

Locating Local Extrema under Interval Uncertainty: Multi-D Case

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Abstract—In many practical situations, we need to locate local maxima and/or local minima of a function which is only known with interval uncertainty. For example, in radioastronomy, components of a radiosource are usually identified by locations at which the observed brightness reaches a local maximum. In clustering, different clusters are usually identified with local maxima of the probability density function (describing the relative frequency of different combinations of values). In the 1-D case, a feasible (polynomial-time) algorithm is known for locating local extrema under interval (and fuzzy) uncertainty. In this paper, we extend this result to the general multi-dimensional case.

I. INTRODUCTION

The problem of locating local extrema is important. In *spectral analysis*, chemical species are identified by locating local maxima of the spectra.

In *radioastronomy*, sources of celestial radio emission and their subcomponents are identified by locating local maxima of the measured brightness of the radio sky. In other words, we are interested in the local maxima of the *brightness distribution*, i.e., of the function $f(t)$ that describes how the intensity y of the signal depends on the position t of the point from which we receive this signal.

Elementary particles are identified by locating local maxima of the experimental curves that describe (crudely speaking) the scattering intensity y as a function of energy t .

In *clustering*, different clusters correspond to a multi-modal distribution, so clusters correspond to local maxima of the probability density function.

In each of these applications, the following problem arises:

- we know that a physical quantity y is a function of one or several ($m \geq 1$) other physical quantities t_1, \dots, t_m , $y = f(t_1, \dots, t_m)$;
- we have n situations, $i = 1, \dots, n$, in each of which we know the values of all m quantities: $v_i = (t_{i1}, \dots, t_{im})$;
- in each of these n situations, we have measured the values $y_1 = f(v_1), \dots, y_n = f(v_n)$ of the quantity y ;
- based on this information, we want to locate the local maxima and/or the local minima of the function f .

Need to take into account interval uncertainty. The observed values $y_i = f(v_i)$ come from measurements, and measurements are never absolutely accurate. The measurement

results \tilde{y}_i are, in general, different from the actual (unknown) values y_i of the corresponding quantity.

In some cases, we know the probabilities of different values of the measurement error $\Delta y_i \stackrel{\text{def}}{=} \tilde{y}_i - y_i$. In many practical cases, however, we only know the upper bound $\varepsilon > 0$ on the (absolute value of) this measurement error: $|\Delta y_i| < \varepsilon$; see, e.g., [12]. In such situations, the only information that we have about the actual (unknown) value y_i of the corresponding quantity is that it belongs to the interval $(\tilde{y}_i - \varepsilon, \tilde{y}_i + \varepsilon)$. We thus need to locate the local maxima and local minima of a function under such interval uncertainty.

Need for guaranteed results. Due to measurement uncertainty, the actual observed values fluctuate, and the function corresponding to the actual measurement results usually has many local maxima and minima. Most of these local maxima and minima are caused by the measurement errors and do not have any physical significance. From the practical viewpoint, we only want to keep those local maxima and minima which reflect the local extrema of the actual dependence, i.e., which are guaranteed to correspond to source components, chemical substances, etc.

Case of fuzzy uncertainty. Often, in addition (or instead) the guaranteed bound ε for the measurement error Δy_i , an expert can provide bounds that contain Δy_i with a certain degree of confidence. Usually, we know several such bounding intervals corresponding to different degrees of confidence. Such a nested family of intervals is also called a *fuzzy set*, because it turns out to be equivalent to a more traditional definition of fuzzy set; see, e.g., [3], [6], [7], [8] (if a traditional fuzzy set is given, then different intervals from the nested family can be viewed as α -cuts corresponding to different levels of uncertainty α).

From the algorithmic viewpoint, the case of fuzzy uncertainty can be reduced to the case of interval uncertainty.

In the case of fuzzy uncertainty, for each degree of confidence α , we must solve the problem corresponding to the α -cut intervals. Thus technically, the fuzzy problem can be reduced to several interval ones. Because of this reduction, we will be concentrating on the algorithms for solving the interval problem.

Locating local extrema under interval uncertainty: what is known. For the case of $m = 1$, when there is only one input variable t_1 , there exist feasible (polynomial-time) algorithms for locating local extrema under interval uncertainty; see, e.g., [14], [15], [16]; see also [2], [4], [5], [9], [10], [11], [13].

Need for considering the general multi-D case. In many practical applications, we need to solve a similar problem in a situation when we have several inputs t_1, \dots, t_m , $m > 1$. For example, in locating components of a radioastronomical source, we start with a 2-D function that describes the intensity as a function of two spatial coordinates. In clustering, we also need to consider local maxima of functions of several variables, etc.

What we do in this paper. In this paper, we describe a polynomial-time algorithm that solves the problem of locating local extrema of functions of several variables.

II. DEFINITIONS AND THE MAIN RESULT

Describing locations. We are interested in the situation in which we measure the value of some quantity – e.g., intensity of an image – at different spatial locations x . In each situation, we have finitely many locations at which measurements were made.

We are interested in finding local minima and local maxima. By definition, a function f has a local minimum at a location x if its value $f(x)$ at this location is smaller than or equal to its values in all the neighboring points. To formally describe this notion, we therefore need to describe which locations are neighbors and which are not. This notion of neighborhood is a symmetric relation on the set G of all locations. In mathematical terms, with this relation, the set of locations becomes a graph. Thus, we arrive at the following description of all the locations.

Definition 1. Let G be a finite undirected graph.

- The vertices of the graph G will be called locations.
- If vertices $x, y \in G$ are connected by an edge, we will call them neighbors and denote it by $x \sim y$.
- Let $S \subseteq G$ be a subset of the graph G . We say locations $x, y \in S$ are S -connected if there exists a S -connecting sequence, i.e., a sequence $x_0 = x \in S$, $x_1 \in S$, ..., $x_{m-1} \in S$, $x_m = y \in S$ for which $x_i \sim x_{i+1}$ for all i .
- We say that a subset $S \subseteq G$ is connected if every two locations $x, y \in S$ are S -connected.
- We say that a function $f : G \rightarrow \mathbb{R}$ has a local minimum at location x if $f(x) \leq f(y)$ for all neighbors y of the location x .
- We say that a function $f : G \rightarrow \mathbb{R}$ has a local maximum at location x if $f(x) \geq f(y)$ for all neighbors y of the location x .

Describing the results of realistic (imprecise) measurements. The above definitions describe local extrema of a precisely known function. In practice, measurements are never absolutely accurate, so we can only find the value $f(x)$ of the desired function with some accuracy $\varepsilon > 0$. In other

words, for each location x , instead of the exact (unknown) value $f(x)$, we get a measurement result $f_0(x)$ (usually, a rational number) which is ε -close to $f(x)$: $|f(x) - f_0(x)| < \varepsilon$. This means that for every location x , the only information that we have about the value $f(x)$ is that it belongs to the interval $(f_0(x) - \varepsilon, f_0(x) + \varepsilon)$. Thus, we arrive at the following definition.

Definition 2. Let G be a graph.

- By a measurement result, we mean a pair $\mathbf{f} = \langle f_0, \varepsilon \rangle$ consisting of a rational-valued function $f_0 : G \rightarrow \mathbb{R}$ and a rational number $\varepsilon > 0$;
 - for each $x \in G$, the value $f_0(x)$ is called a measured value;
 - the value ε is called a measurement accuracy.

A measurement result will also be called an interval-valued function and denoted

$$\mathbf{f}(x) = (f_0(x) - \varepsilon, f_0(x) + \varepsilon).$$

- We say that a function $f : G \rightarrow \mathbb{R}$ is consistent with the interval-valued function $\mathbf{f}(x) = (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$ if $f(x) \in (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$ for every location x . We will denote this consistency by $f \in \mathbf{f}$.

What does it mean to locate a local extremum. A natural idea is to find the smallest connected subset $S \subseteq G$ on which every function $f \in \mathbf{f}$ attains a local extremum.

Definition 3. Let G be a graph, and let \mathbf{f} be an interval-valued function on this graph. We say that a connected set S is a local minimum set of \mathbf{f} if the following two properties are satisfied:

- every function $f \in \mathbf{f}$ attains a local minimum at some location $x \in S$; moreover, each location $x_m \in S$ at which f attains its smallest value on S is a local minimum of f on G ;
- for every smaller set $S' \subset S$, $S' \neq S$, there exists a function $f \in \mathbf{f}$ that does not have any local minimum on the set S .

Comment. For example, if the set S consists of a single location x_0 , the first condition means that for every function $f \in \mathbf{f}$, the value $f(x_0)$ is smaller than or equal to the value at all neighboring points $y \sim x_0$. When the set S consists of several locations x_1, x_2 , etc., different functions $f \in \mathbf{f}$ may attain local minimum at different locations $x_i \in S$.

A similar definition can be given for local maxima.

Definition 4. Let G be a graph, and let \mathbf{f} be an interval-valued function on this graph. We say that a connected set S is a local maximum set of \mathbf{f} if the following two properties are satisfied:

- every function $f \in \mathbf{f}$ attains a local maximum at some location $x \in S$; moreover, each location $x_m \in S$ at which f attains its largest value on S is a local maximum of f on G ;
- for every smaller set $S' \subset S$, $S' \neq S$, there exists a function $f \in \mathbf{f}$ that does not have any local maximum on the set S .

No, we are ready to formulate our main results.

Theorem 1. *There exists a polynomial-time algorithm that, given an interval-valued function on a graph, returns all its local minimum sets.*

Theorem 2. *There exists a polynomial-time algorithm that, given an interval-valued function on a graph, returns all its local maximum sets.*

The corresponding algorithms are easy to describe.

Algorithm 1. By trying all locations $x \in G$, we can find all local minima x_ℓ of the function $f_0(x)$. For each such local minimum, we again try all locations $x \in G$ and find the set $S_\ell = \{x : f_0(x) < f_0(x_\ell) + 2\varepsilon\}$. From this set, we select the subset S'_ℓ consisting of all locations $x \in S_\ell$ which are S_ℓ -connected to x_ℓ . If for all the locations x from the set S'_ℓ , we have $f_0(x) \geq f_0(x_\ell)$, then this set S'_ℓ is returned as one of the desired local minimum sets S .

Algorithm 2. By trying all locations $x \in G$, we can find all local maxima x_ℓ of the function $f_0(x)$. For each such local maximum, we again try all locations $x \in G$ and find the set $S_\ell = \{x : f_0(x) > f_0(x_\ell) - 2\varepsilon\}$. From this set, we select the subset S'_ℓ consisting of all locations $x \in S_\ell$ which are S_ℓ -connected to x_ℓ . If for all the locations x from the set S'_ℓ , we have $f_0(x) \leq f_0(x_\ell)$, then this set S'_ℓ is returned as one of the desired local maximum sets S .

III. PROOFS

1°. One can easily check that local maxima sets of an interval function $\langle f_0, \varepsilon \rangle = (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$ are exactly local minimum sets of the interval function

$$\langle -f_0, \varepsilon \rangle = (-f_0(x) - \varepsilon, -f_0(x) + \varepsilon).$$

Because of this reduction, it is sufficient to prove the result about the local minimum sets.

For this case, we need to prove:

- that the algorithm is indeed polynomial-time,
- that every set generated by this algorithm is indeed a local minimum set, and
- that every local minimum set appears in the list of sets generated by our algorithm.

2°. Let us first prove that our algorithm is indeed polynomial-time, i.e., its number of computational steps does not exceed the polynomial of a size of the input.

2.1°. The input to this algorithm included:

- a graph G , i.e., the list of vertices v_1, \dots, v_n and a description of edges, and
- an interval-valued function, i.e., a list of rational numbers $f_0(v_1), \dots, f_0(v_n)$, and ε .

The edges can be described by an *adjacency matrix*, i.e., by matrix describing, for each i and j , whether the i -th and the j -th locations v_i and v_j are neighbors. Alternatively, the edges can be described by listing, for each location v_i , all its neighbors. In both cases, the description requires $O(n^2)$

values. In all the cases, the size of the input is larger than $C \cdot n$ is smaller than $C \cdot n^2$.

We will prove that the computation time t for our algorithm is bounded by a polynomial of n : $t \leq P(n)$. Since the size s of the input is $\geq C \cdot n$, this will be guarantee that this computation time is also bounded by a polynomial of s .

Let us analyze our algorithm stage-by-stage.

2.2°. At the first stage of our algorithm we test all n locations $v_i \in G$, and for each of them, we check whether this location v_i is a local minimum of the function $f_0(x)$, i.e., whether $f_0(v_i) \leq f_0(y)$ for all the neighbors y of the location v_i . For each location, there are $\leq n$ neighbors, so this testing takes $\leq C \cdot n$ computational steps. We perform this testing for each of n points, so the total number of computational steps on this stage is bounded by $n \cdot (C \cdot n) = O(n^2)$.

As a result of this stage, we get a list of locations which are local minima of the function $f_0(x)$. The number of such locations is smaller than or equal to the total number n of possible locations.

2.3°. On the second step, for each of the local minima x_ℓ of the function $f_0(x)$, we form the set $S_\ell = \{x : f_0(x) < f_0(x_\ell) + 2\varepsilon\}$, i.e., we mark all the locations x for which $f_0(x) < f_0(x_\ell) + 2\varepsilon$. For each local minimum location x_ℓ , this marking can be done by testing all n possible locations x , so this marking takes $O(n)$ steps.

We do this for all $\leq n$ local minima, so the total number of steps is $\leq n \cdot O(n) = O(n^2)$.

2.4°. Now, for each local minimum x_ℓ of the function $f_0(x)$, we need to select only those locations $x \in S_\ell$ which are S_ℓ -connected to x_ℓ . By definition, the connectedness relation is a transitive closure of the neighborhood relation \sim , so we can use standard graph algorithms to find all such such points (see, e.g., [1]):

- we start by marking the location x_ℓ ;
- at each step, we checked whether each unmarked location from set S_ℓ is a neighbor of one of the marked locations from this set, and if yes, we mark it too.

On each iteration, there are $\leq n$ unmarked locations and $\leq n$ marked locations, so this takes $\leq n^2$ steps.

Once no new locations are marked, we stop. At each iteration except for the last one, at least one location is newly marked. So, the number of iterations cannot exceed the number n of locations. Thus, we have n iterations each of which takes $\leq n^2$ steps, to the total of $\leq n^3$ steps.

We do it for all $\leq n$ local minimum x_ℓ , so the total number of computational steps on this stage is $\leq n \cdot n^3 = n^4$. We have therefore proved that the above algorithm is indeed polynomial-time.

3°. Let us now prove that every set S'_ℓ generated by this algorithm is indeed a local minimum set. By definition, this means that the following two properties are satisfied:

- every function $f \in \mathbf{f}$ attains a local minimum at some location $x \in S'_\ell$; moreover, each location $x_m \in S'_\ell$

at which f attains its smallest value on S'_ℓ is a local minimum of f on G ;

- for every smaller set $S' \subset S'_\ell$, $S' \neq S'_\ell$, there exists a function $f \in \mathbf{f}$ that does not have any local minimum on the set S .

Let us prove these two properties one by one.

3.1°. Let us first prove that each location $x_m \in S'_\ell$ at which f attains its smallest value on S'_ℓ is a local minimum of f on G . Once we prove this statement, it will follow that every function $f \in \mathbf{f}$ attains a local minimum at some location $x \in S'_\ell$: indeed, it attains its local minimum at a point x_m at which f attains its smallest value on the set S'_ℓ .

Let x_m be a point at which the function f attains its smallest possible value at the set S'_ℓ . Thus, if $y \in S'_\ell$, then $f(x_m) \leq f(y)$. Let us show that this location x_m is indeed a local minimum of the function f on the graph G , i.e., that $f(x_m) \leq f(y)$ for all neighbors $y \sim x_m$. To prove this statement, let us consider two possible cases: $y \in S_\ell$ and $y \notin S_\ell$.

3.1.1°. Let us first consider the first case, when $y \in S_\ell$. By definition of the set S'_ℓ , the location x_m is S_ℓ -connected to the location x_ℓ , i.e., there is a S_ℓ -connecting sequence $x_\ell \sim x_1 \sim \dots \sim x_m$ of elements from the set S_ℓ . We know that y is an element of the set S_ℓ , and that y is a neighbor of the location x_m . Thus, by adding the element $y \sim x_m$ to the end of the above S_ℓ -connecting sequence, we get a new S_ℓ -connecting sequence $x_\ell \sim x_1 \sim \dots \sim x_m \sim y$ whose existence shows that the location y also belongs to the set S'_ℓ . By our selection of x_m , we know that $f(x_m) \leq f(y)$ for all elements $y \in S'_\ell$. So, for the case when $y \in S_\ell$, the inequality is proven.

3.1.2°. Let us now consider the second case, when $y \notin S_\ell$. By definition of the set S_ℓ as the set of all the locations x for which $f_0(x) < f_0(x_\ell) + 2\varepsilon$, the fact that $y \notin S_\ell$ means that this inequality is not satisfied for the location y , i.e., that $f_0(y) \geq f_0(x_\ell) + 2\varepsilon$.

We know that the function $f(x)$ is consistent with the interval-valued function $(f_0(x) - \varepsilon, f_0(x) + \varepsilon)$. Thus, for $x = y$, we conclude that $f(y) > f_0(y) - \varepsilon$. So,

$$f(y) >> f_0(y) - \varepsilon \geq (f_0(x_\ell) + 2\varepsilon) - \varepsilon = f_0(x_\ell) + \varepsilon,$$

and $f(y) > f_0(x_\ell) - \varepsilon$. Similarly, we conclude that $f_0(x_\ell) + \varepsilon > f(x_\ell)$, hence $f(y) > f_0(x_\ell) + \varepsilon > f(x_\ell)$ and $f(y) > f(x_\ell)$.

The location x_ℓ is in the set S_ℓ and it is S_ℓ -connected to itself, so it belongs to the set S'_ℓ . Since the function f attains its smallest value on S'_ℓ on the location x_m and $x_\ell \in S'_\ell$, this means that $f(x_\ell) \geq f(x_m)$.

From $f(y) > f(x_\ell)$ and $f(x_\ell) \geq f(x_m)$, we can now deduce the desired inequality $f(y) \geq f(x_m)$.

3.1.3°. The desired inequality $f(y) \geq f(x_m)$ is proven in both cases, when $y \in S_\ell$ and when $y \notin S_\ell$. Thus, the location x_m is indeed a local minimum of the function f on the graph G .

3.2°. Let us first prove that for every smaller set $S' \subset S'_\ell$, $S' \neq S'_\ell$, there exists a function $f \in \mathbf{f}$ that does not have any local minimum on the set S .

3.2.1°. According to Algorithm 1, for all values $x \in S'_\ell$, we have $f_0(x_\ell) \leq f_0(x) < f_0(x_\ell) + 2\varepsilon$. Let us denote $M \stackrel{\text{def}}{=} \max_{x \in S'_\ell} f_0(x)$. Then, we have $f_0(x) \leq M$ for all $x \in S'_\ell$ and $M < f_0(x_\ell) + 2\varepsilon$. Thus, $f_0(x_\ell) > M - 2\varepsilon$.

3.2.2°. Let us denote $m \stackrel{\text{def}}{=} \frac{f(x_\ell) + M}{2}$. Let us prove that for all $x \in S'_\ell$, we have $m - \varepsilon < f_0(x) < m + \varepsilon$ (i.e., equivalently, $|f_0(x) - m| < \varepsilon$ and $f_0(x) - \varepsilon < m < f_0(x) + \varepsilon$).

Indeed, for every $x \in S_\ell$, we have $f_0(x) < f_0(x_\ell) + 2\varepsilon$; by definition of M , we have $f_0(x) \leq M$. By adding these two inequalities and dividing both sides by two, we conclude that

$$f_0(x) < \frac{(f_0(x_\ell) + 2\varepsilon) + M}{2} = \frac{f_0(x_\ell) + M}{2} + \varepsilon = m + \varepsilon.$$

Similarly, for all $x \in S'_\ell$, we have $f_0(x) \geq f_0(x_\ell)$. From this inequality and from the above inequality $f_0(x_\ell) > M - 2\varepsilon$, we conclude that $f_0(x) > M - 2\varepsilon$. By adding this inequality and the inequality $f_0(x) \geq f_0(x_\ell)$ and dividing both sides by two, we conclude that

$$f_0(x) > \frac{(M - 2\varepsilon) + f_0(x_\ell)}{2} = \frac{M + f_0(x_\ell)}{2} - \varepsilon = m - \varepsilon.$$

The statement is proven.

3.2.3°. Since the set S' is non-empty, it contains a location v_{in} . Since S' is a proper subset of the set S'_ℓ , there exists a location $v_{\text{out}} \in S'_\ell$ for which $v_{\text{out}} \notin S'$. By definition of the set S'_ℓ , all the locations $x, y \in S'_\ell$ are S'_ℓ -connected; in particular, the locations v_{in} and v_{out} are S'_ℓ -connected by a S_ℓ -connecting chain. At the one end of this chain, we have a location in S' , at the other end, we have a location which is not in S' . Thus, there must exist two neighboring locations $v_+ \sim v_-$ from this connecting chain for which $v_+ \in S'$ and $v_- \notin S'$.

Since all locations from S'_ℓ are S'_ℓ -connected, for every location $x \in S'_\ell$, there exists a S'_ℓ -connecting chain that connects x and v_- ; there may be several such chains. Let $d(x, v_-)$ be the smallest number of neighborhood relations in such a chain.

For $x = v_-$, the shortest S_ℓ -connecting chain consists of a single element v_- , so $d(v_-, v_-) = 0$. For $x \neq v_-$, we have $d(x, v_-) > 0$. In particular, for $x = v_+$, we have $d(v_+, v_-) = 1$.

Let us pick a small value $k > 0$ and define a function f as follows:

- for $x \in S'_\ell$, we take $f(x) = m + k \cdot d(x, v_-)$;
- for all $x \notin S'_\ell$, we take $f(x) = f_0(x)$.

We have shown that $m \in (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$ for all $x \in S'_\ell$. So, for sufficiently small k , we have

$$f(x) \in (f_0(x) - \varepsilon, f_0(x) + \varepsilon)$$

for all x . Let us show, by contradiction, that thus defined function $f(x)$ does not attain a local minimum on any location

from the set S' . Indeed, for all $x \in S'$, we have $d(x, v_-) \geq 1$ and thus, $f(x) \geq m + k$. Let x' be the previous element to x in the shortest S'_ℓ -connecting chain that connects v_- and x . Then, $x \sim x'$ and $d(x', v_-) = d(x, v_-) - 1 < d(x, v_-)$ and thus, $f(x') < f(x)$. So, indeed, no location $x \in S'$ is a local minimum, since for each such location, there exists a neighboring location x' at which the function f has a smaller value. The statement is proven.

4°. To complete the proof of our result, we need to show that every local minimum set S appears in the list of sets generated by our algorithm.

4.1°. By Definition 3, the fact that S is a local minimum set means that for every function $f \in \mathbf{f}$, every location $x_m \in S$ at which f attains its smallest value on the set S is a local minimum of f . In particular, this is true for the function f_0 . Let x_ℓ be the location at which the function f_0 attains its minimum on the set S ; then $f_0(x_\ell) \leq f_0(x)$ for all locations $x \in S$.

4.2°. For every location $x \in S$, there must exist a function $f \in \mathbf{f}$ that attains its smallest value at this location x .

Indeed, otherwise, the set $S - \{x\}$ will be a proper subset of the set S on which each function $f \in \mathbf{f}$ has a local minimum, in contradiction to the second part of the definition of a local minimum set.

In particular, for this function f , we have $f(x) \leq f(x_\ell)$. Since $f \in \mathbf{f}$, we have $f_0(x) - \varepsilon < f(x)$ and $f(x_\ell) < f_0(x_\ell) + \varepsilon$. From

$$f_0(x) - \varepsilon < f(x) \leq f(x_\ell) < f_0(x_\ell) + \varepsilon,$$

we conclude that $f_0(x) - \varepsilon < f_0(x_\ell) + \varepsilon$, and thus, that $f_0(x) < f_0(x_\ell) + 2\varepsilon$. So, the set S is a subset of the set S_ℓ .

4.3°. One can also prove that the set S is *connected*, i.e., that every two locations $x, y \in S$ are connected by an S_ℓ -connecting chain in which every two consequent elements are neighbors. Indeed, otherwise, the set S can be divided into several *connected components*, subsets in which every two locations can be thus connected, and one can prove that each of these components will also be a local minimum set – in contradiction to the second part of the definition of a local minimum set.

Thus, S is a subset of the set S'_ℓ .

4.4°. Let us prove that S coincides with the set S'_ℓ .

By definition of the set S'_ℓ as a connected component of the set S_ℓ , to prove this equality, it is sufficient to prove that if a location $v \in S_\ell$ is a neighbor to one of the locations $v_{\text{in}} \in S$, then v also belongs to the set S . We will prove this by considering two possible cases: $f_0(v) \geq f_0(x_\ell)$ and $f_0(v) < f_0(x_\ell)$.

4.4.1°. When $f_0(v) \geq f_0(x_\ell)$, we can, similarly to Part 3.2 of this proof, define the $M \stackrel{\text{def}}{=} \max_{x \in S_\ell} f_0(x)$ and $m \stackrel{\text{def}}{=} \frac{f_0(x_\ell) + M}{2}$. Then, for a sufficiently small $k > 0$, the following function f is consistent with the given interval-valued function \mathbf{f} :

- for $x \in S$ and for $x = v$, we take $f(x) = m + k \cdot d(x, v)$;
- for all $x \notin S$, we take $f(x) = f_0(x)$.

On the set S , this function f attains the smallest possible value at the location v_{in} (or at any other location neighboring with v) at which $d(x, v) = 1$ and $f(v_{\text{in}}) = m + k$. However, this location is not a local minimum since at the neighboring location v , this function $f(x)$ attains a smaller value

$$f(v) = m < f(v_{\text{in}}) = m + k.$$

This contradicts to the definition of the local minimum set.

4.4.2°. When $f_0(v) < f_0(x_\ell)$, then, for a sufficiently small $k > 0$, the following function f is consistent with the given interval-valued function \mathbf{f} :

- for $x \in S$, we take $f(x) = m + k \cdot d(x, v)$;
- for all $x \notin S$, we take $f(x) = f_0(x)$.

On the set S , this function f attains the smallest possible value at the location v_{in} (or at any other location neighboring with v) at which $d(x, v) = 1$ and $f(v_{\text{in}}) = m + k$. However, this location is not a local minimum since at the neighboring location v , this function $f(x)$ attains a smaller value

$$f(v) < f_0(x_\ell) < m < f(v_{\text{in}}) = m + k.$$

This also contradicts to the definition of the local minimum set.

4.4.3°. So, the set S indeed coincides with the set S'_ℓ .

4.5°. To complete the proof, let us show that this set S'_ℓ will indeed be returned by our algorithm. Indeed, by definition of the location x_ℓ , we have $f_0(x) \geq f_0(x_\ell)$ for all locations $x \in S$. We have just proven that $S = S'_\ell$. Thus, for all the locations x from the set S'_ℓ , we have $f_0(x) \geq f_0(x_\ell)$. According to our algorithm, this means that this set $S = S'_\ell$ will indeed be returned by our algorithm.

The theorem is proven.

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