NEGATIVE RESULTS OF COMPUTABLE ANALYSIS DISAPPEAR IF WE RESTRICT OURSELVES TO RANDOM (OR, MORE GENERALLY, TYPICAL) INPUTS

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Abstract. It is well known that many computational problems are, in general, not algorithmically solvable: e.g., it is not possible to algorithmically decide whether two computable real numbers are equal, and it is not possible to compute the roots of a computable function. We propose to constraint such operations to certain "sets of typical elements" or "sets of random elements".

In our previous papers, we proposed (and analyzed) physics-motivated definitions for these notions. In short, a set $\mathcal{T}$ is a *set of typical elements* if for every definable sequences of sets A_n with $A_n \supseteq A_{n+1}$ and $\bigcap\limits_n A_n = \emptyset$, there exists an N for which $A_N \cap \mathcal{T} = \emptyset$; the definition of a *set of random elements* with respect to a probability measure P is similar, with the condition $\bigcap\limits_n A_n = \emptyset$ replaced by a more general condition $\lim\limits_n P(A_n) = 0$.

In this paper, we show that if we restrict computations to such typical or random elements, then problems which are non-computable in the general case – like comparing real numbers or finding the roots of a computable function – become computable.

1. Physically meaningful computations with real numbers: a brief reminder

In practice, many quantities such as weight, speed, etc., are characterized by real numbers. To get information about the corresponding value x, we perform measurements. Measurements are never absolute accurate. As a result of each measurement, we get a measurement result $\widetilde{x}$; for each measurement, we usually also know the upper bound Δ on the (absolute value of) the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$: $|x - \widetilde{x}| \le \Delta$.

To fully characterize a value x, we must measure it with a higher and higher accuracy. As a result, when we perform measurements with accuracy 2^{-n} with $n = 0, 1, \ldots$, we get a sequence of rational numbers r_n for which $|x - r_n| \le 2^{-n}$.

From the algorithmic viewpoint, we can view this sequence as an oracle that, given an integer n, returns a rational number r_n. Such sequences represent real numbers in computable analysis; see, e.g., [Pou89, Wei00].

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2. First negative result

In computable analysis, several negative results are known. For example, it is known that no algorithm is possible that, given two numbers x and y, would check whether these numbers are equal or not.

3. Computable functions and relative negative results

Similarly, we can define a function $f(x)$ from real numbers to real numbers as a mapping that, given an integer n, a rational number x_m and its accuracy m, produces either a message that this information is insufficient, or a rational number y_n which is 2^{-n}-close to all the values $f(x)$ for $d(x, x_m) \le 2^{-m}$ – and for which, for every x and for each desired accuracy n, there is an m for which a rational number y_n is produced. We can also define a computable function $f(x_1, \ldots, x_k)$ of several real variables (and, even more generally, a function on a computable compact).

Several negative results are known about computable functions as well. For example,

- while there is an algorithm that, given a function $f(x)$ on a computable compact set K (e.g., on a box $[\underline{x}_1, \overline{x}_1] \times \ldots \times [\underline{x}_k, \overline{x}_k]$ in k-dimensional space), produces the values $\max\{f(x) : x \in K\}$,
- no algorithm is possible that would always return a point x at which this maximum is attained (and similarly, with minimum).

4. From the physicists' viewpoint, these negative results seem rather theoretical

From the purely mathematical viewpoint, if two quantities coincide up to 13 digits, they may still turn to be different: for example, they may be 1 and $1 + 10^{-100}$.

However, in the physics practice, if two quantities coincide up to a very high accuracy, it is a good indication that they are actually equal. This is how physical theories are confirmed: if an experimentally observed value of a quantity turned out to be very close to the value predicted based on a theory, this means that this theory is (triumphantly) true. This is, for example, how General Relativity has been confirmed.

This is how discoveries are often made: for example, when it turned out the speed of the waves described by Maxwell equations of electrodynamics is very close to the observed speed of light c, this led physicists to realize that light is formed of electromagnetic waves.

5. How physicists argue

A typical physicist argument is that while numbers like $1 + 10^{-100}$ (or $c \cdot (1 + 10^{-100})$) are, in principle, possible, they are abnormal (not typical).

When a physicist argues that second order terms like $a \cdot \Delta x^2$ of the Taylor expansion can be ignored in some approximate computations because Δx is small, the argument is that

- while abnormally high values of a (e.g., $a = 10^{40}$) are mathematically possible,
- typical (= not abnormal) values appearing in physical equations are usually of reasonable size.

6. How to formalize the physicist's intuition of typical (not abnormal)

A formalization of this intuition was proposed and analyzed in [FK87, Kre09, KF06, KK03, KK03a, KK04, KLK98]. Its main idea is as follows. To some physicist, all the values of a coefficient a above 10 are abnormal. To another one, who is more cautious, all the values above 10 000 are abnormal. Yet another physicist may have another threshold above which everything is abnormal. However, for every physicist, there is a value n such that all value above n are abnormal.

This argument can be generalized as a following property of the set $\mathcal{T}$ of all typical elements. Suppose that we have a monotonically decreasing sequence of sets $A_1 \supseteq A_2 \supseteq \ldots$ for which $\bigcap\limits_n A_n = \emptyset$ (in the above example, A_n is the set of all numbers $\geq n$). Then, there exists an integer N for which $\mathcal{T} \cap A_N = \emptyset$.

We thus arrive at the following definition:

Definition 6.1. We say that $\mathcal{T}$ is a *set of typical elements* if for every definable decreasing sequence $\{A_n\}$ for which $\bigcap\limits_n A_n = \emptyset$, there exists an N for which $\mathcal{T} \cap A_N = \emptyset$.

Mathematical comment. The word "definable" is understood in the usual way. Let $\mathcal{L}$ be a theory, let $P(x)$ be a formula from the language of the theory $\mathcal{L}$, with one free variable x so that the set $\{x \,|\, P(x)\}$ is defined in $\mathcal{L}$. We will then call the set $\{x \,|\, P(x)\}$ *$\mathcal{L}$-definable*.

Our objective is to be able to make mathematical statements about $\mathcal{L}$-definable sets. Thus, we must restrict definability to a *subset* of properties, so that the resulting notion of definability will be defined in ZFC itself (or in whatever language we use). In other words, we must have a stronger theory $\mathcal{M}$ in which the class of all $\mathcal{L}$-definable sets is a countable set. One can prove that such $\mathcal{M}$ always exists; for details, see, e.g., [KF06].

One can prove that the above definition of the set of typical elements is consistent in the following sense:

Theorem 6.2. *For every probability measure P on the universal set X which is defined on all definable subsets of X, and for every real number $\varepsilon > 0$, there exists a set $\mathcal{T}$ of typical elements for which the lower probability is $\geq 1 - \varepsilon$: $\underline{P}(\mathcal{T}) \geq 1 - \varepsilon$.*

Proof. There are countably many monotonically decreasing definable sequences with empty intersection: $\{A_n\}$: $\{A_n^{(1)}\}$, $\{A_n^{(2)}\}$, ... For each k, since the sequence is monotonically decreasing and has an empty intersection, we have $P\left(A_n^{(k)}\right) \to 0$ as $n \to \infty$. Hence, there exists N_k for which $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$. We can now take $\mathcal{T} \stackrel{\text{def}}{=} -\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}$. Since $P\left(A_{N_k}^{(k)}\right) \leq \varepsilon \cdot 2^{-k}$, we have

$$\overline{P}\left(\bigcup_{k=1}^{\infty} A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} P\left(A_{N_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} \varepsilon \cdot 2^{-k} = \varepsilon.$$

Hence, $\underline{P}(\mathcal{T}) = 1 - \overline{P}\left(\bigcup\limits_{k=1}^{\infty} A_{N_k}^{(k)}\right) \geq 1 - \varepsilon$. □

7. Relation to randomness

The above notion of typicality is related to the notion of a random object (see, e.g., [LV08]).

Namely, Kolmogorov and Martin-Löf proposed a new definition of a random sequence, a definition that separates physically random binary sequences – e.g., sequences that appear in coin flipping experiments or sequences that appear in quantum measurements – from sequence that follow some pattern. Intuitively, if a sequence s is random, it satisfies all the probability laws – like the law of large numbers, the central limit theorem, etc. Vice versa, if a sequence satisfies all probability laws, then for all practical purposes we can consider it random. Thus, we can define a sequence to be random if it satisfies all probability laws.

What is a probability law? In precise terms, it is a statement S which is true with probability 1: $P(S) = 1$. So, to prove that a sequence is not random, we must show that it does not satisfy one of these laws.

Equivalently, this statement can be reformulated as follows: a sequence s is *not* random if $s \in C$ for a (definable) set C $(= -S)$ with $P(C) = 0$. As a result, we arrive at the following definition:

Definition 7.1. We say that a sequence is *random* if it does not belong to any definable set of measure 0.

(If we use different languages to formalize the notion "definable", we get different versions of Kolmogorov-Martin-Löf randomness.)

It is easy to prove that this definition is consistent, in the sense that almost all sequences are random. Indeed, every definable set C is defined by a finite sequence of symbols (its definition). Since there are countably many sequences of symbols, there are (at most) countably many definable sets C. So, the complement $-\mathcal{R}$ to the class $\mathcal{R}$ of all random sequences also has probability 0.

Informally, this definition means that (definable) events with probability 0 cannot happen. In practice, physicists also assume that events with a *very small* probability cannot happen. For example, they believe that it is not possible that all the molecules in the originally uniform air move to one side of the room – although, from the viewpoint of statistical physics, the probability of this event is not zero. This fits very well with a commonsense understanding of rare events: e.g., if a coin falls head 100 times in a row (or a casino roulette gets to red 100 times in a row), any reasonable person will conclude that this coin is not fair.

It is not possible to formalize this idea by simply setting a threshold $p_0 > 0$ below which events are not possible – since then, for N for which $2^{-N} < p_0$, no sequence of N heads or tails would be possible at all. However, we know that for each monotonic sequence of properties A_n with $\lim p(A_n) = 0$ (e.g., $A_n =$ "we can get first n heads"), there exists an N above which a truly random sequence cannot belong to A_N. In [FK87, Kre09, KF06, KK03, KK03a, KK04, KLK98], we thus propose the following definition of a set of random elements:

Definition 7.2. We say that $\mathcal{R}$ is a *set of random elements* if for every definable decreasing sequence $\{A_n\}$ for which $\lim P(A_n) = 0$, there exists an N for which $\mathcal{R} \cap A_N = \emptyset$.

Let us show, on the example of coin tossing, that this definition indeed formalizes our intuition. In this case, the universal set is the set of all sequences of Heads (H) and Tails (T): $U = \{\mathrm{H}, \mathrm{T}\}^{\mathbb{N}}$ Here, A_n is the set of all the sequences that start with n heads. The

sequence $\{A_n\}$ is decreasing and definable, and its intersection has probability 0. Therefore, for every set $\mathcal{R}$ of random elements of U, there exists an integer N for which $A_N \cap \mathcal{R} = \emptyset$. This means that if a sequence $s \in \mathcal{R}$ is random and starts with N heads, it must consist of heads only. In physical terms, this means that a random sequence cannot start with N heads. This is exactly what we wanted to formalize.

The above definition is very similar to the definition of the set of typical elements; the only difference is that the condition $\bigcap\limits_n A_n = \emptyset$ is replaced with a more general condition $\lim P(A_n) = 0$. This relation leads to the following relation between these two definitions. Let $\mathcal{R}_K$ denote the set of the elements random in the usual Kolmogorov-Martin-Löf sense. Then the following is true [Kre09]:

Theorem 7.3.

- *Every set of random elements is also a set of typical elements.*
- *For every set of typical elements $\mathcal{T}$, the intersection $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements.*

Proof. If $\cap A_n = \emptyset$ then $P(A_n) \to 0$. Thus, every set of random elements is also a set of typical elements.

Vice versa, let $\mathcal{T}$ be a set of typical elements. Let us prove that $\mathcal{T} \cap \mathcal{R}_K$ is a set of random elements. Indeed, if $P(\cap A_n) = 0$ then for $B_m \stackrel{\text{def}}{=} A_m - \cap A_n$, we have $B_m \supseteq B_{m+1}$ and $\cap B_n = \emptyset$. Thus, by definition of a set of typical elements, we conclude that there exists an integer N for which $B_N \cap \mathcal{T} = \emptyset$). Since $P(\cap A_n) = 0$, we also know that $(\cap A_n) \cap \mathcal{R}_K = \emptyset$. Thus, $A_N = B_N \cup (\cap A_n)$ has no common elements with the intersection $\mathcal{T} \cap \mathcal{R}_K$. □

8. Physically interesting consequences of these definitions

These definitions have useful consequences [FK87, Kre09, KF06, KK03, KK03a, KK04, KLK98].

8.1. **Ill-posed problems.** The first example is related to *inverse problems* (see, e.g., [Tik77]). These problems are related to the main objectives of science: ro provide (ideally) *guaranteed* estimates for physical quantities and (ideally) *guaranteed* predictions for these quantities. The problem with getting such guarantees is that estimation and prediction are *ill-posed problems* in the sense that very small changes in the measurement results can lead to very large changes in the reconstructed state.

One reason for this phenomenon is that measurement devices are inertial. As a result, they suppress high frequencies ω in the measured signal. For such high frequencies ω, signals $\varphi(t)$ and $\varphi(t) + A \cdot \sin(\omega \cdot t)$ are indistinguishable. So, based on the measurements only, we cannot tell whether the actual signal is the original signal $\varphi(t)$ or – for some large A – a very difference signal $\varphi(t) + A \cdot \sin(\omega \cdot t)$.

The existing approaches to this problem are based on some prior assumptions about the actual signal. For example, if we know the actual probability distribution on the set of all possible signals, we can get *statistical* regularization (filtering). If we know bounds on the actual signal's rate of change, e.g., if we know that $|\dot{\varphi}| \le \Delta$ for some known Δ, then we can use *Tikhonov regularization*. Experts can provide other information about the actual signal, in which case we have expert-based regularization. The problem is that we rarely

have this information. We may assume some bounds on the rate of change – but then there is no guarantee that the prediction based on this assumption is correct.

In precise terms, this problem can be formulated as follows. Let S denote the set of all possible states, and let R denote the set of all possible measurement results. In this description, an (ideal) measurement is a continuous 1-1 mapping $f : S \to R$. In principle, we can reconstruct the original state s from the measurement result $r = f(s)$ by applying the inverse function $s = f^{-1}(r)$. However, the inverse function is, in general, not continuous. As a result, very small measurement errors (changes in r) can lead to drastic changes in the reconstructed state $f^{-1}(r)$. It turns out that if we take into account that the actual states should be typical (i.e., belong to a set of typical states), then this problem disappears:

Definition 8.1. A definable metric space S is called *definably separable* if there exists a definable sequence $s_1, \ldots, s_n, \ldots$ which is everywhere dense in S.

Theorem 8.2. *Let S be a definably separable metric space, let $\mathcal{T}$ be a set of all not abnormal elements of S, and let $f : S \to R$ be a continuous 1-1 function. Then, the inverse mapping $f^{-1} : R \to S$ is continuous for every $r \in f(\mathcal{T})$.*

Proof. It is well known that if a f is continuous and 1-1 on a compact, then the inverse function f^{-1} is also continuous. So, to prove our result, it is sufficient to prove that the set $f(\mathcal{T})$ is *pre-compact*, i.e., that its closure is compact. In a metric space, a set X is compact if and only if it is closed and for every ε, it has a finite ε-net.

Since the metric space S is definably separable, there exists a definable sequence $s_1, \ldots, s_n, \ldots$ which is everywhere dense in S. Let us take take $A_n \stackrel{\text{def}}{=} -\bigcup_{i=1}^{n} B_\varepsilon(s_i)$. Since s_i are everywhere dense, we have $\cap A_n = \emptyset$. Hence, there exists N for which $A_N \cap \mathcal{T} = \emptyset$. Since $A_N = -\bigcup_{i=1}^{N} B_\varepsilon(s_i)$, this means $\mathcal{T} \subseteq \bigcup_{i=1}^{N} B_\varepsilon(s_i)$. Hence $\{s_1, \ldots, s_N\}$ is an ε-net for $\mathcal{T}$. □

This continuity means that, in contrast to general ill-defined problem, if we perform measurements accurately enough, we can reconstruct the state of the system with any desired accuracy.

8.2. **Justification of physical induction.** What is physical induction? This is a conclusion that physicists make: if a property P is satisfied in the first N experiments, and this number N is sufficiently large, then the property is satisfied always. It turns out that our definition enables us to formalize this idea.

For every property P and for every system o, we can define a sequence of values $s \stackrel{\text{def}}{=} s_1 s_2 \ldots$, where:

- $s_i = T$ if P holds in the i-th experiment with the system o, and
- $s_i = F$ if $\neg P$ holds in the i-th experiment with the system o.

Let X be the set of all such sequences. It is reasonable to require that if the system is typical, then the resulting sequence s is also typical i.e., belongs to some set $\mathcal{T}$ of typical elements.

Theorem 8.3. *For every set of typical element $\mathcal{T} \subseteq X$, there exists an integer N such that if or some $s \in \mathcal{T}$, we have $s_1 = \ldots = S_N = T$, then $s_m = T$ for all m.*

Proof. Let us consider the following sequence of sets:

$$A_n \stackrel{\text{def}}{=} \{o : s_1 = \ldots = s_n = T \,\&\, \exists m\, (s_m = F)\}.$$

Once can easily check that $A_n \supseteq A_{n+1}$ and $\cup A_n = \emptyset$. Thus, there exists an integer N for which $A_N \cap \mathcal{T} = \emptyset$. This means that if $s \in \mathcal{T}$ and $s_1 = \ldots = s_N = T$, then we cannot have m for which $s_m = F$, i.e., $s_m = T$ for all m. □

In other words, there exists an N such that if for a typical sequence, a property is satisfied in the first N experiments, then it is satisfied always – this is exactly physical induction.

9. New results: when we restrict ourselves to typical elements, algorithms become possible

In this paper, we analyze the computability consequences of the above definitions. Specifically, we show that most negative results of computability analysis disappear if we restrict ourselves to typical elements.

9.1. **Deciding equality.** Our first result is about checking whether two given real numbers are equal or not, the problem which, as we have mentioned, is, in general, algorithmically unsolvable.

Theorem 9.1. *For every set of typical pairs of real numbers $\mathcal{T} \subseteq \mathbb{IR}^2$, there exists an algorithm, that, given computable real numbers $(x, y) \in \mathcal{T}$, decides whether $x = y$ or not.*

Proof. The main idea behind this proof is that we can take the sets

$$A_n = \{(x, y) : 0 < d(x, y) < 2^{-n}\}.$$

Then, we have $A_n \supseteq A_{n+1}$ and $\cap A_n = \emptyset$, so there exists an integer N for which $A_N \cap \mathcal{T} = \emptyset$, i.e., for which, if $(x, y) \in \mathcal{T}$, then either $d(x, y) = 0$ (i.e., $x = y$) or $d(x, y) \geq 2^{-N}$.

Thus, if we know that the pair (x, y) belongs to the set $\mathcal{T}$, we can decide whether $x = y$ by using the following algorithm. We compute $d(x, y)$ with accuracy $2^{-(N+2)}$, i.e., compute d such that $|d(x, y) - d| \leq 2^{-(N+2)}$. Then:

- if $d \geq 2^{-(N+1)}$, then $d(x, y) \geq d - 2^{-(N+2)} \geq 2^{-(N+1)} - 2^{-(N+2)} > 0$, hence $x \neq y$;
- if $d < 2^{-(N+1)}$, then $d(x, y) \leq d + 2^{-(N+2)} \leq 2^{-(N+1)} + 2^{-(N+2)} < -2^{-N}$, hence $x = y$.

□

Comment. This and following results are similar to results of Matheiu Hoyrup on layerwise computability (see, e.g., [Hoy11]: we have computability on each set $\mathcal{K}$, and – as we have mentioned earlier – we can have such sets with probability $P(\mathcal{K}) \geq 1 - \varepsilon$, for any given ε.

9.2. **Finding roots.** As we have mentioned, in general, it is not possible, given a computable function, to compute its root. This becomes possible if we restrict ourselves to *typical* functions:

Theorem 9.2. *Let K be a computable compact, and let X be the set of all functions $f : K \to \mathbb{R}$ that attains 0 value somewhere on K. Then, for every set of typical elements $\mathcal{T} \subseteq X$, there exists an algorithm that, given a function $f \in \mathcal{K}$, computes a point x at which $f(x) = 0$.*

Moreover, we can not only produce a root x, we can actually compute, for any given n, an 2^{-n}-approximations to the corresponding set of roots $\{x : f(x) = 0\}$ in the sense of the Hausdorff distance

$$d_H(A, B) \stackrel{\text{def}}{=} \max\left(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right),$$

where $d(a, B) \stackrel{\text{def}}{=} \inf\limits_{b\in B} d(a, b)$.

Comment. In other words, there exists an algorithm that, given a typical function $f(x)$ on a computable compact K that attains a 0 value somewhere on K, computes a root x – and also computes an 2^{-n}-approximations to the corresponding set of roots.

Proof. To compute the set $R \stackrel{\text{def}}{=} \{x : f(x) = 0\}$ with accuracy $\varepsilon > 0$, let us take an $(\varepsilon/2)$-net $\{x_1, \ldots, x_n\} \subseteq K$. Such a net exists, since K is a constructive compact set.

For each i, we can compute $\varepsilon' \in (\varepsilon/2, \varepsilon)$ for which $B_i \stackrel{\text{def}}{=} \{x : d(x, x_i) \le \varepsilon'\}$ is a computable compact set; see, e.g., [Bis67]. It is possible to algorithmically compute the maximum of a computable function on a computable compact set; thus, we can compute the value $m_i \stackrel{\text{def}}{=} \min\{|f(x)| : x \in B_i\}$. Since $f \in \mathcal{T}$, similarly to the previous proof, we can prove that there exists an integer N for which, for all $f \in \mathcal{T}$ and for all i, we have either $m_i = 0$ or $m_i \ge 2^{-N}$. Thus, by computing each m_i with accuracy $2^{-(N+2)}$, we can check whether $m_i = 0$ or $m_i > 0$.

We claim that $d_H(R, \{x_i : m_i = 0\}) \le \varepsilon$, i.e., that:

- for every point x_i for which $m_i = 0$, there exists an ε-close root x, and
- for every root x, there exists an ε-close point x_i for which $m_i = 0$.

Indeed, if $m_i = 0$, this means that the minimum of a function $|f(x)|$ on the ε'-ball B_i with a center in x_i is equal to 0. Since the set K is compact, this value 0 is attained, i.e., there exists a value $x \in B_i$ for which $f(x) = 0$. From $x \in B_i$, we conclude that $d(x, x_i) \le \varepsilon'$ and, since $\varepsilon' < \varepsilon$, that $d(x, x_i) < \varepsilon$. Thus, x_i is ε-close to the root x.

Vice versa, let x be a root, i.e., let $f(x) = 0$. Since the points x_i form an $(\varepsilon/2)$-net, there exists an index i for which $d(x, x_i) \le \varepsilon/2$. Since $\varepsilon/2 < \varepsilon'$, this means that $d(x, x_i) \le \varepsilon'$ and thus, $x \in B_i$. Therefore, $m_i = \min\{|f(x)| : x \in B_i\} = 0$. So, the root x is ε-close to a point x_i for which $m_i = 0$. □

9.3. **Computing fixed points.** In general, it is not possible, given a computable function from a constructive compact set to itself, to compute its fixed point. This becomes possible if we restrict ourselves to *typical* functions:

Theorem 9.3. *Let K be a computable compact, and let X be the set of all computable functions $f : K \to K$ that have a fixed point x for which $f(x) = x$. Then, for every set of typical elements $\mathcal{T} \subseteq X$, there exists an algorithm that, given a function $f \in \mathcal{K}$, computes a point x at which $f(x) = x$.*

Moreover, we can not only produce such a fixed point, we can actually compute, for any given n, an 2^{-n}-approximations to the corresponding set of all fixed points $\{x : f(x) = x\}$.

Comment. In other words, there exists an algorithm that, given a typical function $f(x)$ on a computable compact K that has a fixed point, computes this fixed point – and also computes an 2^{-n}-approximations to the corresponding set of all fixed points.

Proof. This problem can be reduced to the root finding problem if we take into consideration that that $f(x) = x$ if and only if $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} d(f(x), x)$. □

9.4. **Locating global maxima.** In general, it is not possible, given a computable function, to find a point where it attains its maximum. This becomes possible if we restrict ourselves to *typical* functions:

Theorem 9.4. *Let K be a computable compact, and let X be the set of all functions $f : K \to \mathbb{R}$. Then, for every set of typical elements $\mathcal{T} \subseteq X$, there exists an algorithm that, given a computable function $f \in \mathcal{K}$, computes a point x at which $f(x) = \max\limits_{y} f(y)$.*

Moreover, we can not only produce such a point, we can actually compute, for any given n, an 2^{-n}-approximations to the corresponding set of global maximum locations $\left\{x : f(x) = \max\limits_{y} f(y)\right\}$.

Comment. In other words, there exists an algorithm that, given a typical function $f(x)$ on a computable compact K, computes a point where this function attains its maximum – and also computes an 2^{-n}-approximations to the corresponding set of all such points.

Proof. This problem can be reduced to previous one if we take into consideration the fact that maximum $\max\limits_{y} f(y)$ of a computable function on a computable compact is computable and that that $f(x) = \max\limits_{y} f(y)$ if and only if $g(x) = 0$, where $g(x) \stackrel{\text{def}}{=} f(x) = \max\limits_{y} f(y)$. □

9.5. **Computing minimax strategies.** Is it similarly possible to compute the optimal *minimax* strategies, i.e., find x such that

$$\min_{y} f(x, y) = \max_{z} \min_{y} f(z, y).$$

Indeed, this problem is equivalent to finding location of the maximum of a computable function $g(x) \stackrel{\text{def}}{=} \min\limits_{y} f(x, y)$.

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