Negative Results of Computable Analysis Disappear If We Restrict Ourselves to Random (Or, More Generally, Typical) Inputs

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Real numbers in computable analysis: a brief reminder. In practice, many quantities such as weight, speed, etc., are characterized by real numbers. To get information about the corresponding value x, we perform measurements. Measurements are never absolute accurate. As a result of each measurement, we get a measurement result $\widetilde{x}$; for each measurement, we usually also know the upper bound Δ on the (absolute value of) the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$: $|x - \widetilde{x}| \le \Delta$.

To fully characterize a value x, we must measure it with a higher and higher accuracy. As a result, when we perform measurements with accuracy 2^{-n} with $n = 0, 1, \ldots$, we get a sequence of rational numbers r_n for which $|x - r_n| \le 2^{-n}$.

From the algorithmic viewpoint, we can view this sequence as an oracle that, given an integer n, returns a rational number r_n. Such sequences represent real numbers in computable analysis; see, e.g., [9, 10].

First negative result. In computable analysis, several negative results are known. For example, it is known that no algorithm is possible that, given two numbers x and y, would check whether these numbers are equal or not.

Computable functions and relative negative results. Similarly, we can define a function $f(x)$ from real numbers to real numbers as a mapping that, given an integer n, a rational number x_m and its accuracy m, produces either a message that this information is insufficient, or a rational number y_n which is 2^{-n}-close to all the values $f(x)$ for $d(x, x_m) \le 2^{-m}$ – and for which, for every x and for each desired accuracy n, there is an m for which a rational number y_n is produced. We can also define a computable function $f(x_1, \ldots, x_k)$ of several real variables (and, even more generally, a function on a computable compact).

Several negative results are known about computable functions as well. For example,

- while there is an algorithm that, given a function $f(x)$ on a computable compact set K (e.g., on a box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_k, \overline{x}_k]$ in k-dimensional space), produces the values $\max\{f(x) : x \in K\}$,
- no algorithm is possible that would always return a point x at which this maximum is attained (and similarly, with minimum).

From the physicists' viewpoint, these negative results seem rather theoretical. From the purely mathematical viewpoint, if two quantities coincide up to 13 digits, they may still turn to be different: for example, they may be 1 and $1 + 10^{-100}$.

However, in the physics practice, if two quantities coincide up to a very high accuracy, it is a good indication that they are actually equal. This is how physical theories are confirmed: if an experimentally observed value of a quantity turned out to be very close to the value predicted based on a theory, this means that this theory is (triumphantly) true. This is, for example, how General Relativity has been confirmed.

This is how discoveries are often made: for example, when it turned out the speed of the waves described by Maxwell equations of electrodynamics is very close to the observed speed of light c, this led physicists to realize that light is formed of electromagnetic waves.

How physicists argue. A typical physicist argument is that while numbers like $1 + 10^{-100}$ (or $c \cdot (1 + 10^{-100})$) are, in principle, possible, they are abnormal (not typical).

When a physicist argues that second order terms like $a \cdot \Delta x^2$ of the Taylor expansion can be ignored in some approximate computations because Δx is small, the argument is that

- while abnormally high values of a (e.g., $a = 10^{40}$) are mathematically possible,
- typical (= not abnormal) values appearing in physical equations are usually of reasonable size.

How to formalize the physicist's intuition of typical (not abnormal). A formalization of this intuition was proposed and analyzed in [1, 2, 3, 4, 5, 6, 7]. Its main idea is as follows. To some physicist, all the values of a coefficient a above 10 are abnormal. To another one, who is more cautious, all the values above 10 000 are abnormal. Yet another physicist may have another threshold above which everything is abnormal. However, for every physicist, there is a value n such that all value above n are abnormal.

This argument can be generalized as a following property of the set T of all typical elements. Suppose that we have a monotonically decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \ldots$ for which $\bigcap\limits_n A_n = \emptyset$ (in the above example, A_n is the set of all numbers $\geq n$). Then, there exists an integer N for which $T \cap A_N = \emptyset$.

We thus say that T is a *set of typical elements* if for every definable decreasing sequence $\{A_n\}$ for which $\bigcap\limits_n A_n = \emptyset$, there exists an N for which $T \cap A_N = \emptyset$.

Comment. Of course, to make this definition precise, we must restrict definability to a *subset* of properties, so that the resulting notion of definability will be defined in ZFC itself (or in whatever language we use); for details, see, e.g., [3].

Relation to randomness. The above notion of typicality is related to the randomness. Indeed, a usual definition of a random sequence (see, e.g., [8]) is based on the idea that a sequence is random if it satisfies all the probability laws – like the law of large numbers, the central limit theorem, etc. A probability law is then described as a definable property that is satisfied with probability 1, i.e., as a complement to a definable set S of probability measure 0 ($P(S) = 0$). Thus, we can say that a sequence is random if it does not belong to any definable set of measure 0. (If we use different languages to formalize the notion "definable", we get different versions of Kolmogorov-Martin-Löf randomness.)

Informally, this definition means that (definable) events with probability 0 cannot happen. In practice, physicists also assume that events with a *very small* probability cannot happen. It is not possible to formalize this idea by simply setting a threshold $p_0 > 0$ below which events are not

possible – since then, for N for which $2^{-N} < p_0$, no sequence of N heads or tails would be possible at all. However, we know that for each monotonic sequence of properties A_n with $\lim p(A_n) = 0$ (e.g., A_n = "we can get first n heads"), there exists an N above which a truly random sequence cannot belong to A_N. In [1, 2, 3, 4, 5, 6, 7], we thus propose to describe a set R as a set of random elements if it satisfies the following property: for every definable decreasing sequence $\{A_n\}$ for which $\lim P(A_n) = 0$, there exists an N for which $R \cap A_N = \emptyset$.

It turns out that properties of T and R are related:

- every set of typical elements is also a set of random elements, and
- for every set of random elements R, the difference $R - R_K$, where R_K is the set of the elements random in the usual Komogorov-Martin-Löf sense, is a set of typical elements [2].

Physically interesting consequences of these definitions. These definitions have useful consequences [1, 2, 3, 4, 5, 6, 7].

For example, when the universal set X is a metric space, both sets T and R are *pre-compact* – with the consequence that all inverse problems become well-defined: for any 1-1 continuous function $f : X \to X$, the restriction of the inverse function to T is also continuous. This means that, in contrast to ill-defined problem, if we perform measurements accurately enough, we can reconstruct the state of the system with any desired accuracy.

Another example is a justification of physical induction: crudely speaking, there exists an N such that if for a typical sequence, a property is satisfied in the first N experiments, then it is satisfied always.

New results: when we restrict ourselves to typical elements, algorithms become possible. In this paper, we analyze the computability consequences of the above definitions. Specifically, we show that most negative results of computability analysis disappear if we restrict ourselves to typical elements.

For example, for every set of typical pairs of real numbers $T \subseteq \mathbb{IR}^2$, there exists an algorithm, that, given real numbers $(x, y) \in T$, decides whether $x = y$ or not. To prove it, consider a decreasing sequence of definable sets $A_n = \{(x, y) : 0 < d(x, y) < 2^{-n}\}$. By definition of T, there exists an N such that $A_N \cap T = \emptyset$. Thus, if we compute $d(x, y)$ with accuracy $2^{-(N+1)}$ and get a value $< 2^{-N}$, this means that $x = y$ – otherwise $x \neq y$.

Similar (but somewhat more complex) arguments lead to

- an algorithm that, given a typical function $f(x)$ on a computable compact K, computes a value x at which $f(x)$ attains its maximum,
- an algorithm that, given a typical function $f(x)$ on a computable compact K that attains a 0 value somewhere on K, computes a value x at which $f(x) = 0$,
- etc.

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References

[1] A. M. Finkelstein and V. Kreinovich, "Impossibility of hardly possible events: physical consequences", *Abstracts of the 8 International Congress on Logic, Methodology, and Philosophy of Science*, Moscow, 1987, Vol. 5, Part 2, pp. 23–25.

[2] V. Kreinovich, "Toward Formalizing Non-Monotonic Reasoning in Physics: the Use of Kolmogorov Complexity", *Revista Iberoamericana de Inteligencia Artificial*, 2009, Vol. 41, pp. 4–20.

[3] V. Kreinovich and A. M. Finkelstein, "Towards Applying Computational Complexity to Foundations of Physics", *Notes of Mathematical Seminars of St. Petersburg Department of Steklov Institute of Mathematics*, 2004, Vol. 316, pp. 63–110; reprinted in *Journal of Mathematical Sciences*, 2006, Vol. 134, No. 5, pp. 2358–2382.

[4] V. Kreinovich and I. A. Kunin, "Kolmogorov Complexity and Chaotic Phenomena", *International Journal of Engineering Science*, 2003, Vol. 41, No. 3–5, pp. 483–493.

[5] V. Kreinovich and I. A. Kunin, "Kolmogorov Complexity: How a Paradigm Motivated by Foundations of Physics Can Be Applied in Robust Control", In: A. L. Fradkov and A. N. Churilov (eds.), *Proceedings of the International Conference "Physics and Control" PhysCon'2003*, Saint-Petersburg, Russia, August 20–22, 2003, pp. 88–93.

[6] V. Kreinovich and I. A. Kunin, "Application of Kolmogorov Complexity to Advanced Problems in Mechanics", *Proceedings of the Advanced Problems in Mechanics Conference APM'04*, St. Petersburg, Russia, June 24–July 1, 2004, pp. 241–245.

[7] V. Kreinovich, L. Longpré, and M. Koshelev, "Kolmogorov complexity, statistical regularization of inverse problems, and Birkhoff's formalization of beauty", In: A. Mohamad-Djafari (ed.), *Bayesian Inference for Inverse Problems*, Proceedings of the SPIE/International Society for Optical Engineering, Vol. 3459, San Diego, CA, 1998, pp. 159–170.

[8] M. Li and P. Vitanyi, *An Introduction to Kolmogorov Complexity and Its Applications*, Springer, 2008.

[9] M. B. Pour-El and J. I. Richards, *Computability in Analysis and Physics*, Springer, Berlin, 1989.

[10] K. Weihrauch, *Computable Analysis*, Springer-Verlag, Berlin, 2000.