When Is Busemann Product a Lattice? A Relation Between Metric Spaces and Corresponding Space-Time Models

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Abstract

The causality relation of special relativity is based on the assumption that the speed of all physical processes is limited by the speed of light. As a result, an event (t, x) occurring at moment t at location x can influence an event (y, s) if and only if $s \geq t + \dfrac{d(x, y)}{c}$. We can simplify this formula if we use units of time and distance in which $c = 1$ (e.g., by using a light second as a unit of distance). In this case, the above causality relation takes the form $s \geq t + d(x, y)$. Since the actual space can be non-Euclidean, H. Busemann generalized this ordering relation to the case when points x, y, etc. are taken from an arbitrary metric space X. A natural question is: when is the resulting ordered space – called a Busemann product – a lattice? In this paper, we provide a necessary and sufficient condition for it being a lattice: it is a lattice if and only if X is a real tree, i.e., a metric space in which every two points are connected by exactly one arc, and this arc is geodesic (i.e., metrically isomorphic to an interval on a real line).

1 Formulation of the Problem

Special relativity: brief reminder. To uniquely describe an event, we need to describe the moment of time t at which it occurs and its spatial location x. In other words, an event can be characterized by a pair (t, x), where $t \in \mathbb{R}$ is a real number and x is an element of the metric space X describing the proper physical space.

Such pairs form a *space-time* $\mathbb{R} \times X$. How can we describe the causality relation $\leq$ on this space-time, i.e., the relation $a \leq b$ meaning that an event a can causally influence the event b?

According to the special relativity theory, the speed of all processes is limited by the speed of light c. So, an event (t, x) can influenced an event (s, y) if $t \leq s$ and if it is possible for a signal from x to reach y in time $s - t$. During this time, the signal can cover at most the distance $c \cdot (s - t)$, so this condition can be expressed as $(t, x) \leq (s, y) \Leftrightarrow d(x, y) \cdot c \cdot (s - t)$.

This condition can be simplified even further if, instead of using different units for measuring space and time, we use the same units for both, i.e., if we use, as a unit of distance, the distance c that the light covers in one second. In these new units, the numerical value of the speed of light is 1, so the causality relation takes the following simplified form:

$$(s, y) \geq (t, x) \Leftrightarrow s - t \geq d(x, y). \tag{1}$$

Busemann product. In the special relativity theory, the proper space X is a usual Euclidean space. Starting with general relativity, however, physicists realized that the actual space-time is curved. Thus, it is reasonable to consider space-time models $\mathbb{R} \times X$ with non-Euclidean metric spaces X and causality relation (1). Such models were first considered by H. Busemann [2] and are thus called *Busemann products* of the real line $\mathbb{R}$ and the metric space X (see also [3, 4]).

A natural question: when is the Busemann product $\mathbb{R} \times X$ a lattice? From the viewpoint of ordered space, a natural question is: when is the Busemann product a lattice?

In the simplest case of a 1-D Euclidean space (and thus, 2-D space-time) it is a lattice. Indeed, in this case, $d(x, y) = |x - y|$ and since $|z| = \max(z, -z)$, the relation $s - t \geq d(x, y) = |x - y| = \max(x - y, y - x)$ is equivalent to $s - t \geq x - y$ *and* $s - t \geq y - x$. By moving terms t and x related to the event (t, x) to one side of each of these inequalities, and terms s and y related to the event (s, y) to another side, we get an equivalent form: $s + y \geq t + x$ and $s - y \geq t - x$. So, if instead of the original coordinates t and x, we use new coordinates $u = t + x$ and $v = t - x$, the ordering relation between two events (u, v) and (u', v') takes the form

$$(u', v') \geq (u, v) \Leftrightarrow ((u' \geq u) \,\&\, (v' \geq v)).$$

One can easily check that for this relation, every two elements (u, v) and (u', v') have the greatest lower bound (*meet*) $(u, v) \wedge (u', v') = (\min(u, u'), \min(v, v'))$ and least upper bound (*join*) $(u, v) \vee (u', v') = (\max(u, u'), \max(v, v'))$ – i.e., that it is indeed a lattice.

On the other hand, for the 3-D Euclidean space, the Busemann product – i.e., the causality relation of the Special Relativity theory – is *not* a lattice.

Indeed, for a lattice, the intersection of two *future cones* $a^+ = \{b : b \geq a\}$ and $(a')^+ = \{b : b \geq a'\}$ is also a future cone: namely, the future cone of the join $a \vee a'$. For special relativity, the future cone is, from the geometric viewpoint, an actual cone

$$(s, y_1, y_2, y_3) \geq (t, x_1, x_2, x_3) \Leftrightarrow (s - t) \geq d(x, y) \Leftrightarrow$$

$$(s \geq t \,\&\, (s-t)^2 \geq d^2(x, y) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2) \Leftrightarrow$$

$$(s \geq t \,\&\, (s-t)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 \geq 0).$$

It is also easy to see that the intersection of two geometric cones is, in general, *not* a cone – and thus, this ordered space is not a lattice.

It is therefore reasonable to ask: when is a Busemann product a lattice? The definition of a lattice means that for every two elements, we have a meet and a join. If for every two elements, we have a meet, this is called a *lower semi-lattice*; if for every two elements, we have a join, this is called an *upper semi-lattice*. A lattice is thus an ordered space which is at the same time a lower and an upper semi-lattice. We can therefore also ask: when is a Busemann product a lower semi-lattice? an upper semi-lattice? In this paper, we provide a necessary and sufficient condition for the Busemann product to be a lattice, a lower semi-lattice, and/or an upper semi-lattice.

2 Main Result

The answer comes in terms of *real trees* (*R-trees*), i.e., metric spaces in which every two points x and y are connected by exactly one *arc* – a homeomorphic embedding of an interval into this space, and this arc is *geodesic*, i.e., is formed by points x_α, $\alpha \in [0, d(x, y)]$ for which $d(x_\alpha, x_\beta) = |\alpha - \beta|$; see, e.g., [1].

An example of a real tree is a *hedgehog set* – a collection of several intervals with a common starting point O, in which the distance on each interval is Euclidean, between two points x and y on different intervals is defined as $d(x, y) = d(x, O) + d(O, y)$.

Definition 1. *Let X be a metric space with distance d. A set $\mathbb{R} \times X$ with an ordering relation $(s, y) \geq (t, x) \Leftrightarrow s - t \geq d(x, y)$ is called a Busemann product.*

Theorem. *For each metrics space X, the following conditions are equivalent to each other:*

- *the Busemann product $\mathbb{R} \times X$ is a lattice;*
- *the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice;*
- *the Busemann product $\mathbb{R} \times X$ is an upper semi-lattice;*
- *the space X is a real tree.*

Proof.

1°. In this proof, we will use the following equivalent characterization of real trees: a metric space X is a real tree if and only if the following two conditions are satisfied:

- every two points x and y can be connected by a geodesic arc, and
- for every point x_α on the geodesic arc connecting x and y, and for every other point z, either x_α lies on a geodesic arc connecting x and z, or x_α lies on a geodesic arc connecting y and z.

These conditions are intuitively clear: when we go from x to z in a tree, we may follow the path from x to z for a while, but there is a branching point at which the paths deviate.

- If this branching point is after x_α, then x_α is on the geodesic path from x to y.
- If the branching happens before x_α, then the geodesic path from z to y should go pass x_α – otherwise, the paths from x to y, from y to z, and from x to z would form a loop, which cannot happen in a tree.

2°. First, let us prove that if $\mathbb{R} \times X$ is a lower semi-lattice, then X is a real tree. For the upper semi-lattice, the proof is similar.

3°. Let us first prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x,y))$, there exists a point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x,y) - \alpha$.

Indeed, let us consider the following four points: (α, x), $(d(x,y)-\alpha, y)$, $(-\alpha, x)$, and $(\alpha - d(x,y), y)$. By using the definition of the Busemann product order, we can easily check that each of the first two points follows each of the second two points:

$$(\alpha, x) \geq (-\alpha, x), \quad (\alpha, x) \geq (\alpha - d(x,y), y),$$

$$(d(x,y) - \alpha, y) \geq (-\alpha, x), \quad (d(x,y) - \alpha, y) \geq (\alpha - d(x,y), y).$$

Since the Busemann product $\mathbb{R} \times X$ is a lower semi-lattice, the first two points (α, x) and $(d(x,y)-\alpha, y)$ have a meet, i.e., a point $(s, z) \stackrel{\text{def}}{=} (\alpha, x) \wedge (d(x,y)-\alpha, y)$ which precedes both of them and which follows both of the points $(-\alpha, x)$ and $(\alpha - d(x,y), y)$:

$$(\alpha, x) \geq (s, z), \quad (d(x,y) - \alpha, y) \geq (s, z),$$

$$(s, z) \geq (-\alpha, x), \quad (s, z) \geq (\alpha - d(x,y), y).$$

These orders mean that the following four inequalities are satisfied:

$$\alpha - s \geq d(x,z); \quad d(x,y) - \alpha - s \geq d(y,z);$$

$$s + \alpha \geq d(x,z); \quad s + d(x,y) - \alpha \geq d(y,z).$$

Adding the first and the third inequalities and dividing the sum by two, we conclude that $\alpha \geq d(x, z)$. Similarly, by adding the second and the fourth inequalities and dividing the sum by two, we conclude that $d(x, y) - \alpha \geq d(y, z)$.

We cannot have strict inequality in any of these two inequalities, because if, e.g., $\alpha > d(x, z)$, then by adding it to $d(x, y) - \alpha \geq d(y, z)$, we could conclude that $d(x, y) > d(x, z) + d(y, z)$ – which contradicts to the triangle inequality. Thus, we must have equality, i.e., we must have $d(x, z) = \alpha$ and $d(y, z) = d(x, y) - \alpha$. The statement is proven: the point z is our desired point x_α.

In this case, from $d(x, z) = \alpha$ and $\alpha - s \geq d(x, z)$, we conclude that $s \leq 0$. Similarly, from $d(x, z) = \alpha$ and $\alpha + s \geq d(x, z)$, we conclude that $s \geq 0$. Since $s \leq 0$ and $s \geq 0$, we have $s = 0$. Thus, $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, z)$.

4°. Let us now prove that for every two points $x, y \in X$, and for every $\alpha \in (0, d(x, y))$, there exists only one point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$.

Indeed, we have already shown that one such point exists – the point x_α for which $(\alpha, x) \wedge (d(x, y) - \alpha, y) = (0, x_\alpha)$. Let us assume that for some point $x'_\alpha \neq x_\alpha$, we have $d(x, x'_\alpha) = \alpha$ and $d(x'_\alpha, y) = d(x, y) - \alpha$. Then, by definition of the Busemann product order, we have $(\alpha, x) \geq (0, x'_\alpha)$ and $(d(x, y) - \alpha, y) \geq (0, x'_\alpha)$. By the definition of a meet, we then conclude that $(0, x_\alpha) \geq (0, x'_\alpha)$. By definition of the Busemann product order, this means that $0 \geq d(x_\alpha, x'_\alpha)$, i.e., that $d(x_\alpha, x'_\alpha) = 0$ and $x_\alpha = x'_\alpha$. Uniqueness is proven.

5°. Now, we can conclude that every two points $x, y \in X$ are connected by a geodesic arc.

We have already shown that for every α, there exists a unique point x_α for which $d(x, x_\alpha) = \alpha$ and $d(x_\alpha, y) = d(x, y) - \alpha$. We want to prove that these points x_α form a geodesic arc, i.e., that for every $\alpha < \beta$, we have $d(x_\alpha, x_\beta) = \beta - \alpha$. Indeed, due to Part 1 of this proof, if we take points x_α and y with $d(x_\alpha, y) = d(x, y) - \alpha$, then there exists a point x'_β for which $d(x_\alpha, x'_\beta) = \beta - \alpha$ and

$$d(x'_\beta, y) = d(x_\alpha, y) - (\beta - \alpha) = (d(x, y) - \alpha) - (\beta - \alpha) = d(x, y) - \beta.$$

Due to the triangle inequality,

$$d(x, x'_\beta) \leq d(x, x_\alpha) + d(x_\alpha, x'_\beta) \leq \alpha + (\beta - \alpha) = \beta,$$

so $d(x, x'_\beta) \leq \beta$. We cannot have $d(x, x'_\beta) < \beta$, since then we would have

$$d(x, y) \leq d(x, x'_\beta) + d(x'_\beta, y) < \beta + (d(x, y) - \beta) < d(x, y),$$

i.e., $d(x, y) < d(x, y)$, a contradiction. Thus, we have $d(x, x'_\beta) = \beta$ and $d(x'_\beta, y) = d(x, y) - \beta$. Due to Part 2 of our proof, this means that $x'_\beta = x_\beta$. Thus, $d(x_\alpha, x'_\beta) = \beta - \alpha$ implies that $d(x_\alpha, x_\beta) = \beta - \alpha$. The statement is proven.

6°. Let us now prove that for every $x, y \in X$, for every $\alpha \in (0, d(x, y))$, and for every point $z \in X$, we have either $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$ or $d(y, z) =$

$d(y,x_\alpha)+d(x_\alpha,z)$. In other words, x_α either lies on a geodesic arc connecting x and z or on a geodesic arc connecting y and z. This would mean that X is a real tree.

Indeed, we know that $(\alpha,x)\wedge(d(x,y)-\alpha,y)=(0,x_\alpha)$. Let us find s for which $(\alpha,x)\ge(-s,z)$ and $(d(x,y)-\alpha,y)\ge(-s,z)$. The first desired relation means that $\alpha+s\ge d(x,z)$, i.e., that

$$s\ge d(x,z)-\alpha=d(x,z)-d(x,x_\alpha).$$

The second relation means that $d(x,y)-\alpha+s\ge d(y,z)$, i.e., that

$$s\ge d(y,z)-(d(x,y)-\alpha)=d(y,z)-d(y,x_\alpha).$$

So, if we take

$$s=\max(d(x,z)-d(x,x_\alpha),d(y,z)-d(y,x_\alpha)),$$

both inequalities will be satisfied and thus, we will have $(\alpha,x)\ge(-s,z)$ and $(d(x,y)-\alpha,y)\ge(-s,z)$.

By definition of the meet, this means that $(0,x_\alpha)\ge(-s,z)$, i.e., that $s\ge d(x_\alpha,z)$. The value s is defined as the largest of the two expressions, so it is equal to one of them.

If s is equal to the first expression $s=d(x,z)-d(x,x_\alpha)$, then the above inequality $s\ge d(x_\alpha,z)$ takes the form $d(x,z)-d(x,x_\alpha)\ge d(x_\alpha,z)$, i.e., equivalently, $d(x,z)\ge d(x,x_\alpha)+d(x_\alpha,z)$. Since by the triangle inequality, we have $d(x,z)\le d(x,x_\alpha)+d(x_\alpha,z)$, we thus conclude that $d(x,z)=d(x,x_\alpha)+d(x_\alpha,z)$.

If s is equal to the second expression $s=d(y,z)-d(y,x_\alpha)$, then the above inequality $s\ge d(x_\alpha,z)$ takes the form $d(y,z)-d(y,x_\alpha)\ge d(x_\alpha,z)$, i.e., equivalently, $d(y,z)\ge d(y,x_\alpha)+d(x_\alpha,z)$. Since by the triangle inequality, we have $d(y,z)\le d(y,x_\alpha)+d(x_\alpha,z)$, we thus conclude that $d(y,z)=d(y,x_\alpha)+d(x_\alpha,z)$.

The statement is proven.

7°. To complete our proof, we need to show that if X is a real tree, then the Busemann product is a lattice.

Let us assume that the metric space X is a real tree, and let us consider two points (t,x) and (s,x) in the Busemann product $\mathbb{R}\times X$. Let us show that the meet of these points exists (for the join, the proof is similar).

7.1°. If $t-s\ge d(x,y)$, then $(t,x)\ge(s,y)$, so the smallest point (s,y) is the desired meet.

7.2°. If $s-t\ge d(x,y)$, then $(s,y)\ge(t,x)$, so the smallest point (t,x) is the desired meet.

7.3°. Let us now consider the remaining case when $d(x,y)>|t-s|$. In this case, $-d(x,y)\le t-s\le d(x,y)$ hence $0\le t-s+d(x,y)\le 2d(x,y)$ and thus, $0\le\alpha\le d(x,y)$, where we denoted $\alpha\stackrel{\rm def}{=}\dfrac{t-s+d(x,y)}{2}$. We will prove that in this case, the desired meet is the element (t_0, x_α), where $t_0 \stackrel{\text{def}}{=} \frac{t+s-d(x,y)}{2}$ and x_α is a point on the geodesic arc connecting x and y for which $d(x, x_\alpha) = \alpha$.

Note that indeed $(t, x) \geq (t_0, x_\alpha)$ and $(s, y) \geq (t_0, x_\alpha)$.

We need to prove that for every q and z, if $(t, x) \geq (q, z)$ and $(s, y) \geq (q, z)$ then $(t_0, x_\alpha) \geq (q, z)$. By the property of a real tree,

- either x_α lies on a geodesic arc connecting x and z,
- or x_α lies on a geodesic arc connecting y and z.

Without losing generality, let us consider the first case, in which $d(x, z) = d(x, x_\alpha) + d(x_\alpha, z)$. We know that $(t, x) \geq (q, z)$ and $(s, y) \geq (q, z)$, i.e., that $t - q \geq d(x, z)$ and $s - q \geq d(y, z)$. We need to prove that $(t_0, x_\alpha) \geq (q, z)$, i.e., that $t_0 - q \geq d(x_\alpha, z)$. Since we are in the first case, we have $d(x_\alpha, z) = d(x, z) - d(x, x_\alpha) = d(x, z) - \alpha$. By definition of α, this means that

$$d(x_\alpha, z) = d(x, z) - \frac{t - s + d(x, y)}{2}.$$

Substituting this expression for $d(x_\alpha, z)$ and the definition of t_0 into the desired inequality $t_0 - q \geq d(x_\alpha, z)$, we get an equivalent inequality

$$\frac{t + s - d(x, y)}{2} - q \geq d(x, z) - \frac{t - s + d(x, y)}{2} = d(x, z) + \frac{s - t - d(x, y)}{2}.$$

By canceling identical terms $s/2$ and $-d(x, y)/2$ on both sides, and by moving $t/2$ into the left-hand side of this inequality, we get an equivalent inequality $t - q \geq d(x, z)$ which we assumed to be true. The statement is proven, and so is the theorem.

3 Open Questions

Case of quasimetrics. In the main text, we only considered metric spaces X, in which $d(x, y) = d(y, x)$, but a similar construction of a Busemann product order can be described for a *quasimetric*, i.e., a function which is not necessarily symmetric; see, e.g., [4]. It is desirable to extend our results to such quasimetrics.

More general Busemann products. The space $\mathbb{R} \times X$ is not just a ordered space: similarly to the case of special relativity, it can be equipped by a function describing proper time [2]:

$$\tau((t, x), (s, y)) = \sqrt[\alpha]{\max((s - t)^\alpha - d^\alpha(x, y), 0)}.$$

This function – called *kinematic metric* – satisfies the following two conditions:

- if $\tau(a, b) > 0$ then $a \geq b$, and

- the *anti-triangle inequality:* if $a \geq b \geq c$, then $\tau(a, c) \geq \tau(a, b) + \tau(b, c)$.

For each ordered space E with a function τ that satisfies these two conditions, and for each metric space X, we can define a Busemann product as the following ordering relation of $E \times X$:

$$(t, x) \geq (s, y) \Leftrightarrow \tau(t, s) \geq d(x, y).$$

It is desirable to analyze when this order is a lattice.

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References

[1] M. Bestvina, "R-trees in topology, geometry, and group theory", In: R. J. Daverman and R. B. Sher (eds.), *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, pp. 55–91.

[2] H. Busemann, *Timelike spaces*, PWN Publishers, Warszawa, 1967.

[3] V. Kreinovich, "Space-time is 'square times' more difficult to approximate than Euclidean space", *Geombinatorics*, 1996, Vol. 6, No. 1, pp. 19–29.

[4] H.-P. A. Künzi and V. Kreinovich, "Static Space-Times Naturally Lead to Quasi-Pseudometrics", *Theoretical Computer Science*, 2008, Vol. 405, No. 1–2, pp. 64–72.