

Mamdani Approach to Fuzzy Control, Logical Approach, What Else?

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Abstract—In fuzzy control, two approaches are mainly used: *Mamdani's approach*, in which we represent the knowledge base as a disjunction of statements $A_i(x) \& B_i(u)$ corresponding to individual rules, and *logical approach*, in which the knowledge base is represented as a conjunction of the rules themselves $A_i(x) \rightarrow B_i(u)$. Both approaches are known not to be perfect, so a natural question arises: what other approaches are possible? In this paper, we describe all possible approaches; alternative approaches use an “exclusive or” operation and correspond, e.g., to the fuzzy transform idea.

I. INTRODUCTION

Need for fuzzy control. In many application areas, we do not have the exact control strategies, but we have human operators who are skilled in the corresponding control. Human operators are often unable to describe their knowledge in a precise quantitative form. Instead, they describe their knowledge in terms of control rules, rules that formulate their expertise by using words from natural language. These rules usually have the form “If $A_i(x)$ then $B_i(u)$ ”, they relate properties of the input x with properties of the corresponding control u . For example, a rule may say “If a car in front is somewhat too close, break a little bit”.

Fuzzy control is a set of techniques for transforming these rules into a precise control strategy; see, e.g., [1], [3].

Mamdani approach to fuzzy control. Historically the first – and still most widely used – idea of fuzzy control was described by E. Mamdani. Mamdani argued that for a given input x , a control value u is reasonable if:

- either the first rule is applicable, i.e., its condition $A_1(x)$ is satisfied and its conclusion $B_1(u)$ is satisfied,
- or the second rule is applicable, i.e., its condition $A_2(x)$ is satisfied and its conclusion $B_2(u)$ is satisfied,
- etc.

Thus, in Mamdani's approach, the condition $R(x, u)$ meaning that the control u is reasonable for the input x takes the following form

$$(A_1(x) \& B_1(u)) \vee (A_2(x) \& B_2(u)) \vee \dots \quad (1)$$

For a given input x_0 , to get a desired control value $u(x_0)$, we must now apply an appropriate defuzzification procedure to the resulting membership function $R(x_0, u)$.

Logical approach to fuzzy control. An alternative (more recent) approach to fuzzy control is to simply state that all the rules are valid, i.e., that the following statement holds:

$$(A_1(x) \rightarrow B_1(u)) \& (A_2(x) \rightarrow B_2(u)) \& \dots \quad (2)$$

For example, we can interpret $A \rightarrow B$ as $\neg A \vee B$, in which case the formula (2) has the form

$$(\neg A_1(x) \vee B_1(u)) \& (\neg A_2(x) \vee B_2(u)) \& \dots, \quad (3)$$

or, equivalently, the form

$$(A'_1(x) \vee B_1(u)) \& (A'_2(x) \vee B_2(u)) \& \dots, \quad (4)$$

where $A'_i(x)$ denotes $\neg A_i(x)$.

Both approaches have a universality property. Both Mamdani's and logical approaches to fuzzy control have a universality (universal approximation) property [2], [3], [6] meaning that an arbitrary control strategy can be, with arbitrary accuracy, approximated by controls generated by this approach.

Corresponding crisp universality property. One of the reasons why the corresponding fuzzy controls have the universal approximation property is that the corresponding crisp formulas (1) and (2) have the following universal property: for finite sets X and U , an arbitrary relation $C(x, u)$ on $X \times U$ can be represented both in the form (1) and in the form (2), for appropriate properties $A_i(x)$ and $B_i(u)$.

Indeed, an arbitrary crisp property $C(x, u)$ can be described by the set $C \subseteq X \times U$ of all the pairs (x, u) that satisfy this property. Once this set is given, we can represent the corresponding property in the form (1) by taking

$$C(x, u) \Leftrightarrow \vee_{(x_0, u_0) \in C} ((x = x_0) \& (u = u_0)) \quad (5)$$

and in the form (2) (equivalent to (4)) by taking

$$C(x, u) \Leftrightarrow \&_{(x_0, u_0) \notin C} ((x = x_0) \rightarrow (u \neq u_0)). \quad (6)$$

(The proof of this equivalence is given in the special Proofs section.)

Comment. This universality property is well known and actively used, e.g., in digital design: when we design, e.g., a vending machine, then to implement a general logical condition in terms of “and”, “or”, and “not”-gates, we first represent this condition in Conjunctive Normal Form (CNF) or in a Disjunctive Normal Form (DNF). These forms correspond exactly to our formulas (1) and (4) (equivalent to (2)), and the possibility to transform each logical condition into one of these forms is our universality property.

Fuzzy control: what other approaches are possible? Both Mamdani's and logical approaches are actively used in fuzzy control. The fact that both approaches are actively used means

that both have advantages and disadvantages, i.e., none of these two approaches is perfect. Since both are not perfect, it is reasonable to analyze what other approaches are possible.

In this paper, we start this analysis by analyzing what type of crisp forms like (1) and (2) are possible.

II. DEFINITIONS AND THE MAIN RESULT

In the above two representations, we used $\&$, $\vee$, and $\rightarrow$. These logical connectives are examples of *binary operations* in the following precise sense.

Definition 1. By a binary operation, we mean a function

$$f : \{0, 1\} \times \{0, 1\} \rightarrow \{0, 1\}$$

that transforms two Boolean values a and b into a new Boolean value $f(a, b)$.

Comment. In this paper, as usual, we identify “false” with 0 and “true” with 1.

We are looking for general representations of the type

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots, \quad (7)$$

for arbitrary pairs of binary operations; we denoted these general binary operations $\odot$ and $\ominus$. For example, in the above representations, we used $\ominus = \vee$ and $\ominus = \&$; we want to find all other binary operations for which such a representation is possible.

It is important to notice that the operation $\ominus$ is used to combine different rules. Therefore, the result of this operation should not depend on the order in which we present the rules. Thus, this operation should be commutative and associative.

So, we arrive at the following definitions.

Definition 2. We say that a pair of binary operations $(\odot, \ominus)$ in which the operation $\ominus$ is commutative and associative has a universality property if for every two finite sets X and U , an arbitrary relation $C(x, u)$ can be represented in the form (7) for appropriate relations $A_i(x)$ and $B_i(u)$.

Discussion. One can easily check that if the pair $(\odot, \ominus)$ has the universality property, then the pair $(\odot', \ominus)$, where $a \odot' b \stackrel{\text{def}}{=} \neg a \odot b$, also has the universality property: indeed, each statement of the type $A_i(x) \odot B_i(u)$ can be equivalently represented as $A'_i(x) \odot' B_i(u)$ for $A'_i(x) \stackrel{\text{def}}{=} \neg A_i(x)$.

Similarly, if the pair $(\odot, \ominus)$ has the universality property, then the pair $(\odot, \ominus')$, where $a \odot' b \stackrel{\text{def}}{=} a \odot \neg b$, also has the universality property: indeed, each statement of the type $A_i(x) \odot B_i(u)$ can be equivalently represented as

$$A_i(x) \odot' B'_i(u)$$

for $B'_i(u) \stackrel{\text{def}}{=} \neg B_i(u)$.

Finally, if the pair $(\odot, \ominus)$ has the universality property, then the pair $(\odot', \ominus')$, where $a \odot' b \stackrel{\text{def}}{=} \neg a \odot \neg b$, also has the universality property: indeed, each statement of the type $A_i(x) \odot B_i(u)$ can be equivalently represented as

$$A'_i(x) \odot' B'_i(u)$$

for $A'_i(x) \stackrel{\text{def}}{=} \neg A_i(x)$ and $B'_i(u) \stackrel{\text{def}}{=} \neg B_i(u)$.

Thus, from the viewpoint of universality, the relations $\odot$ and $\odot'$ are similar. So, we arrive at the following definition.

Definition 3. We say that binary operations $\odot$ and $\odot'$ are similar if the relation $\odot'$ has one of the following forms:

$$a \odot' b \stackrel{\text{def}}{=} \neg a \odot b, \quad a \odot' b \stackrel{\text{def}}{=} a \odot \neg b, \quad \text{or} \quad a \odot' b \stackrel{\text{def}}{=} \neg a \odot \neg b.$$

Definition 4. We say that pairs $(\odot, \ominus)$ and $(\odot', \ominus)$ are similar if the operations $\odot$ and $\odot'$ are similar.

The above discussion can be formulated as follows:

Proposition 1. If the binary operations $\odot$ and $\odot'$ are similar, then the following two statements are equivalent to each other:

- the pair $(\odot, \ominus)$ has the universality property;
- the pair $(\odot', \ominus)$ has the universality property.

Comment. One can easily check that the similarity relation is symmetric and transitive, i.e., in mathematical terms, that it is an equivalence relation. Thus, to classify all pairs with the universality property, it is sufficient to consider equivalence classes of binary operations $\odot$ with respect to the similarity relation.

Discussion. Our definition of a universal property requires that the rule-combining operation $\ominus$ be commutative and associative. It turns out that there are only six such operations.

Proposition 2. Out of all binary operations, only the following six are commutative and associative:

- the “zero” operation for which $f(a, b) = 0$ for all a and b ;
- the “one” operation for which $f(a, b) = 1$ for all a and b ;
- the “and” operation for which $f(a, b) = a \& b$;
- the “or” operation for which $f(a, b) = a \vee b$;
- the “exclusive or” operation for which $f(a, b) = a \oplus b$;
- the “equivalence” operation $a \equiv b \stackrel{\text{def}}{=} a \oplus \neg b$.

Comments.

- The proof of this proposition is given in the following section.
- The “exclusive or” operation is actively used in digital design: e.g., when we add two binary numbers which end with digits a and b , the last digit of the sum is $a \oplus b$ (and the carry is $a \& b$). In view of this, “exclusive or” is also called *addition modulo 2*.
- It is interesting to mention that the “exclusive or” operation $a \oplus b$ and the “equivalence” operation $a \equiv b$ are similar to each other – in the sense of our Definition 3. This is the only such pair. For example, an operation $a \& \neg b$ is similar to the “and” operation $a \& b$, but while the “and” operation is commutative, the operation $a \& \neg b$ is not commutative.
- Due to Proposition 2, it is sufficient to consider only these six operations $\ominus$. The following theorem provides a full classification of all resulting pairs of operations.

Theorem. Every pair of operations with the universality property is similar to one of the following pairs: $(\vee, \&)$, $(\&, \vee)$, $(\oplus, \vee)$, $(\oplus, \&)$, $(\equiv, \vee)$, $(\equiv, \&)$, and all these six pairs of operations have the universality property.

Discussion. Thus, in addition to the Mamdani and logical approaches, we have four other possible pairs with the universality property.

What is their meaning? As we will see from the proof, for each operation $\odot$, the combination

$$(A_1(x) \odot B_1(u)) \equiv (A_2(x) \odot B_2(u)) \equiv \dots \equiv (A_n(x) \odot B_n(u))$$

is equal to

$$(A_1(x) \odot B_1(u)) \oplus (A_2(x) \odot B_2(u)) \oplus \dots \oplus (A_n(x) \odot B_n(u))$$

for odd n and to the negation of this relation for even n .

Thus, for odd n , the use of the “equivalence” operation $\equiv$ to combine the rules is equivalent to using the original “exclusive or” operation $\oplus$. For even n , in order to represent an arbitrary property $C(x, u)$, we can use the $\oplus$ combination to represent its negation $\neg C(x, u)$ – this is equivalent to representing the original property by $\equiv$.

Thus, in essence, in addition to forms (1) and (2), we only have two more forms:

$$(A_1(x) \& B_1(u)) \oplus (A_2(x) \& B_2(u)) \oplus \dots \quad (8)$$

and

$$(A_1(x) \vee B_1(u)) \oplus (A_2(x) \vee B_2(u)) \oplus \dots \quad (9)$$

The meaning of these forms is that, crudely speaking, we restrict ourselves to the cases when exactly one rule is applicable. The case of *fuzzy transforms* (*f-transforms*, for short) [4], [5], where we consider rules “if $A_i(x)$ then $B_i(u)$ ” for which $\sum_{i=1}^n A_i(x) = 1$, can be therefore viewed as a natural fuzzy analogue of these cases.

III. PROOFS

0°. Let us first prove the formula (6). Indeed, for negation $\neg C(x, u)$, the formula (5) takes the form

$$\neg C(x, u) \Leftrightarrow \bigvee_{(x_0, u_0) \notin C} ((x = x_0) \& (u = u_0)).$$

$$\text{Thus, } C(x, u) \Leftrightarrow \neg(\neg C(x, u)) \Leftrightarrow$$

$$\neg \left(\bigvee_{(x_0, u_0) \notin C} ((x = x_0) \& (u = u_0)) \right).$$

Applying de Morgan laws, we can move the negations inside the right-hand side formula and conclude that

$$C(x, u) \Leftrightarrow \bigwedge_{(x_0, u_0) \notin C} (\neg(x = x_0) \vee (u \neq u_0)).$$

Since $\neg A \vee B$ is the same as $A \rightarrow B$, we get exactly the desired formula (6).

1°. In order to prove Proposition 2 and Theorem, let us first recall all possible binary operations. By definition, to describe a binary operation, one needs to describe four Boolean values: $f(0, 0)$, $f(0, 1)$, $f(1, 0)$, and $f(1, 1)$. Each of these four quantities can have two different values: 0 and 1; thus, totally, we have $2^4 = 16$ possible operations.

A natural way to classify these operations is to describe how many 1s we have as values $f(a, b)$. Out of 4 values, we can have 0, 1, 2, 3, and 4 ones. Let us describe these cases one by one.

1.1°. When we have zero 1s, this means that all the values $f(a, b)$ are zeros. Thus, in this case, we have a binary operation that always returns zero: $f(a, b) = 0$ for all a and b . It is easy to show that this operation cannot lead to the universality property:

- if we use this operation as $\odot$, then the formula (7) turns into a constant $0 \odot 0 \odot \dots$ independent on x and u ; thus, it cannot have the universality property;
- if we use this operation as $\ominus$, then the formula (7) turns into a constant 0, and thus, also cannot have the universality property.

1.2°. Similarly, when we have four 1s, this means that $f(a, b) = 1$ for all a and b , and we do not have a universality property.

1.3°. When we have a single one, this means that we have an operation similar to “and”. Indeed, if $f(1, 1) = 1$ and all other values $f(a, b)$ are 0s, this means that $f(a, b)$ is true if and only if a is true and b is true, i.e., that $f(a, b) \Leftrightarrow a \& b$. Similarly, if $f(1, 0) = 1$, then $f(a, b) \Leftrightarrow a \& \neg b$; if $f(0, 1) = 1$, then $f(a, b) \Leftrightarrow \neg a \& b$; and if $f(0, 0) = 1$, then $f(a, b) \Leftrightarrow \neg a \& \neg b$.

1.4°. Similarly, we can prove that when we have three ones, this means that we have an operation similar to “or”.

2°. To complete our classification, it is sufficient to describe all the cases when we have exactly two 1s. By enumerating all possible binary operations, we can check that in this case, we have six options: $f(a, b) = a$, $f(a, b) = \neg a$, $f(a, b) = b$, $f(a, b) = \neg b$, $f(a, b) = a \oplus b$, and $f(a, b) = a \equiv b$. By analyzing these operations $f(a, b)$ one by one and by testing commutativity $f(a, b) = f(b, a)$ and associativity $f(a, f(b, c)) = f(f(a, b), c)$ for all possible values a, b , and c , we can describe all commutative and associative operations – i.e., prove Proposition 2.

3°. Arguments similar to the ones that we just gave enables us to prove the statement listed after the formulation of the Theorem: that for any statements $S_1, \dots, S_n$:

- for odd n , we have

$$S_1 \equiv \dots \equiv S_n \Leftrightarrow S_1 \oplus \dots \oplus S_n;$$

- for even n , we have

$$S_1 \equiv \dots \equiv S_n \Leftrightarrow \neg(S_1 \oplus \dots \oplus S_n).$$

Indeed, as we have mentioned, $S \equiv S'$, i.e., $S \oplus \neg S'$, is equivalent to $S \oplus S' \oplus 1$. Thus, for every n , due to associativity and commutativity of both operations $\oplus$ and $\equiv$, we have:

$$S_1 \equiv S_2 \equiv S_3 \equiv \dots \equiv S_n \Leftrightarrow$$

$$(\dots ((S_1 \equiv S_2) \equiv S_3) \equiv \dots \equiv S_n \Leftrightarrow$$

$$(\dots ((S_1 \oplus S_2 \oplus 1) \oplus S_3 \oplus 1) \oplus \dots \oplus S_n \oplus 1 \Leftrightarrow$$

$$\begin{aligned}
& S_1 \oplus S_2 \oplus 1 \oplus S_2 \oplus 1 \oplus \dots \oplus S_n \oplus 1 \Leftrightarrow \\
& (S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n) \oplus (1 \oplus 1 \oplus \dots \oplus 1)(n-1 \text{ times}). \\
& \text{Here, } 1 \oplus 1 = 0, \text{ So, for odd } n, \text{ when } n-1 \text{ is even, we have} \\
& (1 \oplus 1 \oplus \dots \oplus 1)(n-1 \text{ times}) = (1 \oplus 1) \oplus (1 \oplus 1) \oplus \dots \oplus (1 \oplus 1) = \\
& 0 \oplus 0 \oplus \dots \oplus 0 = 0
\end{aligned}$$

and thus

$$\begin{aligned}
& S_1 \equiv S_2 \equiv S_3 \equiv \dots \equiv S_n \Leftrightarrow \\
& (S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n) \oplus (1 \oplus 1 \oplus \dots \oplus 1)(n-1 \text{ times}) \Leftrightarrow \\
& S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n.
\end{aligned}$$

Similarly, for even n , when $n-1$ is odd, we have

$$\begin{aligned}
& (1 \oplus 1 \oplus \dots \oplus 1)(n-1 \text{ times}) = (1 \oplus 1) \oplus (1 \oplus 1) \oplus \dots \oplus (1 \oplus 1) \oplus 1 = \\
& 0 \oplus 0 \oplus \dots \oplus 0 \oplus 1 = (0 \oplus 0 \oplus \dots \oplus 0) \oplus 1 = 0 \oplus 1 = 1
\end{aligned}$$

and thus

$$\begin{aligned}
& S_1 \equiv S_2 \equiv S_3 \equiv \dots \equiv S_n \Leftrightarrow \\
& (S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n) \oplus (1 \oplus 1 \oplus \dots \oplus 1)(n-1 \text{ times}) \Leftrightarrow \\
& (S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n) \oplus 1 \Leftrightarrow \neg(S_1 \oplus S_2 \oplus S_3 \oplus \dots \oplus S_n).
\end{aligned}$$

4°. Due to Proposition 2, we only have to consider the six operations $\ominus$ described in this Proposition when checking the universality property.

We have already shown, in Part 1 of this proof, that pairs with $\ominus = 0$ and $\ominus = 1$ do not have the universality property.

We have also shown that a pair $(\odot, \equiv)$ has a universality property if and only if the pair $(\odot, \oplus)$ has the universality property.

Thus, it is sufficient to consider only three possible operations $\ominus$: $\&$, $\vee$, and $\oplus$. In the following text, we will analyze these three cases one by one, and for each of these three cases, we will consider all possible operations $\odot$.

5°. For $\odot$, as we have mentioned in the discussion from the main text, it is sufficient to consider only one operation from each class of operations which are similar to each other. According to the general classification (Part 1 of this proof), this leaves us with the operations 0, 1, $\&$, $\vee$, $\oplus$, and the degenerate operations – i.e., operations $f(a, b)$ which only depend on one of the two variables a or b .

5.1°. We have already shown (in Part 1) that we cannot have $\odot = 0$ or $\odot = 1$. Thus, for $\odot$, we have to consider cases when $\odot = \&$, when $\odot = \vee$, when $\odot = \oplus$, and when $a \odot b$ is one of the four “degenerate” operations $a \odot b = a$, $a \odot b = \neg a$, $a \odot b = b$, and $a \odot b = \neg b$.

5.2°. Let us prove that the pairs $(\odot, \ominus)$ for which $a \odot b = a$, $a \odot b = \neg a$, $a \odot b = b$, or $a \odot b = \neg b$, cannot have the universality property.

Indeed, e.g., for $a \odot b = a$, each expression $A_i(x) \odot B_i(u)$ has the form $A_i(x)$. Thus, the $\ominus$ -combination of these expressions $A_1(x) \ominus A_2(x) \ominus \dots \ominus A_n(x)$ does not depend on u at all

and thus, cannot represent any property $C(x, u)$ that actually depends on u . Similarly, for $a \odot b = \neg a$, we get the expression $\neg A_1(x) \ominus \neg A_2(x) \ominus \dots \ominus \neg A_n(x)$ which also does not depend on u and thus, cannot represent any property $C(x, u)$ that actually depends on u .

For $a \odot b = b$ and $a \odot b = \neg b$, we get, correspondingly, expressions

$$B_1(u) \ominus B_2(u) \ominus \dots \ominus B_n(u)$$

and $\neg B_1(u) \ominus \neg B_2(u) \ominus \dots \ominus \neg B_n(u)$ which do not depend on x and thus, cannot represent any property $C(x, u)$ that actually depends on x .

5.3°. Because of what we have proved in Parts 5.1 and 5.2, it is sufficient to consider only three operations $\odot$ for combining the premise $A_i(x)$ and the conclusion $B_i(u)$ of each rule: $\&$, $\vee$, and $\oplus$.

6°. Let us first consider the case when $\ominus = \&$. In accordance with Part 5 of this proof, it is sufficient to analyze the universality property for the three subcases when $\odot = \&$, $\odot = \vee$, and $\odot = \oplus$. Let us consider these subcases one by one.

6.1°. When $\ominus = \&$ and $\odot = \&$, the general expression

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots$$

takes the form

$$(A_1(x) \& B_1(u)) \& (A_2(x) \& B_2(u)) \& \dots$$

Due to commutativity and associativity of the “and” operation, this expression is equivalent to

$$(A_1(x) \& A_2(x) \& \dots) \& (B_1(u) \& B_2(u) \& \dots),$$

i.e., to $A(x) \& B(u)$, where $A(x) \stackrel{\text{def}}{=} A_1(x) \& A_2(x) \& \dots$ and

$$B(u) \stackrel{\text{def}}{=} B_1(u) \& B_2(u) \& \dots$$

One can easily see that not every property $C(x, u)$ can be represented as $A(x) \& B(u)$. Indeed, let us take arbitrary sets X and U with at least two elements each, and let $x_0 \in X$ and $u_0 \in U$ be arbitrary elements from these sets. Let us prove, by contradiction, that the property $(x = x_0) \vee (u = u_0)$ cannot be represented in the form $A(x) \& B(u)$. Indeed, let us assume that for some properties $A(x)$ and $B(u)$, for every $x \in X$ and $u \in U$, we have

$$((x = x_0) \vee (u = u_0)) \Leftrightarrow (A(x) \& B(u)). \quad (10)$$

In particular, for $x = x_0$ and $u = u_1 \neq u_0$, the left-hand side of this equivalence (10) is true, hence the right-hand side $A(x_0) \& B(u_1)$ is true as well. Thus, both statements $A(x_0)$ and $B(u_1)$ are true.

Similarly, for $x = x_1 \neq x_0$ and $u = u_0$, the left-hand side of the equivalence (10) is true, hence the right-hand side $A(x_1) \& B(u_0)$ is true as well. Thus, both statements $A(x_1)$ and $B(u_0)$ are true.

Since $A(x_1)$ and $B(u_1)$ are both true, the conjunction $A(x_1) \& B(u_1)$ is also true, so due to (10), we would conclude

that $(x_1 = x_0) \vee (u_1 = u_0)$, which is false. The contradiction proves that the representation (10) is indeed impossible and thus, the pair $(\&, \&)$ does not have the universality property.

6.2°. For $\ominus = \&$ and $\odot = \vee$, the universality property is known to be true – this is one of the two basic cases with which we started our analysis.

6.3°. Let us prove that the subcase $\ominus = \&$ and $\odot = \oplus$ does not lead to the universality property.

In this case, the general expression

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots$$

takes the form

$$(A_1(x) \oplus B_1(u)) \& (A_2(x) \oplus B_2(u)) \& \dots \quad (11)$$

Let us prove, by contradiction, that for every $x_0 \in X$ and $u_0 \in U$, the property $C(x, u) \Leftrightarrow x \neq x_0 \vee u \neq u_0$ cannot be represented in the form (11). Indeed, let us assume that this representation is possible, for some properties $A_i(x)$ and $B_i(u)$.

For the above property $C(x, u)$, the set S of all the values for which this property is true contains all the pairs (x, u) from $X \times U$ except for the pair (x_0, u_0) . Due to equivalence, this same set S is also the set of all the pairs for which the formula (11) holds.

Due to the known properties of the “and” operations, the set S of all the values (x, u) for which the formula (11) holds is equal to the intersection of the sets

$$S_i = \{(x, u) : A_i(x) \oplus B_i(u)\}.$$

Thus, each of the sets S_i is a superset of the set S : $S \subseteq S_i$. By our construction, the set S is missing only one element; thus, it has only two supersets: itself and the set $X \times U$ of all the pairs. If all the sets S_i coincided with $X \times U$, then their intersection would also be equal to $X \times U$, but it is equal to $S \neq X \times U$. Thus, at least for one i , we have $S_i = S$. For this i , we have the equivalence

$$((x \neq x_0) \vee (u \neq u_0)) \Leftrightarrow (A_i(x) \oplus B_i(u)). \quad (12)$$

Let us now reduce this equivalence to the case when $A_i(x_0)$ is true (i.e., when $A_i(x_0) = 1$). Specifically, if $A_i(x_0)$ is false ($A_i(x_0) = 0$), then, since $A \oplus B \Leftrightarrow \neg A \oplus \neg B$, we can replace the original equivalence with the new one

$$((x \neq x_0) \vee (u \neq u_0)) \Leftrightarrow (A'_i(x) \oplus B'_i(u)),$$

where $A'_i(x) \stackrel{\text{def}}{=} \neg A_i(x)$ and $B'_i(u) \stackrel{\text{def}}{=} \neg B_i(u)$, and $A'_i(x_0) = \neg A_i(x_0) = \text{“true”}$. So, we can assume that $A_i(x_0) = 1$.

Now, for $x = x_0$ and $u = u_0$, the left-hand side of the equivalence (12) is false, hence the right-hand side $A_i(x_0) \oplus B_i(u_0)$ is false as well. Since we assumed that $A_i(x_0) = 1$, by the properties of “exclusive or”, we thus conclude that $B_i(u_0) = 1$.

For $x = x_1 \neq x_0$ and $u = u_0$, the left-hand side of the equivalence (12) is true, hence the right-hand side $A_i(x_1) \oplus B_i(u_0)$ is true as well. Since, as we have already proven, $B_i(u_0)$ is true, we conclude that $A_i(x_1)$ is false.

Similarly, for $x = x_0$ and $u = u_1 \neq u_0$, the left-hand side of the equivalence (12) is true, hence the right-hand side $A_i(x_0) \oplus B_i(u_1)$ is true as well. Since $A_i(x_0)$ is true, we conclude that $B_i(u_1)$ is false.

Now, for $x = x_1$ and $u = u_1$, both formulas $A_i(x_1)$ and $B_i(u_1)$ are false, hence their combination $A_i(x_1) \oplus B_i(u_1)$ is also false. So, due to (12), we would conclude that the statement $(x_1 \neq x_0) \vee (u_1 \neq u_0)$ is false, but this statement is actually true. The contradiction proves that the representation (12) is indeed impossible, and so, the pair $(\oplus, \&)$ does not have the universality property.

7°. Let us now consider the case when $\ominus = \vee$. In accordance with Part 5 of this proof, it is sufficient to analyze the universality property for the three subcases when $\odot = \&$, $\odot = \vee$, and $\odot = \oplus$. Let us consider them one by one.

7.1°. For $\ominus = \vee$ and $\odot = \&$, the universality property is known to be true – this is one of the two basic cases with which we started our analysis.

7.2°. When $\ominus = \vee$ and $\odot = \vee$, the general expression

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots$$

takes the form

$$(A_1(x) \vee B_1(u)) \vee (A_2(x) \vee B_2(u)) \vee \dots$$

Due to commutativity and associativity of the “or” operation, this expression is equivalent to

$$(A_1(x) \vee A_2(x) \vee \dots) \vee (B_1(u) \vee B_2(u) \vee \dots),$$

i.e., to $A(x) \vee B(u)$, where $A(x) \stackrel{\text{def}}{=} A_1(x) \vee A_2(x) \vee \dots$ and

$$B(u) \stackrel{\text{def}}{=} B_1(u) \vee B_2(u) \vee \dots$$

Let us prove, by reduction to a contradiction, that a property $(x = x_0) \& (u = u_0)$ cannot be represented as $A(x) \vee B(u)$:

$$((x = x_0) \& (u = u_0)) \Leftrightarrow (A(x) \vee B(u)). \quad (13)$$

Indeed, for $x = x_0$ and $u = u_1 \neq u_0$, the left-hand side of (13) is false, hence the right-hand side $A(x_0) \vee B(u_1)$ is false as well. Thus, both statements $A(x_0)$ and $B(u_1)$ are false.

Similarly, for $x = x_1 \neq x_0$ and $u = u_0$, both statements $A(x_1)$ and $B(u_0)$ are false. Since $A(x_0)$ and $B(u_0)$ are both false, the disjunction $A(x_0) \vee B(u_0)$ is also false, so due to (13), we would conclude that the statement

$$(x_0 = x_0) \& (u_0 = u_0)$$

is false, while in reality, this statement is true. The contradiction proves that the pair $(\vee, \vee)$ does not have the universality property.

7.3°. Let us prove that the subcase $\ominus = \vee$ and $\odot = \oplus$ does not lead to the universality property.

In this case, the general expression

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots$$

takes the form

$$(A_1(x) \oplus B_1(u)) \vee (A_2(x) \oplus B_2(u)) \vee \dots \quad (14)$$

Let us prove, by contradiction, that for every $x_0 \in X$ and $u_0 \in U$, the property $C(x, u) \Leftrightarrow (x = x_0 \& u = u_0)$ cannot be represented in the form (14). To prove it, let us assume that this property can be represented in this form, for some properties $A_i(x)$ and $B_i(u)$, and let us show that this assumption leads to a contradiction.

For the above property $C(x, u)$, the set S of all the values for which this property is true consists of a single pair (x_0, u_0) . Due to equivalence, this same set S is also the set of all the pairs for which the formula (14) holds.

Due to the known properties of the “or” operations, the set S of all the values (x, u) for which the formula (14) holds is equal to the union of the sets $S_i = \{(x, u) : A_i(x) \oplus B_i(u)\}$. Thus, each of the sets S_i is a subset of the set S : $S_i \subseteq S$. By our construction, the set S consists of only one element; thus, it has only two subsets: itself and the empty set. If all the sets S_i coincided with the empty set, then their intersection would also be equal to the empty set $\emptyset$, but it is equal to $S \neq \emptyset$. Thus, at least for one i , we have $S_i = S$. For this i , we have the equivalence

$$((x = x_0) \& (u = u_0)) \Leftrightarrow (A_i(x) \oplus B_i(u)). \quad (15)$$

Similarly to Part 6.3 of this proof, we can reduce this equivalence to the case when $A_i(x_0)$ is true, i.e., when $A_i(x_0) = 1$. So, in the remaining part of this subsection, we assume that $A_i(x_0) = 1$.

Now, for $x = x_0$ and $u = u_0$, the left-hand side of the equivalence (15) is true, hence the right-hand side $A_i(x_0) \oplus B_i(u_0)$ is true as well. Since we assumed that $A_i(x_0) = 1$, by the properties of “exclusive or”, we thus conclude that $B_i(u_0) = 0$.

For $x = x_1 \neq x_0$ and $u = u_0$, the left-hand side of the equivalence (15) is false, hence the right-hand side of this equivalence $A_i(x_1) \oplus B_i(u_0)$ is false as well. Since, as we have already proven, $B_i(u_0)$ is false, we conclude that $A_i(x_1)$ is false.

Similarly, for $x = x_0$ and $u = u_1 \neq u_0$, the left-hand side of the equivalence (15) is false, hence the right-hand side $A_i(x_0) \oplus B_i(u_1)$ is false as well. Since $A_i(x_0)$ is true, we conclude that $B_i(u_1)$ is true.

Now, for $x = x_1$ and $u = u_1$, $A_i(x_1)$ is true and $B_i(u_1)$ are false, hence their combination $A_i(x_1) \oplus B_i(u_1)$ is true. So, due to (15), we would conclude that the statement $(x_1 = x_0) \& (u_1 = u_0)$ is true, but this statement is actually false. The contradiction proves that the representation (15) is indeed impossible, and so, the pair $(\oplus, \vee)$ does not have the universality property.

8°. The last case is when $\ominus = \oplus$. Similarly to the previous two cases, we will analyze the three subcases when $\odot = \&$, $\odot = \vee$, and $\odot = \oplus$ one by one.

8.1°. The following explicit formula enables us to show that the pair $(\&, \oplus)$ has the universality property:

$$C(x, u) \Leftrightarrow \oplus_{(x_0, u_0) \in C} ((x = x_0) \& (u = u_0)). \quad (16)$$

Indeed, we know that a similar formula (5) holds with “or” instead of “exclusive or”. Here, the properties

$$(x = x_0) \& (u = u_0)$$

corresponding to different pairs (x_0, u_0) are mutually exclusive, and thus, for these properties, “or” coincides with “exclusive or”.

8.2°. To prove that the pair $(\vee, \oplus)$ has the universality property, we need the following auxiliary result:

$$((x = x_0) \& (u = u_0)) \Leftrightarrow ((x = x_0) \vee (u = u_0)) \oplus (x = x_0) \oplus (u = u_0). \quad (17)$$

Indeed, this can be proven by considering all four possible cases: $x = x_0$ and $u = u_0$, $x = x_0$ and $u \neq u_0$, $x \neq x_0$ and $u = u_0$, $x \neq x_0$ and $u \neq u_0$. Thus, the expression (10) can be reformulated in the following equivalent form:

$$C(x, u) \Leftrightarrow \oplus_{(x_0, u_0) \in C} (((x = x_0) \vee (u = u_0)) \oplus (x = x_0) \oplus (u = u_0)).$$

Hence, the pair $(\vee, \oplus)$ indeed has the universality property.

8.3°. When $\ominus = \oplus$ and $\odot = \oplus$, the general expression

$$(A_1(x) \odot B_1(u)) \ominus (A_2(x) \odot B_2(u)) \ominus \dots$$

takes the form

$$(A_1(x) \oplus B_1(u)) \oplus (A_2(x) \oplus B_2(u)) \oplus \dots$$

Due to commutativity and associativity of the “exclusive or” operation, this expression is equivalent to

$$(A_1(x) \oplus A_2(x) \oplus \dots) \oplus (B_1(u) \oplus B_2(u) \oplus \dots),$$

i.e., to $A(x) \oplus B(u)$, where $A(x) \stackrel{\text{def}}{=} A_1(x) \oplus A_2(x) \oplus \dots$ and

$$B(u) \stackrel{\text{def}}{=} B_1(u) \oplus B_2(u) \oplus \dots$$

We have already shown, in Parts 6.3 and 7.3 of this proof, that not every property $C(x, u)$ can be represented in the form $A(x) \oplus B(u)$: for example, the property $(x = x_0) \& (u = u_0)$ cannot be thus represented. So, the pair $(\oplus, \oplus)$ does not have the universality property. The theorem is proven.

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REFERENCES

- [1] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
- [2] V. Kreinovich, G. C. Mouzouris, and H. T. Nguyen, “Fuzzy rule based modeling as a universal approximation tool”, In: H. T. Nguyen and M. Sugeno (eds.), *Fuzzy Systems: Modeling and Control*, Kluwer, Boston, MA, 1998, pp. 135–195.
- [3] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, Boca Raton, Florida: Chapman & Hall/CRC, 2006.
- [4] I. Perfilieva, Fuzzy transforms: theory and applications, *Fuzzy Sets and Systems*, 2006, Vol. 157, pp. 993–1023.
- [5] I. Perfilieva, “Fuzzy transforms: a challenge to conventional transforms”, *Advances in Imaging and Electron Physics*, 2007, Vol. 47, pp. 137–196.
- [6] I. Perfilieva and V. Kreinovich, “A New Universal Approximation Result For Fuzzy Systems, Which Reflects CNF–DNF Duality”, *International Journal of Intelligent Systems*, 2002, Vol. 17. No. 12, pp. 1121–1130.