Any Improved Trapezoidal Approximation to Intersection (Fusion) of Trapezoidal Fuzzy Numbers Leads to Non-Associativity: A Theorem

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***Abstract*— In some cases, our uncertainty about about a quantity can be described by an interval of its possible values. If we have two or more pieces of interval information about the same quantity, then we can conclude that the actual value belongs to the intersection of these intervals.**

In general, we may need a fuzzy number to represent our partial knowledge. A fuzzy number can be viewed as a collection of intervals (α-cuts) corresponding to different degrees $\alpha \in [0,1]$. In practice, we can only store finitely many α-cuts. Usually, we only store the lower and upper α-cuts (corresponding to $\alpha = 0$ and $\alpha = 1$) and use linear interpolation – i.e., use trapezoidal fuzzy numbers. However, the intersection of two trapezoidal fuzzy numbers is, in general, not trapezoidal. One possible approach is to simply take an intersection of lower and alpha α-cuts, but this approach underestimates the resulting membership function. A more accurate approach is to use, e.g., Least Squares Method to provide a better linear approximation to the resulting membership function. However, as we will show, this approximation method makes the corresponding "knowledge fusion" operation non-associative. Moreover, we prove that any improved trapezoidal approximation to intersection (fusion) of trapezoidal fuzzy numbers leads to non-associativity.

I. Introduction

Data fusion: case of interval uncertainty. In some cases, our uncertainty about about a quantity can be described by an interval of its possible values; see, e.g., [2].

A typical case of interval uncertainty is when all our information about a quantity comes from measurements. Measuring instruments are never absolutely accurate. As a result, the measured value $\widetilde{x}$ of a quantity is usually somewhat different from the actual (unknown) value x of this quantity. For crude measurements, the difference $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ (or, to be more precise, its absolute value $|\Delta x|$) is larger, for accurate measurements this difference $|\Delta x|$ is smaller, but it is always non-zero.

Often, the only information we have about the accuracy of the measuring instrument is the upper bound Δ on this measurement error: $|\Delta x| \le \Delta$. In this case, based on the measurement result $\widetilde{x}$, the only information that we can deduce about the actual (unknown) value x is that x belongs to the interval $X \stackrel{\text{def}}{=} [\widetilde{x} - \Delta, \widetilde{x} + \Delta]$.

Often, we have two or more pieces of interval information $X_1, \ldots, X_n$ about the same quantity – e.g., coming from two or more different measurements. In this case, we can conclude that the actual value x belongs to the intersection

$$X = X_1 \cap \ldots \cap X_n$$

of the corresponding intervals.

The intersection $X = [\underline{x}, \overline{x}]$ is easy to describe in terms of the bounds $X_i = [\underline{x}_i, \overline{x}_i]$ of the corresponding intervals:

$$\underline{x} = \max(\underline{x}_1, \ldots, \underline{x}_n);$$

$$\overline{x} = \min(\overline{x}_1, \ldots, \overline{x}_n).$$

In full accordance with common sense, this operation has the following properties:

- if we combine the same piece of information with itself, the result remains the same $X \cap X = X$; in mathematical terms, this interval "data fusion" operation $\cap$ is idempotent;
- the result of combining several pieces of knowledge does not depend on the order in which we combine them; in particular,

  $$X_1 \cap (X_2 \cap X_3) = (X_1 \cap X_2) \cap X_3;$$

  in mathematical terms, this interval "data fusion" operation $\cap$ is associative.

Need for fuzzy uncertainty. In many cases, the information about a quantity comes from expert estimates. Expert are often "fuzzy" in their estimates. For example, an expert can say that the value of the quantity is small (without providing us an exact meaning of "small"). In such situations, a natural way to represent this expert information is by using fuzzy numbers; see, e.g., [1], [3].

Data fusion for fuzzy numbers. If we have several fuzzy numbers $X_1, \ldots, X_n$ describing the same quantity, then it is natural to take their intersection

$$X = X_1 \cap \ldots \cap X_n$$

as the fuzzy number that combines (fuses) all these pieces of information.

In terms of membership functions $X_i(x)$, the intersection is usually interpreted as the minimum:

$$X(x) = \min(X_1(x), \ldots, X_n(x)).$$

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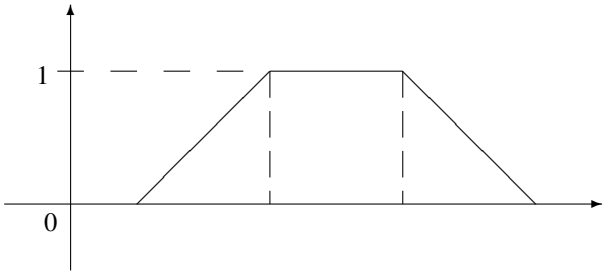
Computer representation of fuzzy numbers: trapezoidal numbers. A fuzzy number X can be viewed as a collection of intervals (α-cuts) $X(\alpha)$ corresponding to different degrees $\alpha \in [0,1]$: $X(\alpha) \stackrel{\text{def}}{=} \{x : X(x) \geq \alpha\}$ for $\alpha > 0$ and $X(\alpha) \stackrel{\text{def}}{=} \overline{\{x : X(x) > 0\}}$ for $\alpha = 0$, where $\overline{S}$ means the closure of the set S.

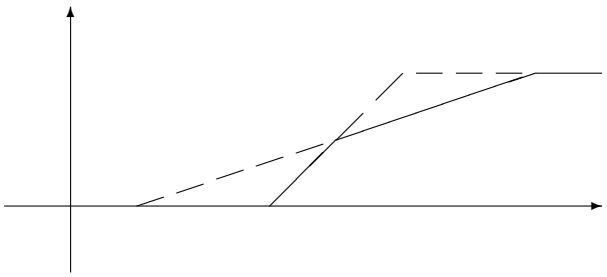
In terms of α-cuts, an intersection simply means that for every α, we take the intersection of the corresponding α-cuts:

$$X(\alpha) = X_1(\alpha) \cap \ldots \cap X_n(\alpha).$$

In practice, we can only store finitely many α-cuts. Usually, we only store the lower and upper α-cuts (corresponding to $\alpha = 0$ and $\alpha = 1$) and use linear interpolation – i.e., use trapezoidal fuzzy numbers.



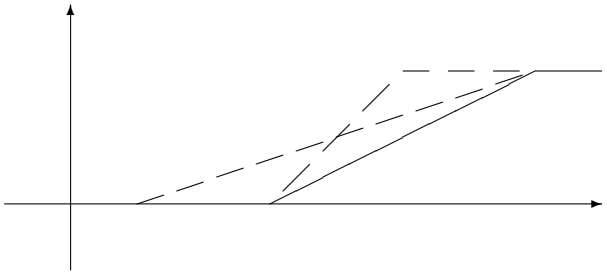
Fusing trapezoidal fuzzy numbers: a problem. The intersection of two trapezoidal fuzzy numbers is, in general, not trapezoidal. It is sufficient to illustrate this fact on the example of the left-hand sides of the intersecting numbers; it is easy to describe a similar example for the right-hand side:



Usual solution. A usual solution to the above problem is simply:

- take an intersection of lower and alpha α-cuts, and then
- as the result of the fusion, take the trapezoidal number corresponding to this intersection.

The resulting trapezoidal number is described in the following picture:

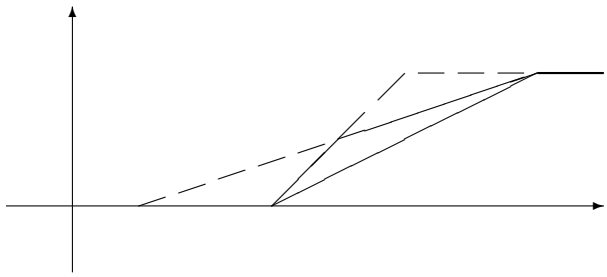


Limitation of the traditional approach. The main problem with this approach is that, as one can easily see, this approach underestimates the resulting membership function.

It is therefore desirable to come up with a different trapezoidal approximation to intersection (fusion) of trapezoidal fuzzy numbers, an approximation that would provide a more accurate approximation to the actual intersection.

Mathematical comment. The above underestimation can be easily explained. Indeed, for both trapezoidal functions, the membership function is linear in the domain where it is different from 0 and 1. Since we use minimum as the intersection, the combined membership function is a minimum of two linear functions.

A minimum of two linear functions is always concave; see, e.g., [4]. Thus, by definition of a concave function, its value are always above the straight line that connects its endpoints:

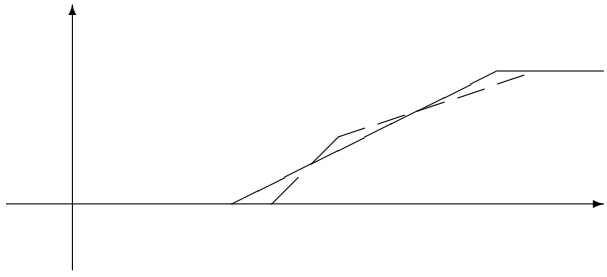


II. How to Improve the Traditional Trapezoidal Approximation to Intersection: Idea and Its Limitations

Analysis of the problem. In the standard approach, we simply take the intersection of the α-cuts corresponding to $\alpha = 0$ and to $\alpha = 1$, and then linearly extrapolate the resulting function. The limitation of this approach is that for the intermediate values $\alpha \in (0,1)$, this approach does not provide a good approximation to the α-cuts of the intersection.

The resulting idea. The above analysis naturally leads to the following idea:

- first, compute the actual intersection
- second, approximate the non-0 and non-1 parts of this intersection by linear functions; for this approximation, we can use, e.g., the usual Least Squares Method; see, e.g., [5].



A somewhat unexpected problem: non-associativity. We tried the above Least Squares approach to combine two trapezoidal membership functions X_1 and X_2 into a single trapezoidal membership function $X_1 \otimes X_2$ that approximates the intersection $X_1 \cap X_2$. We saw that there is an unexpected problem with this approach: the resulting operation is no longer associative.

Specifically, when we take three different pieces of knowledge X_1, X_2, and X_3, then we can combine them in several different ways:

- we can first combine X_1 and X_2 into a trapezoidal function $X_1 \otimes X_2$, and then combine the result $X_1 \otimes X_2$ with X_3 into a new membership function

$$(X_1 \otimes X_2) \otimes X_3;$$

- alternatively, we can first combine X_2 and X_3 into a trapezoidal function $X_2 \otimes X_3$, and then combine X_1 with the result $X_2 \otimes X_3$ into a new membership function

$$X_1 \otimes (X_2 \otimes X_3).$$

In both ways, we combine all three pieces of knowledge. So, intuitively, we expect the results to be equal

$$(X_1 \otimes X_2) \otimes X_3 = X_1 \otimes (X_2 \otimes X_3).$$

In mathematical terms, we expect the combination operation $\otimes$ to be associative.

In practice, however, the Least Squares-based operation $\otimes$ is not associative.

Comment. A numerical example is presented later in the text, in the proof of our main result.

Non-associativity is indeed a problem. One may argue that the above non-associativity is not always a problem. For example, if from the very beginning, we have all three pieces of knowledge X_1, X_2, and X_3, then we can simply take the intersection $X_1 \cap X_2 \cap X_3$ of all three of then, and then use the Least Squares (or any other approximation technique) to find the trapezoidal function that approximates this intersection.

In many practical situations, however, pieces of knowledge are not available from the very beginning, they arrive one by one. If we first have X_1, after which X_2 arrives, then we approximate the resulting knowledge by a trapezoidal membership function $X_1 \otimes X_2$. If after this, a new piece of knowledge X_3 arrives, then combine this piece of knowledge with the current state $X_1 \otimes X_2$ of our knowledge base, resulting in $(X_1 \otimes X_2) \otimes X_3$.

Alternatively, if we first have X_2, after which X_3 arrives, then we approximate the resulting knowledge by a trapezoidal membership function $X_2 \otimes X_3$. If after this, a new piece of knowledge X_1 arrives, then combine this piece of knowledge with the current state $X_2 \otimes X_3$ of our knowledge base, resulting in $X_1 \otimes (X_2 \otimes X_3)$.

In both cases, eventually, we learn the same three pieces of knowledge X_1, X_2, and X_3. It is therefore desirable in both case, we should end up in the same state of our knowledge-base, i.e., that we should have

$$(X_1 \otimes X_2) \otimes X_3 = X_1 \otimes (X_2 \otimes X_3).$$

Resulting question. Associativity seems to be desirable. The Least Squares method does not lead to associativity. Since there are many other approximation techniques beyond the Least Squares, a natural question is: can we retain associativity by using one of these alternative techniques?

What we do in this paper. In this paper, we prove that non-associativity is inevitable: no matter what approximation method we select, the resulting approximation operation is not associative.

Thus, the only way to avoid non-associativity is to use the traditional (underestimating) trapezoidal description of the intersection of two trapezoidal membership functions.

III. Properties of the Approximation Operation: Motivations

Plan of work. We would like to prove that every "reasonable" approximation operation $\otimes$ is not associative. To prove this result, we will first describe properties that are reasonable to require of any operation that approximates an intersection with a trapezoid. We will do describe these properties in this section.

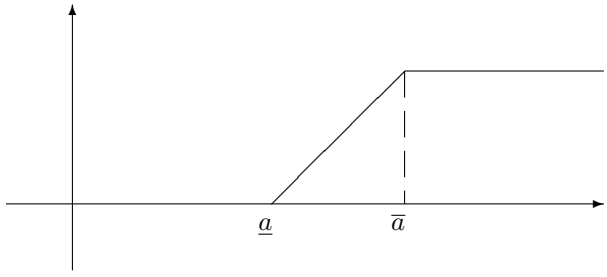
Then, in the next section, we will prove that every operation $\times$ that satisfies these reasonable properties is not associative.

First idea: we only consider left sides. As we have mentioned, it is sufficient to separately consider the left and the right sides of the corresponding membership functions. Since the right sides are similar to the left ones, we will only analyze the left sides.

In any case, this restriction to left sides is sufficient for our purpose: if the combination of left sides is not associative, then the whole combination operation is not associative either.

From the mathematical viewpoint, the left side of a trapezoidal membership is an interval. To describe the left side of a trapezoidal membership function μ, it is sufficient to describe two values:

- the last value $\underline{a}$ for which $\mu(\underline{a}) = 0$ and
- the first value $\overline{a}$ for which $\mu(\overline{a}) = 1$:



The value $\underline{a}$ is always smaller than or equal to the value $\overline{a}$. Thus, these values form an interval $a = [\underline{a}, \overline{a}]$. Vice versa, once we have an interval, $a = [\underline{a}, \overline{a}]$, we can form a (left side of the) trapezoidal function by selection a function $\mu(x)$ which is:

- equal to 0 for $x \le \underline{a}$;
- equal to 1 for $x \ge \overline{a}$, and
- is linear on the interval $[\underline{a}, \overline{a}]$.

Since left sides of trapezoidal membership functions are thus in natural 1-1 correspondence with intervals, we will – whenever it is more convenient – talk about the operation $\otimes$ on intervals.

First property: commutativity. The result of combining two pieces of knowledge should not depend on the order in which these two pieces of knowledge are listed, i.e., we should have $a \otimes b = b \otimes a$ for all intervals a and b.

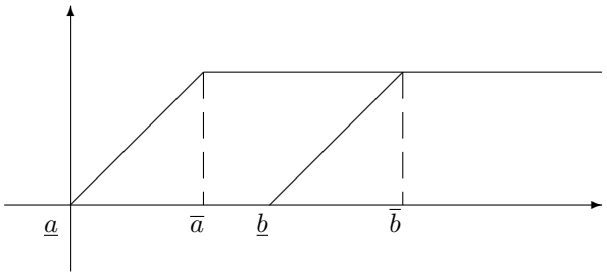
Idempotence: $a \otimes a = a$. If the new piece of knowledge X_2 is identical to our original knowledge X_1 ($X_1 = X_2$), this means that this new piece of knowledge does not provide us with any new information. Thus, we do not gain anything new by combining these two pieces of knowledge X_1 and X_2.

In other words, the result $X_1 \otimes X_1$ of combining every piece of knowledge with itself should be identical to that original piece of knowledge: $X_1 \otimes X_1 = X_1$. This means that for every interval a, we must have $a \otimes a = a$, i.e., in mathematical terms, that the interval operation $\otimes$ is *idempotent*.

Case when the intersection is already a trapezoidal membership function. A slightly more general case where we are sure what we want as a result $a \otimes b$ is when the intersection $a \cap b$ of the corresponding membership functions is itself trapezoidal. In this case, the trapezoidal approximation to this intersection membership function should, of course, coincide with this intersection.

One can easily see that the case when the intersection of the left sides corresponding to the intervals $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ is itself trapezoidal is when this intersection coincides with one of the original membership functions, i.e., when we have one of the following two cases.

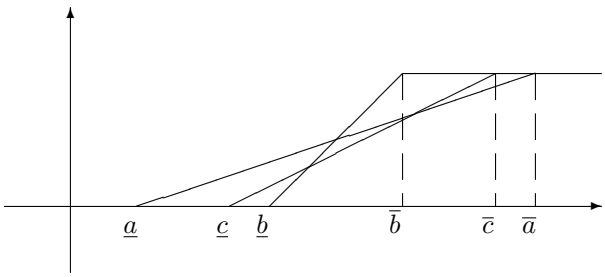
In the first case, we have $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$. In this case, the intersection is equal to the membership function described by the interval b:



In the second case, we have $\underline{b} \le \underline{a}$ and $\overline{a} \le \overline{b}$. In this case, the intersection is equal to the membership function described by the interval a.

For convenience, we will denote the relation "$\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$" between the two intervals $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ as $a \le b$. In terms of this notation, the above requirement is that if $a \le b$, then $a \otimes b = b$.

The requirement that the combination operation should be an improved approximation. For the non-trivial case, when $a \not\le b$ and $b \not\le a$, we would like to require that the upper endpoint of the resulting interval $c = a \otimes$ must be strictly in between the upper endpoints of the combined intervals, and the lower endpoint of c must also be strictly in between the corresponding endpoints:



Thus, we would like to require that in this case, $a \not\le a \otimes b$, $a \otimes b \not\le a$, $b \not\le a \otimes b$, and $a \otimes b \not\le b$.

Now, we are ready to formulate our main result.

IV. Definitions and the main Result

Notation. *For two intervals $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$, we say that $a \le b$ if $\underline{a} \le \underline{b}$ and $\overline{b} \le \overline{b}$.*

Definition. *By an* improved trapezoidal approximation to the intersection of trapezoidal fuzzy numbers*, we mean an operation $\otimes$ that transforms any two intervals a and b into an interval $a \otimes b$, and which satisfies the following properties:*

- *it is* commutative*, i.e., $a \otimes b = b \otimes a$ for all intervals a and b;*
- *it is* idempotent*, i.e., $a \otimes a$ for all intervals a;*
- *it* preserves trapezoidal intersection*, i.e., if $a \le b$, then $a \otimes b = b$;*
- *it improves over the usual approximation, i.e., if $a \not\le b$ and $b \not\le b$, then $a \not\le a \otimes b$, $a \otimes b \not\le a$, $b \not\le a \otimes b$, and $a \otimes b \not\le b$.*

Proposition. *Every improved trapezoidal approximation to the intersection of trapezoidal fuzzy numbers is non-associative.*

Comment. In other words, for every such operation $\otimes$, there exist trapezoidal membership functions X_1, X_2, and X_3 for which

$$(X_1 \otimes X_2) \otimes X_3 \ne X_1 \otimes (X_2 \otimes X_3).$$

In terms of intervals, for every operation $\otimes$, there exist intervals a, b, and c, for which

$$(a \otimes b) \otimes c \ne a \otimes (b \otimes c).$$

Proof. Let us pick two intervals a and b for which $a \not\leq b$ and $b \not\leq a$. For example, we can take $a = [1, 4]$ and $b = [2, 3]$. Let us show that associativity is violated for $c = b$. We will prove this by reduction to a contradiction. Let us assume that associativity is satisfied for these three intervals, i.e., that

$$(a \otimes b) \otimes b = a \otimes (b \otimes b).$$

Due to idempotence, we have $b \otimes b = b$, so the above equality leads to

$$(a \otimes b) \otimes b = a \otimes b.$$

So, for $d \stackrel{\text{def}}{=} a \otimes b$, we have $d \otimes b = d$ and thus, $d \leq d \otimes b$. One of our requirements is that the operation $\otimes$ is improving, meaning that $a \not\leq d$ and $d \not\leq a$ imply that $d \not\leq d \otimes b$. Thus, the fact that $d \leq d \otimes b$ means that we have either $a \leq d$ or $d \leq a$.

However, since we assumed that $a \not\leq b$ and $b \not\leq a$, we can use the same improvement property to conclude that $a \not\leq d = a \otimes b$ and $d = a \otimes b \not\leq a$. The contradiction proves that the above equality cannot be true. So, associativity is indeed violated. The proposition is proven.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation grants HRD-0734825 and DUE-0926721, by Grant 1 T36 GM078000-01 from the National Institutes of Health, by Grant MSM 6198898701 from MŠMT of Czech Republic, and by Grant 5015 "Application of fuzzy logic with operators in the knowledge based systems" from the Science and Technology Centre in Ukraine (STCU), funded by European Union.

The authors are greatly thankful to Ronal R. Yager for valuable discussions.

REFERENCES

[1] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Upper Saddle River, New Jersey: Prentice Hall, 1995.

[2] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, Philadelphia, Pennsylvania: SIAM Press, 2009.

[3] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Boca Raton, Florida: Chapman & Hall/CRC, 2006.

[4] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1996.

[5] J. Wolberg, *Data Analysis Using the Method of Least Squares: Extracting the Most Information from Experiments*, Springer Verlag, Berlin Heidelberg New York, 2005.