $S_\prec$.

Let $\bar{t}$ denote the supremum (least upper bound) of the set of all the values t for which $\gamma(t) \in S_\succ$. By definition of the supremum, there exists a sequence $t^{(k)} \to \bar{t}$ for which $\gamma(t^{(k)}) \in S_\succ$. Similarly to Part 11, we can use continuity to prove that in the limit, $\gamma(\bar{t}) \in S_\succ$ or $\gamma(\bar{t}) \in S_\sim$. Since the path avoids the set $S_\sim$, we thus get $\gamma(\bar{t}) \in S_\succ$.

Similarly, since $\gamma(1) \notin S_\succ$, there exists a sequence $t^{(k)} \downarrow \bar{t}$ for which $\gamma(t^{(k)}) \in S_\prec$. We can therefore conclude that in the limit, $\gamma(\bar{t}) \in S_\succ$ or $\gamma(\bar{t}) \in S_\sim$ – a contradiction with our previous conclusion that $\gamma(\bar{t}) \in S_\succ$.

This contradiction shows that the linear space $S_\sim$ cannot have dimension $< n-1$ and thus, that this space have dimension $n-1$.

14°. Every $(n-1)$-dimensional linear subspace of an n-dimensional superspace separates the superspace into two half-spaces. Let us show that one of these half-spaces is $S_\succ$ and the other is $S_\prec$.

Indeed, if one of the subspaces contains two tuples Z and Z' for which $Z \in S_\succ$ and $Z' \in S_\prec$, then the line segment $\gamma(t) = t \cdot Z + (1-t) \cdot Z'$ containing these two points also belongs to the same subspace, i.e., avoids the set $S_\sim$. Thus, similarly to Part 13, we would get a contradiction.

So, if one point from a half-space belongs to $S_\succ$, all other points from this subspace also belong to the set $S_\succ$. Similarly, if one point from a half-space belongs to $S_\prec$, all other points from this subspace also belong to the set $S_\prec$.

15°. Every $(n-1)$-dimensional linear subspace of an n-dimensional space has the form $\alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n = 0$ for some real values α_i, and the corresponding half-spaces have the form $\alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n > 0$ and $\alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n < 0$.

The set $S_\succ$ coincides with one of these subspaces. If it coincides with the set of all tuples Z for which $\alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n < 0$, then we can rewrite it as $(-\alpha_1) \cdot Z_1 + \ldots + (-\alpha_n) \cdot Z_n > 0$, i.e., as $\alpha'_1 \cdot Z_1 + \ldots + \alpha'_n \cdot Z_n > 0$ for $\alpha'_i = -\alpha_i$.

Thus, without losing generality, we can conclude that the set $S_\succ$ coincides with the set of all the tuples Z for which $\alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n > 0$. We have mentioned that

$$y' = (y'_1, \ldots, y'_n) \succ y = (y_1, \ldots, y_n) \Leftrightarrow (Z_1, \ldots, Z_n) \in S_\succ,$$

where $Z_i = \ln\left(\frac{y'_i}{y_i}\right)$. Thus,

$$y' \succ y \Leftrightarrow \alpha_1 \cdot Z_1 + \ldots + \alpha_n \cdot Z_n = \alpha_1 \cdot \ln\left(\frac{y'_1}{y_1}\right) + \ldots + \alpha_n \cdot \ln\left(\frac{y'_n}{y_n}\right) > 0.$$

Since $\ln\left(\frac{y'_i}{y_i}\right) = \ln(y'_i) - \ln(y_i)$, the last inequality is equivalent to

$$\alpha_1 \cdot \ln(y'_1) + \ldots + \alpha_n \cdot \ln(y'_n) > \alpha_1 \cdot \ln(y_1) + \ldots + \alpha_n \cdot \ln(y_n).$$

Let us take exp of both sides; then, due to the monotonicity of the exponential function, we get an equivalent inequality

$$\exp(\alpha_1 \cdot \ln(y_1') + \ldots + \alpha_n \cdot \ln(y_n')) > \exp(\alpha_1 \cdot \ln(y_1) + \ldots + \alpha_n \cdot \ln(y_n)).$$

Here,

$$\exp(\alpha_1 \cdot \ln(y_1') + \ldots + \alpha_n \cdot \ln(y_n')) = \exp(\alpha_1 \cdot \ln(y_1')) \cdot \ldots \cdot \exp(\alpha_n \cdot \ln(y_n')),$$

where for every i, $e^{\alpha_i \cdot z_i} = (e^{z_i})^{\alpha_i}$, with $z_i \stackrel{\text{def}}{=} \ln(y_i')$, implies that

$$\exp(\alpha_i \cdot \ln(y_i')) = (\exp(\ln(y_i')))^{\alpha_i} = (y_i')^{\alpha_i},$$

so

$$\exp(\alpha_1 \cdot \ln(y_1') + \ldots + \alpha_n \cdot \ln(y_n')) = (y_1')^{\alpha_1} \cdot \ldots \cdot (y_n')^{\alpha_n}$$

and similarly,

$$\exp(\alpha_1 \cdot \ln(y_1) + \ldots + \alpha_n \cdot \ln(y_n)) = y_1^{\alpha_1} \cdot \ldots \cdot y_n^{\alpha_n}.$$

Thus, the condition $y' \succ y$ is equivalent

$$\prod_{i=1}^n y_i^{\alpha_i} > \prod_{i=1}^n (y_i')^{\alpha_i}.$$

Similarly, we prove that

$$(y_1, \ldots, y_n) \sim y' = (y_1', \ldots, y_n') \Leftrightarrow \prod_{i=1}^n y_i^{\alpha_i} = \prod_{i=1}^n (y_i')^{\alpha_i}.$$

The condition $\alpha_i > 0$ follows from our assumption that the pre-ordering is monotonic.

The theorem is proven.

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