Use of Grothendieck’s Inequality in Interval Computations: Quadratic Terms are Estimated Accurately Modulo a Constant Factor

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In interval computations, one of the most widely used methods of efficiently computing an enclosure \( Y \) the range \( y = f(x_1, \ldots, x_n) \) of a given function \( f(x_1, \ldots, x_n) \) on a given box \( x = x_1 \times \ldots \times x_n \) is the Mean Value (MV) method:
\[
Y = f(\tilde{x}_1, \ldots, \tilde{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(\tilde{x}) \cdot [-\Delta_i, \Delta_i],
\]
where \( \tilde{x}_i \) is a midpoint of the \( i \)-th interval, \( \Delta_i \) is its radius, and the ranges of the derivatives \( f_i \triangleq \frac{\partial f}{\partial x_i} \) can be estimated, e.g., by using straightforward interval computations; see, e.g., [5].

This method has excess width \( O(\Delta^2) \), where \( \Delta \triangleq \max \Delta_i \).

Can we come up with more accurate enclosures? We cannot get too drastic an improvement, since even for quadratic functions \( f(x_1, \ldots, x_n) \), computing the interval range is NP-hard (see, e.g., [4,7]) – and therefore (unless P=NP), a feasible algorithm with excess width \( O(\Delta^2 + \epsilon) \) is impossible. What we can do is try to decrease the overestimation of the quadratic term. It turns out that such a possibility follows from an inequality proven by A. Grothendieck in 1953 [2].

Specifically, the MV method is based on the 1st order Mean Value Theorem (MVT):
\[
f(\tilde{x} + \Delta x) = f(\tilde{x}) + \sum f_i(\tilde{x} + \eta) \cdot \Delta x_i \quad \text{for some } \eta_i \in [-\Delta_i, \Delta_i] \quad [3].
\]

Instead, we propose to estimate the range by adding estimates for ranges of linear, quadratic, and cubic terms in the 3rd order MVT:
\[
f(\tilde{x} + \Delta x) = f(\tilde{x}) + \sum f_i(\tilde{x}) \cdot \Delta x_i + \sum f_{ij}(\tilde{x}) \cdot \Delta x_i \cdot \Delta x_j + \sum f_{ijk}(\tilde{x} + \eta) \cdot \Delta x_i \cdot \Delta x_j \cdot \Delta x_k.
\]

The range of the cubic term is estimated via straightforward interval computations; the resulting estimate is of order \( O(\Delta^3) \). The range of the linear term can be explicitly described as \([\tilde{y} - \Delta, \tilde{y} + \Delta]\), where \( \tilde{y} \triangleq f(\tilde{x}) \) and \( \Delta = \sum |f_i(\tilde{x})| \cdot \Delta_i \).

So, the remaining problem is: how accurately can we find the range \([-Q, Q]\)
of the quadratic term $\sum_{i,j=1}^{n} a_{ij} \cdot \Delta x_i \cdot \Delta x_j$ (where $a_{ij} \overset{\text{def}}{=} f_{ij}(\bar{x})$), on the box $[-\Delta_1, \Delta_1] \times \ldots \times [-\Delta_n, \Delta_n]$.

By re-scaling, we conclude that $Q$ is equal to the maximum of the function $B(z) \overset{\text{def}}{=} \sum_{i,j=1}^{n} b_{ij} \cdot z_i \cdot z_j$ (where $b_{ij} \overset{\text{def}}{=} a_{ij} \cdot \Delta_i \cdot \Delta_j$), over values $z_i \in [-1, 1]$.

Grothendieck’s inequality enables us to estimate the maximum $Q'$ of the related bilinear function $b(z,t) \overset{\text{def}}{=} \sum_{i,j=1}^{n} b_{ij} \cdot z_i \cdot t_j$ when $z_i, t_j \in \{-1, 1\}$: namely, we can feasibly compute $Q''$ for which $K_G^{-1} \cdot Q'' \leq Q' \leq Q''$, where $K_G \in [1, 1.782]$ (see, e.g., [1,6]). One can easily see that $Q'$ is equal to the maximum of $b(z,t)$ when $z_i, t_j \in [-1, 1]$. Since $B(z) = b(z,z)$, we have, $Q \leq Q'$; on the other hand, since $b(z,t) = B((z + t)/2) - B((z - t)/2)$, we have $Q' \leq 2Q$. Thus, $Q'/2 \leq Q \leq Q'$ and so, $\frac{Q''}{2K_G} \leq Q \leq Q''$.

Hence, by computing $Q''$, we can feasibly estimate the quadratic term $Q$ accurately modulo a small constant factor $2K_G \leq 3.6$.

References:


