Propagating Range (Uncertainty) and Continuity Information Through Computations: From Real-Valued Intervals to General Sets

Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
vladik@utep.edu

Abstract

One of the main problems of interval computations is to find an enclosure $Y \supseteq f(X_1, \ldots, X_n)$ for a range of a given function $f(x_1, \ldots, x_n)$ over given intervals $X_1, \ldots, X_n$. Most of the techniques for estimating this range are based on propagating the range through computations. Specifically, we follow the computations of $f(x_1, \ldots, x_n)$ step-by-step: we start with ranges $X_1, \ldots, X_n$ of the inputs, and then we sequentially compute the enclosures for the ranges of all intermediate results, until, on the last computation step, we get the desired enclosure Y. A similar propagation of "decorations" – information about continuity – enables us to make conclusions about the continuity of the resulting function $f(x_1, \ldots, x_n)$. In this paper, we show that the interval propagation results can be naturally extended to the general case of arbitrary sets. For this general case, we provide necessary and sufficient conditions for such a propagation.

1 Computations: From Real Values to General Sets

How to describe quantities. Usually, the values of physical quantities are described by real numbers. However, some physical quantities require a more complex description : e.g., some quantities are characterized by a vector (e.g., force or velocity), some by a function (e.g., a current value of a field) or by a geometric shape. In view of this possibility, we will assume that the set S of possible values of each quantity is not necessarily a set of real numbers, it can be a general set.

Functional dependencies are ubiquitous. In many practical situations, quantities are dependent on each other. Often, we know a function $y = f(x_1, \ldots, x_n)$ that relates quantities $x_1, \ldots, x_n$ with a quantity y. Once we know this function and we know the values of $x_1, \ldots, x_n$, we can therefore find the corresponding value of y.

Functional dependencies can be complex. In simple cases, we know an explicit relation between x_i and y. In more complex situations, the relation is more complex: instead of a *single* explicit expression of y in terms of $x_1, \ldots, x_n$, we have a *sequence* of such expressions in which we first determine some intermediate quantities z_j in terms of x_i, then other intermediate quantities z_k in terms of z_j, and finally, y in terms of the the intermediate quantities z_j (and maybe also in terms of the original quantities x_i).

We start with the values $x_1, \ldots, x_n$; for convenience, let us denote the first intermediate value by x_{n+1}, the second by x_{n+2}, etc., until we reach the desired value $y = x_{n+N}$. Thus, we arrive at the following definition.

Definition 1. *Let n and N be natural numbers, and let $S_1, \ldots, S_n$ be sets. By a* computation scheme *f of length N with n inputs, we means a sequence of tuples t_{n+j} $(j = 1, \ldots, N)$ each of which consists of:*

- *a set S_{n+j};*
- *a finite sequence of positive integers $a(j,1) < \ldots < a(j,k(j)) < n+j$; and*
- *a function $f_{n+j} : S_{a(j,1)} \times \ldots \times S_{a(j,k(j))} \to S_{n+j}$.*

For each sequence of elements $x_1 \in S_1, \ldots, x_n \in S_n$, the result *$f(x_1, \ldots, x_n)$ of applying the computation scheme f to these values is defined as x_{n+N}, where, once the values $x_1, \ldots, x_{n+j-1}$ are defined, the next value x_{n+j} is defined as $f_{n+j}(x_{a(j,1)}, \ldots, x_{a(j,k(j))})$.*

Example. The expression $f(x_1) = x_1 \cdot (1 - x_1)$ can be described by the following computation scheme: first, we compute $x_2 = 1 - x_1$, then we compute $y = x_3 = x_1 \cdot x_2$. In this case:

- $S_1 = S_2 = S_3 = \mathbb{R}$,
- on the first intermediate step, we have a function of one variable, i.e., $k(2) = 1$; here, $a(2,1) = 1$, and the corresponding function has the form $f_2(a) = 1 - a$;
- on the second computation step, we have a function of two variables $k(3) = 2$; here, $a(2,1) = 1$, $a(2,2) = 2$, and the corresponding function has the form $f_3(a,b) = a \cdot b$.

Intermediate results as functions of the input. For each N-step computation scheme and for each $i < N$, the value x_{n+j} computed on the j-th step is a function of the inputs $x_1, \ldots, x_n$. Let us denote this function by $g_{n+j}(x_1, \ldots, x_n)$; then $g_{n+N}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n)$. The function g_{n+j} appears if we "truncate" the original computation scheme on the j-th step.

The original values $x_1, \ldots, x_n$ can also be viewed as functions of the n input variables $x_1, \ldots, x_n$, namely, as *projection functions* $g_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = x_i$.

In terms of these functions, each computation step takes the form

$$x_{n+j} = g_{n+j}(x_1, \ldots, x_n) = f_{n+j}(g_{a(j,1)}(x_1, \ldots, x_n), \ldots, g_{a(j,k(j))}(x_1, \ldots, x_n)).$$

2 Propagating Range (Uncertainty) Through Computations: From Real-Valued Intervals to General Sets

Need to take uncertainty into account. In practice, we rarely know the exact values of the quantities $x_1, \ldots, x_n$. Usually, we only have partial information about these values – in the sense that we may have several different values which are consistent with the available information. For each i, let $X_i \subseteq S_i$ denote the set of possible values of x_i which are consistent with the known information about x_i.

In interval computations (see, e.g., [1]), we usually assume that the set S_i is the set of real numbers, and the set X_i is an interval; however, it is also possible that the set X_i is more general, e.g., it may be a multi-interval: a union of finitely many intervals. When S_i is a multi-dimensional Euclidean space, the set X_i can be a box, an ellipsoid, or a more general (convex or non-convex) set.

In general, different values $x_i \in X_i$ lead to different values $y = f(x_1, \ldots, x_n)$. It is therefore desirable to find the *range* of possible values, i.e., the set

$$f(X_1, \ldots, X_n) \stackrel{\text{def}}{=} \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\},$$

or at least an *enclosure* $Y \supseteq f(X_1, \ldots, X_n)$ for this range.

Propagating range through computations: main idea. Most of the techniques for estimating the desired range are based on propagating the range through computations. Specifically, we follow the computations of $f(x_1, \ldots, x_n)$ step-by-step:

- we start with ranges $X_1, \ldots, X_n$ of the inputs,
- we sequentially compute the enclosures X_{n+j} for the ranges of all intermediate results,
- finally, on the last computation step, we get the desired enclosure $Y = X_{n+N}$.

On each intermediate step j of the original computation procedure, we apply a function f_{n+j} to the results $g_{a(j,k)}(x_1, \ldots, x_n)$ of the previous steps. For the above procedure to work, we need to make sure that on each such intermediate step, we have a procedure that transforms enclosures for the ranges $g_{a(j,k)}(X_1, \ldots, X_n)$ into an enclosure for the range of the result. Let us formulate the desired property in precise terms.

Sets of a certain type. In practice, not all sets correspond to uncertainty. Let us consider a class $\mathcal{C}$ of sets – e.g., intervals, boxes, ellipsoids, multi-intervals (finite unions of boxes), etc., and let us only consider sets $X_1, \ldots, X_n$ from this class.

Definition 2. *Let $\mathcal{C}$ be a class of sets. Sets from the class $\mathcal{C}$ will be called* $\mathcal{C}$-sets. *For each set X, the class of all subsets of X which are $\mathcal{C}$-sets will be denoted by $2^X_{\mathcal{C}}$.*

Definition 3. *Let $g : T_1 \times \ldots \times T_m \to Y$ be a function. We say that a mapping $G : 2^{T_1}_{\mathcal{C}} \times \ldots \times 2^{T_m}_{\mathcal{C}} \to 2^Y_{\mathcal{C}}$ is a* $\mathcal{C}$-set extension *of the function g if for every set Z and for every sequence of functions $h_1 : Z \to T_1, \ldots, h_m : Z \to T_m$, if sets $X_1, \ldots, X_m$ are enclosures for $h_1(Z), \ldots, h_m(Z)$, then $G(X_1, \ldots, X_m)$ is an enclosure for the range $h(Z)$ of the function $h(z) \stackrel{\text{def}}{=} g(h_1(z), \ldots, h_m(z))$.*

Discussion. In other words, if $X_1 \supseteq h_1(Z), \ldots, X_m \supseteq h_m(Z)$, then

$$G(X_1, \ldots, X_m) \supseteq h(Z).$$

Straightforward set computations: propagating uncertainty via computations. We can now use the idea explored in interval computations. Namely, for each computation scheme f and for all inputs sets $X_1, \ldots, X_n$, once we know set enclosures F_{n+j} for all the functions f_{n+j}, we can replace each computation $f_{n+j}(x_{a(j,1)}, \ldots, x_{a(j,k(j))})$ with elements x_k by the corresponding computation with sets. As a result, we get the desired enclosure for the range $f(X_1, \ldots, X_n)$.

Definition 4. *Let f be a computation scheme and let $X_1, \ldots, X_n$ be $\mathcal{C}$-sets. For each j, let F_{n+j} be a $\mathcal{C}$-set extension of the corresponding function f_{n+j}. We can then sequentially define the sets $X_{n+1}, \ldots, X_{n+N}$ as follows: once the sets $X_1, \ldots, X_{n+j-1}$ are defined, the next set X_{n+j} is defined as $X_{n+j} = F_{n+j}(X_{a(j,1)}, \ldots, X_{a(j,k(j))})$. The last set X_{n+N} is called the* result of propagating uncertainty via the computation scheme *and denoted by $F(X_1, \ldots, X_n)$.*

Proposition 1. *The result $F(X_1, \ldots, X_n)$ of propagating uncertainty via the computation scheme is an enclosure for the range $f(X_1, \ldots, X_n)$:*

$$F(X_1, \ldots, X_n) \supseteq f(X_1, \ldots, X_n).$$

Proof. The proof is by induction over j. For $j = 1, \ldots, n$, the sets X_j actually coincide with the corresponding ranges. Once we prove the enclosure result for all the steps $1, \ldots, j-1$, the enclosure for j directly follows from Definition 3.

Necessary and sufficient conditions for range propagation. Our description of a $\mathcal{C}$-enclosure (Definition 3) involves quantifiers over all possible functions; checking that some property holds for all possible functions may be difficult. It is therefore desirable to come up with a simpler equivalent definition. This equivalent definition is provided by the following result.

Proposition 2. *A mapping $G : 2^{T_1}_{\mathcal{C}} \times \ldots \times 2^{T_m}_{\mathcal{C}} \to 2^{Y}_{\mathcal{C}}$ is a $\mathcal{C}$-set extension of a function $g : T_1 \times \ldots \times T_m \to Y$ if and only if for every every sequence of $\mathcal{C}$-sets $X_i \subseteq S_i$ and for each sequence of elements $x_i \in X_i$, the value $g(x_1, \ldots, x_m)$ is contained in $G(X_1, \ldots, X_m)$.*

Proof. Let us first prove that if the containment property is satisfied, then G is a $\mathcal{C}$-set extension. Let us assume that the sets $X_1, \ldots, X_m$ are enclosures for the ranges $h_1(Z), \ldots, h_m(Z)$. We want to prove that in this case, the set $G(X_1, \ldots, X_m)$ is an enclosure for the range $h(Z)$ of the function $h(z) \stackrel{\text{def}}{=} g(h_1(z), \ldots, h_m(z))$, i.e., that for every $z \in Z$, we have $h(z) = g(h_1(z), \ldots, h_m(z)) \in G(X_1, \ldots, X_n)$.

Indeed, since each set X_i is an enclosure for the range $h_i(Z)$, we have $h_i(z) \in h_i(Z) \subseteq X_i$ and thus, $h_i(z) \in X_i$. Since $x_i \stackrel{\text{def}}{=} h_i(z) \in X_i$ for all i, the containment property implies that $g(x_1, \ldots, x_m) \in G(X_1, \ldots, X_n)$. Since $x_i = h_i(z)$, this means that $g(h_1(z), \ldots, h_m(z)) \in G(X_1, \ldots, X_n)$ – exactly what we wanted to prove. The implication is proven.

Vice versa, let us prove that every $\mathcal{C}$-set extension has a containment property. Indeed, let X_i be $\mathcal{C}$-sets and let $x_i \in X_i$. In this case, we need to prove that $g(x_1, \ldots, x_m) \in G(X_1, \ldots, X_m)$. To prove this containment, let us take $Z = X_1 \times \ldots \times X_m$ and $h_i(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) = x_i$. In this case, $h_i(Z) = X_i$, so X_i is an enclosure for $h_i(z)$ and thus, since G is a $\mathcal{C}$-set extension, we conclude that $G(X_1, \ldots, X_n)$ is an enclosure for the range $h(Z)$ of the function $h(z) = g(h_1(z), \ldots, h_m(z))$. Enclosure means that $h(z) = g(h_1(z), \ldots, h_m(z)) \in G(X_1, \ldots, X_n)$ for all z. Substituting $h_i(z) = x_i$ into this formula, we conclude that $h(x_1, \ldots, x_n) \in G(X_1, \ldots, X_n)$. The statement is proven.

Corollary. *Let a class $\mathcal{C}$ be such that for all $\mathcal{C}$-sets $X_1, \ldots, X_m$, the range $g(X_1, \ldots, X_m)$ is also a $\mathcal{C}$-set. In this case, the function G generating such a range is the narrowest of all possible $\mathcal{C}$-set extensions.*

Discussion. In other words, if we want to get the narrowest possible enclosures for the range, we should use the range $G(X_1, \ldots, X_m) = g(X_1, \ldots, X_m)$. The condition that the range is always a $\mathcal{C}$-set is satisfied, e.g., if we consider continuous functions over intervals: the range of every such function is also an interval.

3 Propagating Continuity Information Through Computations

Importance of continuity information. In some cases, it is important to check whether a function $f(x_1, \ldots, x_n)$ is continuous. For example, it is useful to determine when the system of equations has a solution: when each range S_i is an interval, then Brouwer's fixed point theorem says that if f is a continuous function and $f(S_1 \times \ldots \times S_n) \subseteq S_1 \times \ldots \times S_n$, then there exists a point $x = (x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$ for which $x = f(x)$; see, e.g., [1].

In other cases, it may be beneficial to know that a function is not continuous. For example, in physical applications, discontinuity may be an indication of a phase transition. Such a discontinuity is not always easy to detect by simply looking at the formulas, since the formulas used in computations may use discontinuous functions like the sign $\text{sign}(x)$, $\arctan(x)$, etc., and still lead to a continuous expression.

Comment. In the following text, we will consider non-trivial Hausdorff spaces, i.e., topological spaces that have at least two different points and in which every two points can be separated by open neighborhoods.

Continuity information can also be propagated. The possibility to propagate continuity information follows from the fact that a composition of continuous functions is always continuous; see, e.g., [2].

For such a propagation, on each intermediate step j, we need to keep not only the enclosure X_j for the corresponding function $g_{n+j}(x_1, \ldots, x_n)$, but also an information re whether this intermediate function is continuous or not. For each function $g : T_1 \times \ldots \times T_m \to Y$, let a value c mean that f is *c*ontinuous, and d means that f is *d*iscontinuous. The corresponding variable will be called the *continuity* of the given function. The set of all continuity values will be denoted by $C^* = \{c, d\}$.

For some functions, we know whether they are continuous or not; for other functions, we do not have this information, so a function can be either continuous or discontinuous. In general, our information about the function's continuity can be described by a non-empty set $C \subseteq C^*$ of values c or d which are consistent with our knowledge. For each function g and each sequence of sets $T_1, \ldots, T_m$, there are only three options:

- the first option is that we know that the function is continuous; in this case, $C = \{c\}$;
- the second option is that we know that the function is discontinuous; in this case, $C = \{d\}$;
- the third option is that we do not know whether the function is continuous or not; in this case, $C = \{c, d\}$.

The class consisting of these three non-empty sets is $\{\{c\}, \{d\}, \{c, d\}\}$.

The original projection functions $h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n)$ are continuous, i.e., $C_1 = \ldots = C_n = \{c\}$. On each intermediate step j, in addition to computing the range X_{n+j}, we should also compute the corresponding set C_{n+j}. Finally, on the last computation step, we get the set C_{n+N}. If this set consists of only c, we conclude that the function f is continuous; if the set C_{n+N} consists of only d, we conclude that the function f is discontinuous.

For the above procedure to work, we need to make sure that on each such intermediate step j, we have a procedure that transforms enclosures and continuity information of all relevant previous intermediate functions into a continuity information for the current function g_{n+j}. Let us formulate the desired property in precise terms.

Definition 5. *Let $T_1, \ldots, T_m, Y$ be topological spaces, and let*

$$g : T_1 \times \ldots \times T_m \to Y$$

be a function. We say that a mapping

$$p : 2^{T_1}_{\mathcal{C}} \times C^* \times \ldots \times 2^{T_m}_{\mathcal{C}} \times C^* \to \{\{c\}, \{d\}, \{c, d\}\}$$

is a continuity propagator *corresponding to g if for every topological space Z and for every sequence of functions $h_1 : Z \to T_1, \ldots, h_m : Z \to T_m$, once sets $X_1, \ldots, X_m$ are enclosures for $h_1(Z), \ldots, h_m(Z)$, and c_i are continuities of the functions h_i, then the continuity c_h of the function $h(z) \stackrel{\text{def}}{=} g(h_1(z), \ldots, h_m(z))$ is contained in the set $p(X_1, c_1, \ldots, X_m, c_m)$.*

Discussion. In other words, if for every i, we have $X_i \supseteq h_i(Z)$, then $c_h \in p(X_1, c_1, \ldots, X_m, c_m)$. If we do not know the continuity of some of the inputs, then we have to consider all possible values of this continuity. In other words, if we only know the sets C_i that contain the actual (unknown) values c_i, then we can conclude that $c_h \in p(X_1, C_1, \ldots, X_m, C_m)$, where we denoted

$$p(X_1, C_1, \ldots, X_m, C_m) \stackrel{\text{def}}{=} \bigcup_{c_i \in C_i} p(X_1, c_1, \ldots, X_m, c_m),$$

where the union is taken over all possible combinations $c_i \in C_i$.

Propagating continuity information via computations. We can now use the same idea as for finding the enclosure for the range. Namely, for each computation scheme f and for all inputs sets $X_1, \ldots, X_n$, once we know set enclosures F_{n+j} for all the functions f_{n+j}, we can replace each computation $f_{n+j}(x_{a(j,1)}, \ldots, x_{a(j,k(j))})$ with elements x_k by the corresponding computation with sets – and simultaneously compute the set C_{n+j}. As a result, we get not only the desired enclosure for the range $f(X_1, \ldots, X_n)$, we also get the continuity information about the function $f(x_1, \ldots, x_n)$.

Definition 6. *Let f be a computation scheme and let $X_1, \ldots, X_n$ be topological spaces which are $\mathcal{C}$-sets. For each j, let F_{n+j} be a $\mathcal{C}$-set extension of the corresponding function f_{n+j}, and let p_{n+j} be a continuity propagator for this function. Let us define $C_1 = \ldots = C_n = \{c\}$. We can then sequentially define the sets $X_{n+1}, C_{n+1}, \ldots, X_{n+N}, C_{n+N}$ as follows: once the sets $X_1, C_1, \ldots, X_{n+j-1}, C_{n+j-1}$ are defined, the next set X_{n+j} is defined as $X_{n+j} = F_{n+j}(X_{a(j,1)}, \ldots, X_{a(j,k(j))})$, and the next set C_{n+j} as $p_{n+j}(X_{a(j,1)}, C_{a(j,1)}, \ldots, X_{a(j,k(j))}, C_{a(j,k(j))})$. The last set C_{n+N} is called the* result of propagating continuity information via the computation scheme *and denoted by C_f.*

Proposition 3. *For each computation scheme, the continuity c_f of the corresponding function is contained in the result C_f of propagating continuity information via the computation scheme: $c_f \in C_f$.*

Comment. In other words, if $C_f = \{c\}$, this means that the function $f(x_1, \ldots, x_n)$ is continuous. Similarly, if $C_f = \{d\}$, this means that the function $f(x_1, \ldots, x_n)$ is discontinuous.

Proof. The proof is by induction over j. For $j = 1, \ldots, n$, the sets X_j actually coincide with the corresponding ranges. Once we prove the continuity result for all the steps $1, \ldots, j-1$, the continuity result for j directly follows from Definition 5.

Necessary and sufficient conditions for range propagation. Our description of a continuity propagator (Definition 5) involves quantifiers over all possible functions; checking that some property holds for all possible functions may be difficult. It is therefore desirable to come up with a simpler equivalent definition. This equivalent definition is provided by the following result.

Definition 7. *For a function $g : X_1 \times \ldots \times X_m \to Y$, we say that the i-th variable is a* dummy *variable if the function does not depend on this variable, i.e., that for all possible values $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}, x_i, x'_i \in X_i, x_{i+1} \in X_{i+1}, \ldots, x_m \in X_m$, we have*

$$g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) = g(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_m).$$

Examples. For a constant function, all inputs are dummy variables. For a function $g(x_1, x_2, x_3) = x_1^2 + x_2$, the variable x_3 is a dummy variable.

Definition 8. *For a function $g : X_1 \times \ldots \times X_m \to Y$, the i-th variable is called* essential *if it is not a dummy variable.*

Definition 9. *We say that a function $g(x_1, \ldots, x_m)$ is* continuously reversible *from variables $x_{i_1}, \ldots, x_{i_k}$ to a variable x_j if:*

- *given the value of $y = f(x_1, \ldots, x_n)$ and the values of these variables $x_{i_1}, \ldots$, we can uniquely reconstruct the value of x_j, and*
- *the corresponding dependence $x_j = H(y, x_{i_1}, \ldots, x_{i_k})$ is continuous.*

Example. The function $f(x_1, x_2) = x_1 + x_2$ is continuously reversible with respect to each of the variables: e.g., if we know $y = x_1 + x_2$ and we know x_1, we can uniquely reconstruct x_2 as $y - x_1$, and the corresponding dependence $x_2 = y - x_1$ is a continuous function of the variables y and x_1.

Proposition 4. *A mapping*

$$p : 2_{\mathcal{C}}^{T_1} \times C^* \times \ldots \times 2_{\mathcal{C}}^{T_m} \times C^* \to \{\{c\}, \{d\}, \{c, d\}\}$$

corresponding to the function $g : T_1 \times \ldots \times T_m \to Y$ is a continuity propagator if and only if it satisfies the following three properties for all $\mathcal{C}$-sets $X_i \subseteq T_i$ and for all values $c_i \in C^*$:

- *if the function $g : X_1 \times \ldots \times X_m \to Y$ is continuous, then*

$$c \in p(X_1, c, \ldots, X_m, c);$$

- *if the function g is continuously reversible from all the variables for which $c_i = c$ to one of the variables for which $c_j = d$, then*

$$d \in p(X_1, c_1, \ldots, X_m, c_m);$$

- *in all other cases, $p(X_1, c_1, \ldots, X_m, c_m) = \{c, d\}$.*

Comment. So, if we want to get the narrowest possible enclosures for the actual continuity, we should take:

- $p(X_1, c, \ldots, X_m, c) = \{c\}$ if the function $g : X_1 \times \ldots \times X_m \to Y$ is continuous;
- $p(X_1, c_1, \ldots, X_m, c_m) = \{d\}$ if the function g is continuously reversible from all the variables for which $c_i = c$ to one of the variables for which $c_j = d$; and
- $p(X_1, c_1, \ldots, X_m, c_m) = \{c, d\}$ in all other cases.

Proof.

1°. Let us first prove that if the mapping g satisfies the above three properties, then it is a continuity propagator. According to the definition of a continuity propagator, this means that if we have m functions $h_i : Z \to T_i$ with continuities c_i and m enclosures $X_i \supseteq h_i(Z)$, then the continuity c_h of the function $h(z) = g(h_1(z), \ldots, h_m(z))$ is contained in the set $p(X_1, c_1, \ldots, X_m, c_m)$.

1.1°. In the first case, $c_i = c$ means that all functions $h_i(z)$ are continuous. Since the function $g(x_1, \ldots, x_n)$ is also continuous, this means that their composition $h(z) = g(h_1(z), \ldots, h_m(z))$ is continuous as well, i.e., $c_h = c$. Thus, $c \in p(X_1, c_1, \ldots, X_m, c_m)$ implies that $c_h \in p(X_1, c_1, \ldots, X_m, c_m)$.

1.2°. In the second case, the functions $h_{i_1}, \ldots, h_{i_k}$ are all continuous, while the function h_j is discontinuous. We assume that $h_j(z)$ is a continuous function of the value $h(z)$ and the values $h_{i_1}, \ldots, h_{i_k}$. Thus, $h(z)$ cannot be continuous – otherwise, $h_j(z)$ would also be continuous, as a composition of continuous functions. Hence, the function $h(z)$ is discontinuous: $c_h = d$. So, in this case, $d \in p(X_1, c_1, \ldots, X_m, c_m)$ implies that $c_h \in p(X_1, c_1, \ldots, X_m, c_m)$.

1.3°. In the third case, when $p(X_1, c_1, \ldots, X_m, c_m) = \{c, d\}$, the property $c_h \in p(X_1, c_1, \ldots, X_m, c_m)$ automatically holds.

2°. Let us now prove that if the mapping g is a continuity propagator, then it must satisfy the above three properties. In other words:

- unless we are in the first case, the result p of the mapping must contain d, and
- unless we are in the second case, the result of the mapping must contain c.

2.1°. Let us first consider the case when the condition of the first property is not satisfied, i.e., when:

- either the function g is discontinuous,
- or it is continuous but $c_i = d$ for some essential variable x_i.

Let us show that in both subcases, it is possible to have a situation in which the resulting function $h(z)$ is discontinuous.

If the function $g(x_1, \ldots, x_n)$ itself is discontinuous, then we take $Z = X_1 \times \ldots \times X_m$ and projections $h_i(x_1, \ldots, x_i, \ldots, x_m) = x_i$. Then, the resulting function h is simply the original discontinuous function $g(x_1, \ldots, x_m)$.

Let us now assume that we have a sequence $c_1, \ldots, c_n$ for which $c_i = d$ for some essential variable x_i. By definition of an essential variable, there are values $x_1, \ldots, x_{i-1}, x_i, x'_i, x_{i+1}, \ldots, x_m$ for which

$$g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) \neq g(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_m).$$

Let us now take $Z = [0, 2]$ with the usual topology, and for values $z \in [0, 1]$, define:

- $h_j(z) = x_j$ for all $j \neq i$ and
- $h_i(z) = x_i$ for $z \in [0, 0.5]$ and $h_i(z) = x'_i$ for $x \in (0.5, 1]$.

On the remaining part of the interval $[0, 2]$, we take:

- $h_j(z) = x_j$ when $c_j = c$ and
- $h_j(z) = \text{const} \neq x_j$ when $c_j = d$.

In this case, all the functions h_j have the desired continuity, while their composition $h(z)$ jumps from $g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ to $g(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_m) \neq g(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m)$ when z is close to 0.5, so the function $h(z)$ is discontinuous.

2.2°. Let us now consider the case when the condition of the second property is not satisfied, i.e., if for some sequence c_i, the function g is not continuously reversible from all the variables for which $c_i = c$ to each of the variable x_j for which $c_j = d$. Let us show that in this case, it is possible that the function $h(z)$ is continuous.

Without losing generality, we can assume that the "continuous" variables (for which $c_i = c$) are numbered first, i.e., that these variables are $x_1, \ldots, x_k$, and that, therefore, the "discontinuous" variables (for which $c_i = d$) are $x_{k+1}, \ldots, x_m$. In this case, as Z, we will get a disjoint union of components $Z_{k+1}, \ldots, Z_m$ corresponding to all discontinuous variables. For each discontinuous variable x_j, the function g is not continuously reversible from the continuous variables $x_1, \ldots, x_k$ to this variable x_j. By definition of continuous reversibility, this means:

- either than we cannot always uniquely reconstruct the value x_j based on the values of all continuous variables $x_1, \ldots, x_k$ and the value $y = g(x_1, \ldots, x_m)$,
- or that we can uniquely reconstruct x_j, but the corresponding reconstruction mapping is not continuous.

Let us consider these two possibilities one by one.

The first possibility – non-uniqueness – means that there exist values $x_1, \ldots, x_k$ and values $x_{k+1}, \ldots, x_m$ and $x'_{k+1}, \ldots, x'_m$ for which $x_j \neq x'_j$ but $g(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) = g(x_1, \ldots, x_k, x'_{k+1}, \ldots, x'_m)$. We therefore take $Z_j = [2j, 2j + 1]$, and define:

- $h_i(z) = x_i$ for $i \leq k$ and
- for $i > k$, we take $h_i(z) = x_i$ for $x \leq 2j + 0.5$ and $h_i(z) = x'_i$ for $x > 2j + 0.5$.

For this choice, the functions $h_1(z), \ldots, h_k(z)$ are constant and hence continuous, the function $h_j(z)$ is discontinuous, but the composition $h(z)$ is constant and hence continuous on Z_j.

The second possibility – uniqueness but discontinuity – means that the corresponding mapping $H(y, x_1, \ldots, x_k)$ is discontinuous. In this case, we take

$$Z = \{(g(x_1, \ldots, x_m), x_1, \ldots, x_k) \,|\, x_i \in X_i\} \subseteq Y \times X_1 \ldots \times X_k,$$

with $h_i(z) = x_i$ (continuous projections) for $i \le k$, $h_j(z) = H(y, x_1, \ldots, x_k)$, and $h_i(z)$ appropriate values for all other i – values for which $g(x_1, \ldots, x_m) = y$. Here, h_j is discontinuous, but the composition $h(z)$ is simply equal to y and is, therefore, continuous.

Once we combine these pieces Z_i into a single disjoint set Z, we conclude that each of the functions $h_1(z), \ldots, h_k(z), h(z)$ is continuous on each of the disjoint pieces and thus, continuous on their union. On the other hand, each of the functions $h_i(z)$ $(i = k+1, \ldots, m)$ is discontinuous on at least one piece – namely, on the corresponding piece Z_i, and is, therefore, not continuous on the union. So, we have the desired case when $c_1 = \ldots = c_k = c$, $c_{k+1} = \ldots = c_m = d$ and $c_h = d$. The proposition is proven.

Discussion. In other words, on each computation step j, if we want to make the most informative conclusion about the continuity of the function

$$x_{n+j} = g_{n+j}(x_1, \ldots, x_n) = f_{n+j}(g_{a(j,1)}(x_1, \ldots, x_n), \ldots, g_{a(j,k(j))}(x_1, \ldots, x_n)),$$

then we should do the following:

- if the function f_{n+j} is known to be continuous on the corresponding range, and all the functions $g_{a(j,k)}(x_1, \ldots, x_n)$ corresponding to its essential (non-dummy) variables are known to be continuous, we conclude that the resulting composition function $g_{n+j}(x_1, \ldots, x_n)$ is also continuous;
- if the function is continuously reversible from the set of all continuous variables to one of the discontinuous variables, then the resulting composition function $g_{n+j}(x_1, \ldots, x_n)$ is discontinuous;
- in all other cases, we report the set $C_{n+j} = \{c, d\}$ meaning that we cannot tell whether the corresponding function $g_{n+j}(x_1, \ldots, x_n)$ is continuous or not.

Comment. The fact that the composition of continuous functions is continuous is well known. What we show is that in all other situations – with the exception of continuously reversible functions – no conclusion can be made because in principle, the resulting composition can be both continuous and discontinuous.

4 What If We Are Only Interested in Detecting Continuity?

In many practical situations, we are only interested in knowing whether continuity can be confirmed or not; in such situations, when the continuity cannot be confirmed, we are not interested in spending time on confirming discontinuity. In terms of our symbols c and d, this means that we are interested only in two cases:

- when the continuity is confirmed, i.e., when $C = \{c\}$; and
- when the continuity has not been confirmed – but could still be, in which case $C = \{c, d\}$.

In terms of Definition 5, this means that we are interested in continuity propagators whose possible values are $\{c\}$ or $\{c, d\}$. In such situations, from the above Proposition 4, we can deduce the following simplified version:

Proposition 5. *A mapping*

$$p : 2_{\mathcal{C}}^{T_1} \times C^* \times \ldots \times 2_{\mathcal{C}}^{T_m} \times C^* \to \{\{c\}, \{c, d\}\}$$

corresponding to the function $g : T_1 \times \ldots \times T_m \to Y$ is a continuity propagator if and only if it satisfies the following three properties for all $\mathcal{C}$-sets $X_i \subseteq T_i$ and for all values $c_i \in C^*$:

- *if the function $g : X_1 \times \ldots \times X_m \to Y$ is continuous, then*

$$c \in p(X_1, c, \ldots, X_m, c);$$

- *in all other cases, $p(X_1, c_1, \ldots, X_m, c_m) = \{c, d\}$.*

Comment 1. So, if we want to get the narrowest possible $\{c\}, \{c, d\}$-valued enclosures for the actual continuity, we should take:

- $p(X_1, c, \ldots, X_m, c) = \{c\}$ if the function $g : X_1 \times \ldots \times X_m \to Y$ is continuous; and
- $p(X_1, c_1, \ldots, X_m, c_m) = \{c, d\}$ in all other cases.

Comment 2. In other words, in such situations, on each computation step j, if we want to make the most informative conclusion about the continuity of the function

$$x_{n+j} = g_{n+j}(x_1, \ldots, x_n) = f_{n+j}(g_{a(j,1)}(x_1, \ldots, x_n), \ldots, g_{a(j,k(j))}(x_1, \ldots, x_n)),$$

then we should do the following:

- if the function f_{n+j} is known to be continuous on the corresponding range, and all the functions $g_{a(j,k)}(x_1, \ldots, x_n)$ corresponding to its essential (non-dummy) variables are known to be continuous, we conclude that the resulting composition function $g_{n+j}(x_1, \ldots, x_n)$ is also continuous;
- in all other cases, we report the set $C_{n+j} = \{c, d\}$ meaning that we cannot tell whether the corresponding function $g_{n+j}(x_1, \ldots, x_n)$ is continuous or not.

Acknowledgments

This work was supported in part by the National Science Foundation grants HRD-0734825 and DUE-0926721, by Grant 1 T36 GM078000-01 from the National Institutes of Health. The author is helpful to all the participants of the 2011 Dagstuhl Seminar on Uncertainty Modeling and Analysis with Intervals: Foundations, Tools, Applications, especially to Baker Kearfott, Arnold Neumaier, John Pryce, Nathalie Revol, and Jürgen Wolff von Gudenberg, for valuable discussions.

References

[1] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylviania, 2009.

[2] J. Pryce, *Motion to Propose a "Discontinuous" Decoration Bit. Version 3*, Motion 22, IEEE Interval Standard Working Group P1788, 2010.