Modal Intervals as a New Logical Interpretation of the Usual Lattice Order Between Interval Truth Values

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Abstract*—In the traditional fuzzy logic, we use numbers from the interval $[0, 1]$ to describe possible expert's degrees of belief in different statements. Comparing the resulting numbers is straightforward: if our degree of belief in a statement A is larger than our degree of belief in a statement B, this means that we have more confidence in the statement A than in the statement B. It is known that to get a more adequate description of the expert's degree of belief, it is better to use not only numbers a from the interval $[0, 1]$, but also subintervals $[\underline{a}, \overline{a}] \subseteq [0, 1]$ of this interval. There are several different ways to compare intervals. For example, we can say that $[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}]$ if every number from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to every number from the interval $[\underline{b}, \overline{b}]$. However, in interval-valued fuzzy logic, a more frequently used ordering relation between interval truth values is the relation $[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow \underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b}$. This relation makes mathematical sense – it make the set of all such interval truth values a lattice – but, in contrast to the above relation, it does not have a clear logical interpretation. Since our objective is to describe logic, it is desirable to have a reasonable logical interpretation of this lattice relation. In this paper, we use the notion of modal intervals to provide such a logical interpretation.

I. Formulation of the Problem

Traditional numerical truth values in fuzzy logic. In the traditional approach to fuzzy logic (see, e.g., [3], [8]), the degree of confidence ("truth value") of each statement is characterized by a number from the interval $[0, 1]$:

- the value 1 means that the expert is absolutely confident in this statement;
- the value 0 means that the expert is absolutely confident that this statement is false; and
- values between 0 and 1 describe typical situations when the expert has some degree of confidence in the statement, but he or she is not absolutely sure that this statement is true.

These degrees of confidence are easy to compare: if our degree of confidence d in a statement S is larger than our confidence d' in a statement S', this means that we have more confidence in the statement S than in the statement S'.

Need to go beyond numerical truth values. Zadeh's ides of using numerical values to describe degrees of confidence has led to many successful practical applications of fuzzy techniques [3], [8]. Many of these applications start with eliciting the corresponding degrees of confidence form the experts.

There are many different ways to elicit such degrees. For example, we can ask an expert to mark his or her confidence of a scale, e.g., on a scale from 0 to 5, 0 meaning no confidence at all, and 5 meaning absolute confidence. If an expert marks his or her confidence by 3, then we estimate the corresponding degree of confidence as 3/5 = 0.6.

Another possibility is to poll several experts; if out of 10 experts, 7 believe that the statement is true, we take 7/10 as our degree of confidence in this statement. There are many other ways of eliciting the corresponding degrees.

At first glance, all these techniques provide a number that measures the expert's degree of confidence – the same way as the height in inches or centimeters measures the person's height. However, there is a big difference between these two types of measurements: if we measure the height of a person again and again, by using different rules, we get (more or less) the same value – describing the actual height of this person. In contrast, if we slightly different versions of the same elicitation techniques, we get somewhat different values.

For example, if we ask a person to mark his or her confidence on a scale from 0 to 5, then possible marks are 0, 1, 2, 3, 4, and 5, and the resulting degrees of confidence are 0/5 = 0, 1/5 = 0.2, 2/5 = 0.4, 3/5 = 0.6, 4/5 = 0.8, and 5/5 = 1.0. To get a better estimate, we can use a more detailed scale, e.g., the scale from 0 to 6. However, with the new scale, we get numbers 0/6 = 0, 1/6, 2/6 = 1/3, 3/6 = 0.5, 4/6 = 2/3, 5/6, and 6/6 = 1.0. With the exception of 0 and 1, none of the previous values can appear in this new scale. So if, e.g., a person selected 3 on a scale from 0 to 5, and we got 0.6 as the degree of confidence, on a new scale, we may get values 3/6 = 0.5 or 4/6 = 2/3 = 0.66..., but never the exact same value 0.6.

To avoid this problem, we could ask the expert to make his or her degree of confidence on a scale, for example, from 0 to 100, but this runs into a different problem: that people are rarely able to meaningfully distinguish between, e.g., values of 70 and 71 on this scale.

Similarly, when we poll 10 experts, we can only get values 0, 0.1, 0.2, ..., 1.0. If we want to get a more accurate estimate, we can ask one more expert, but the resulting values 0, 1/11, 2/11, ..., 1 are all different from the previous values – with the exception, of course, of the values 0 and 1.

In other words, the numerical values depend not only on the actual expert's degree of confidence, they also depend on the technique that was used to elicit these degrees. For example, the same value 0.5 coming from an on-a-scale-from-0-to-something elicitation can mean different things.

- It can mean that we got 1 on a scale from 0 to 2. In this scale, we basically consider three different options: 0 if we are confident that the statement is false, 2 if we are confident that the statement is true, and 1 in cases when we are uncertain. Thus, the fact that the expert selected 1 simply means that the expert is not certain about this statement, and it does not tell us much about the degree of this uncertainty.
- On the other hand, this same value 0.5 could mean that the expert selected 5 on a scale from 0 to 10. This is a completely different story. Here, the expert had 9 values describing uncertainty to choose from: 1, 2, ..., 9, and the fact that the expert selected the midpoint 5 and not any other value means that this expert probably has as many reasons to believe in the original statement as in its negation.

When we make decisions based on the expert's degrees of confidence in different statements, it is definitely desirable to take into account the difference between the above two situations. Since in both situations, we have the exact same numerical value 0.5 of the expert's uncertainty, this means that we need to go beyond the numerical truth values.

Interval truth values. A natural way to go beyond numerical truth values is to use *interval* truth values, when the expert's degree of confidence is described not by a number d from the interval [0,1], but rather by a subinterval $[\underline{d}, \overline{d}]$ of this interval [3], [5], [6], [8].

Indeed, when a person select 3 on a scale from 0 t0 5, this does not necessarily mean that his or her degree of confidence corresponds exactly to the value 3, it simply means that this degree is closer to 3 than to other marks (0, 1, 2, 4, and 5) on scale. Values which are closer to 3 than to all other integers are easy to describe: they form an interval $[2.5, 3.5]$. Based on our scale-from-0-to-5 request, we do not get the actual expert's degree of confidence, we only conclude that this actual (unknown) degree is between $2.5/5 = 0.5$ and $3.5/5 = 0.7$, i.e., that this degree is in the interval $[0.5, 0.7]$. It is therefore reasonable to return this interval as the available information about the expert's degree of confidence in a given statement.

Need to order interval truth values. The ultimate purpose of processing expert knowledge – and, in particular, processing degrees of belief in different statements – is to make decisions. Let us consider a simple example. Suppose that we want to achieve a certain objective. We know of two possible actions each of which can lead to this objective with some confidence, and we need to select the most promising action.

When the degree of confidence is described by a number, this problem is easy to solve: for each of the actions, we estimate the degree of confidence that this particular action will lead to the desired objective, and we select the action for which this degree is the largest possible.

However, when we use intervals to describe degrees of belief, it is not always clear which of the two actions is better. For example, suppose that for one of the actions, we have no information about its possible consequences. In this case, the interval-valued degree of belief is the whole interval $[0, 1]$. Suppose also that for the second action, we have some arguments for and against the success of this action, and we have exactly as many arguments for as we have arguments against. In this case, it is reasonable to take the midpoint 0.5 between 0 ("false") and 1 ("true") as the degree of belief in the second statement. Which one should we prefer?

How to extend functions and operations from numbers to intervals: general idea (a particular case of Zadeh's extension principle). We have an ordering relation between *numbers* a from the interval $[0, 1]$. We need to extend this relation to *subintervals* $[\underline{a}, \overline{a}]$ from this interval.

This is a problem typical in fuzzy techniques: we start with a function $f(x_1, \ldots, x_n)$ defined for real numbers, and we need to extend it to intervals $X_1, \ldots, X_n$ – or, more generally, to fuzzy numbers $X_1, \ldots, X_n$. A natural way towards such extension was developed by Lotfi Zadeh himself and is therefore known as Zadeh's extension principle.

With respect to intervals (and crisp sets in general) Zadeh's extension principle means the following. Suppose that we do not know the exact values x_i of the inputs. For each input i, we only know the set X_i of possible values. Then, a number y is a possible value of the function $f(x_1, \ldots, x_n)$ if and only if there are possible values $x_i \in X_n$ for which $y = f(x_1, \ldots, x_n)$. So, as an answer, we return the set Y of all such numbers y, i.e., the set

$$\{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

This set is known as a *range* of the functions $f(x_1, \ldots, x_n)$ on intervals $X_1, \ldots, X_n$, and it is usually denoted by $f(X_1, \ldots, X_n)$. The task of computing such a range for different functions and different intervals constitutes so-called *interval computations*; see, e.g., [2], [7].

Let us show how the above idea can help us expand the ordering between numbers to ordering between intervals.

Possible representations of an ordering relation. It turns out that what exactly extension to intervals we get depends on how we represent the order. To show this, let us consider three possible representations.

- The first is the standard representation, in which $\leq$ is a functions that transforms two numbers a and b into the truth value of the relation $a \leq b$. In other words, this function returns 1 if $a \leq b$ and it returns 0 if $a \not\leq b$.

We call this representation standard, since our ultimate objective is to process all this in computers, and this is how ordering is represented in the computers.

- Instead of the ordering relation itself, we can consider functions $\max(a, b)$ and $\min(a, b)$. Each of these functions is also computer supported. Each of these functions describe the ordering:
  – once we have the function $\max(a, b)$, we can reconstruct the relation $a \le b$ as $b = \max(a, b)$;
  – similarly, once we have the function $\min(a, b)$, we can reconstruct the relation $a \le b$ as $a = \min(a, b)$.

Let us show how these representations lead to different interval extensions.

Zadeh's extension principle approach applied to the original ordering relation. The original function $\le$ starts with two real numbers a and b and produces a (crisp) truth value, i.e., a number from the set $\{0, 1\}$ of crisp truth values. According to the general definition of Zadeh's extension principle, when we start with sets $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$ of possible values of a and b, we thus get a *set* $\le (\mathbf{a}, \mathbf{b})$ of truth values, i.e., a subset of the set $\{0, 1\}$. Based on the definition, we can distinguish three possible situations:

- if every element $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than every element $b \in [\underline{b}, \overline{b}]$, then the set $\le (\mathbf{a}, \mathbf{b})$ consists of only one value 1 (corresponding to "true");
- if none of the elements $a \in [\underline{a}, \overline{a}]$ is smaller than or equal than any element $b \in [\underline{b}, \overline{b}]$, then the set $\le (\mathbf{a}, \mathbf{b})$ consists of only one value 0 (corresponding to "false");
- in all other case, the set $\le (\mathbf{a}, \mathbf{b})$ contains both values 1 ("true") and 0 ("false'), i.e., we have $\le (\mathbf{a}, \mathbf{b}) = \{0, 1\}$.

In other words, here, $\mathbf{a} \le \mathbf{b}$ if and only every element $a \in \mathbf{a}$ is smaller than or equal to every element $b \in \mathbf{b}$:

$$\forall a \in \mathbf{a}\, \forall b \in \mathbf{b}\, (a \le b).$$

This relation is easy to describe in terms of the endpoints of the intervals $\mathbf{a}$ and $\mathbf{b}$: namely, an element a is smaller than or equal to every element of the interval $[\underline{b}, \overline{b}]$ if and only if it is smaller than or equal to the smallest of these elements, i.e., the element $\underline{b}$.

Thus, the above condition is satisfied if and only if every element a of the interval $\mathbf{a}$ is smaller than or equal to $\underline{a}$.

Similarly, every element a from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to $\underline{b}$ if and only if the largest of possible values of a, i.e., the element $\overline{a}$, is smaller than or equal to $\underline{b}$. Thus,

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow \overline{a} \le \underline{b}.$$

Zadeh's extension principle applied to the function $\max(a, b)$. The function $\max(a, b)$ is non-strictly increasing in a and b, meaning that if $a \le a'$ and $b \le b'$, then $\max(a, b) \le \max(a', b')$. Thus, when a is in the interval $[\underline{a}, \overline{a}]$, and b is in the interval $[\underline{b}, \overline{b}]$, we can conclude that:

- the smallest possible value of $\max(a, b)$ is attained when both a and b attain their smallest possible values, i.e., when $a = \underline{a}$ and $b = \underline{b}$; the corresponding value of the function $\max(a, b)$ is equal to $\max(\underline{a}, \underline{b})$;
- the largest possible value of $\max(a, b)$ is attained when both a and b attain their largest possible values, i.e., when $a = \overline{a}$ and $b = \overline{b}$; the corresponding value of the function $\max(a, b)$ is equal to $\max(\overline{a}, \overline{b})$.

Thus, the range $\max([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$ of the function $\max(a, b)$ on the intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ is equal to

$$\max([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\max(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})].$$

As we have mentioned, we can now define the relation $\mathbf{a} \le \mathbf{b}$ between intervals as $\mathbf{b} = \max(\mathbf{a}, \mathbf{b})$. According to the above formula, this ordering relation has the following form:

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow [\underline{b}, \overline{b}] = [\max(\underline{a}, \underline{b}), \max(\overline{a}, \overline{b})] \Leftrightarrow$$

$$\underline{b} = \max(\underline{a}, \underline{b})\, \&\, \overline{b} = \max(\overline{a}, \overline{b}) \Leftrightarrow$$

$$\underline{a} \le \underline{b}\, \&\, \overline{a} \le \overline{b}.$$

This relation – actively used in interval-valued fuzzy logic – is different from what we get by applying Zadeh's extension principle to the original ordering relation.

Zadeh's extension principle applied to the function $\min(a, b)$. The function $\min(a, b)$ is also non-strictly increasing in a and b, meaning that if $a \le a'$ and $b \le b'$, then $\min(a, b) \le \min(a', b')$. Thus, when a is in the interval $[\underline{a}, \overline{a}]$, and b is in the interval $[\underline{b}, \overline{b}]$, we can conclude that:

- the smallest possible value of $\min(a, b)$ is attained when both a and b attain their smallest possible values, i.e., when $a = \underline{a}$ and $b = \underline{b}$; the corresponding value of the function $\max(a, b)$ is equal to $\min(\underline{a}, \underline{b})$;
- the largest possible value of $\min(a, b)$ is attained when both a and b attain their largest possible values, i.e., when $a = \overline{a}$ and $b = \overline{b}$; the corresponding value of the function $\max(a, b)$ is equal to $\min(\overline{a}, \overline{b})$.

Thus, the range $\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}])$ of the function $\min(a, b)$ on the intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ is equal to

$$\min([\underline{a}, \overline{a}], [\underline{b}, \overline{b}]) = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})].$$

As we have mentioned, we can now define the relation $\mathbf{a} \le \mathbf{b}$ between intervals as $\mathbf{a} = \min(\mathbf{a}, \mathbf{b})$. According to the above formula, this ordering relation has the following form:

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow [\underline{a}, \overline{a}] = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})] \Leftrightarrow$$

$$\underline{a} = \min(\underline{a}, \underline{b})\, \&\, \overline{a} = \min(\overline{a}, \overline{b}) \Leftrightarrow$$

$$\underline{a} \le \underline{b}\, \&\, \overline{a} \le \overline{b}.$$

This relation is exactly the same as we obtained from the function $\max(a, b)$, and it is therefore different from what we get by applying Zadeh's extension principle to the original ordering relation.

Comment. Operations $\max(a, b)$ and $\min(a, b)$ form a lattice, so the corresponding ordering relation can be called a *lattice relation*.

Problem: how to interpret the lattice order in logical terms? Our objective is to develop the corresponding logic. It is therefore desirable to have a logical interpretation of the resulting ordering between intervals. For the first ordering relation – obtained by applying Zadeh's extension principle directly to the order between real numbers – we have a straightforward logical interpretation. However, for the lattice order, we do not have such a direct logical interpretation.

What we do in this paper. In this paper, we show that *modal intervals* – a practice-motivated generalization of intervals – provide the desired logical explanation for the lattice order. To provide such an explanation, we first need to recall what are modal intervals.

II. Modal Intervals: A Brief Reminder

This section provides a brief description of modal intervals as described in [1], [4].

Traditional interval computations: reminder. Let us assume that a quantity z depends on quantities $x = (x_1, \ldots, x_n)$, and that we know the exact form of this dependence, i.e., we know a continuous function $z = f(x) = f(x_1, \ldots, x_n)$. In practice, we often do not know the exact values of the quantities x_i, we only know the intervals $X_i = [\underline{x}_i, \overline{x}_i]$ that contain these values.

These intervals may come from *measurements:* when the measurement result is $\widetilde{x}_i$ and we know the upper bound Δ_i on (absolute value of) the measurement error $\Delta x_i \stackrel{\text{def}}{=} \widetilde{x}_i - x_i$, this means that the actual (unknown) value x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$. These intervals can also come from *manufacturing tolerances*, when we recommend the value $\widetilde{x}_i$ of the corresponding quantity but allow deviations $\pm\Delta_i$ from this recommended value. In this case also, the resulting the resulting quantity x_i can take any value from the interval $[\widetilde{x}_i - \Delta_i, \widetilde{x}_i + \Delta_i]$.

In both cases, the only information that we have about z is that z belongs to the interval

$$Z = \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\} =$$

$$\left[\min_{x \in X} f(x), \max_{x \in X} f(x)\right],$$

where we denoted $X \stackrel{\text{def}}{=} X_1 \times \ldots X_n$. This interval Z is called the *result of applying the function* f to the intervals $X_1, \ldots, X_n$ and denoted by $f(X_1, \ldots, X_n)$.

In many practical situations, it is desirable to make the interval Z as narrow as possible. For example, z may be the direction of the airplane flight, and we want to maintain this direction as accurately as possible. In the above setting, if we want to decrease the width Z, we have to decrease the width of the original intervals – e.g., measure the values x_i more accurately, or impose stricter tolerances on the manufacturing process.

Logical reformulation of the traditional interval computation. First, we need to make sure that for all possible combinations of $x_i \in X_i$, the value $z = f(x_1, \ldots, x_n)$ is contained in the interval Z. In other words, we want to make sure that

$$\forall x_1 \in X_1 \ldots \forall x_n \in X_n \, \exists z \in Z \, (z = f(x_1, \ldots, x_n)).$$

Second, we need to make sure that Z is the narrowest interval with this property. These two requirements guarantee that Z is equal to the above range: $Z = f(X)$.

Beyond the main problem of (traditional) interval computations – possibility of controlled variables: formulation of the problem. In the traditional approach, we have no control over the values of the input variables x_i, we only know that these values belong to the corresponding intervals X_i. In practice, often, the desired value $z = f(x, u)$ depends not only the variables $x = (x_1, \ldots, x_n)$ over which we have no control, it also depends on the additional variables $u = (u_1, \ldots, u_m)$ that we *can* control. Specifically, for each of these additional variables u_j, there is a range U_j, and we can set up any value within this range. We can use these additional variables to narrow down the range $Z = [\underline{z}, \overline{z}]$ of the values z that can be achieved.

In precise terms, we want to select an interval $Z = [\underline{z}, \overline{z}]$ for which, for each combination $x \in X$, there exists a control u that would lead to the value $f(x, u) \in Z$. Among all such intervals Z, we want to select the one which is the narrowest. In other words, we want to make sure that

$$\forall x \in X \, \exists u \in U \, (f(x, u) \in Z),$$

i.e., that

$$\forall x \in X \, \exists u \in U \, \exists z \in Z \, (z = f(x, u)),$$

and that Z is the narrowest interval with this property.

How can we find such an interval Z?

Possibility of controlled variables: towards a solution to the problem. For each $x \in X$, the set of all possible values $f(x, u)$ forms an interval

$$F(x) \stackrel{\text{def}}{=} \left[\min_{u \in U} f(x, u), \max_{u \in U} f(x, u)\right].$$

The existence of a control u for which one of these values is from the interval Z is equivalent to requiring that that the intervals $F(x)$ and Z have a common point. One can easily check that the two intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$ have a common point if and only if $\underline{a} \le \overline{b}$ and $\underline{b} \le \overline{a}$. For intervals $F(x)$ and Z, this means that we must have

$$\min_{u \in U} f(x, u) \le \overline{z} \text{ and } \underline{z} \le \max_{u \in U} f(x, u).$$

These two inequalities much hold for every $x \in X$. For $\overline{z}$, this means that the value $\overline{z}$ must be larger than or equal to $\min\limits_{u \in U} f(x, u)$ for all $x \in X$. This is equivalent to requiring that $\overline{z}$ is larger than or equal to the largest of these values, i.e., that

$$\overline{z} \ge \max_{x \in X} \min_{u \in U} f(x, u).$$

Similarly, the requirement that $\underline{z}$ must be smaller than or equal to $\max\limits_{u \in U} f(x, u)$ for all $x \in X$ is equivalent to requiring that $\underline{z}$

is smaller than or equal to the smallest of these values, i.e., that

$$\underline{z} \le \min_{x\in X} \max_{u\in U} f(x,u).$$

Among all the intervals that satisfy these two inequalities, we need to find the narrowest. It turns out that the selection of the narrowest interval depends on the relation between the two bounds. If

$$\min_{x\in X} \max_{u\in U} f(x,u) \le \max_{x\in X} \min_{u\in U} f(x,u),$$

then the narrowest interval is when $\overline{z}$ is equal to its lower bound and $\underline{z}$ is equal to its upper bound, i.e., when

$$Z = [\underline{z}, \overline{z}] = \left[\min_{x\in X} \max_{u\in U} f(x,u), \max_{x\in X} \min_{u\in U} f(x,u)\right].$$

On the other hand, if the opposite inequality is satisfied, i.e., if

$$\min_{x\in X} \max_{u\in U} f(x,u) > \max_{x\in X} \min_{u\in U} f(x,u),$$

then we can have intervals Z with the desired property which have width 0: namely, for any value z between these two bounds, i.e., for any value z from the interval

$$Z = \left[\max_{x\in X} \min_{u\in U} f(x,u), \min_{x\in X} \max_{u\in U} f(x,u)\right],$$

the one-point interval $Z' = [z, z]$ satisfies the desired property.

Thus, we arrive at the following solution.

Case of controlled variables: solution. Once we have a function $f(x,u)$ and the ranges X and U, we compute the two values

$$z^- = \min_{x\in X} \max_{u\in U} f(x,u) \text{ and } z^+ = \max_{x\in X} \min_{u\in U} f(x,u).$$

If $z^- \le z^+$, then the interval $Z = [z^-, z^+]$ is the narrowest interval for which

$$\forall x \in X\, \exists z \in Z\, \exists u \in U\, (z = f(x,u)).$$

If $z^- > z^+$, then we have many such narrowest intervals – namely, every interval $[z, z]$ for $z \in [z^+, z^-]$ is a one. This can be described as follows:

$$\forall x \in X\, \forall z \in Z\, \exists u \in U\, (z = f(x,u)).$$

Comment. The above solution is presented in [1], where the pair consisting of the values z^- and z^+ is called an f^*-extension of the original function $f(x,u)$.

Reformulation in terms of modal intervals. In [1], logical terms are used to distinguish between intervals X_i over which we have no control and intervals U_j in which we can select whichever value $u_i \in U_i$ we choose. To guarantee that the value z of the desired quantity is within the given range, we need to make sure that this property holds *for all* possible values $x_i \in X_i$, while for the controlled intervals, it is sufficient to require that *there exist* values $u_j \in U_j$ that make this property true. To emphasize this distinction, the authors of [1] treat each interval as a pair of the interval itself and of the corresponding quantifier:

- a traditional interval X_i is considered as a pair $\langle X_i, \forall\rangle$, while
- a controlled interval is considered as a pair $\langle U_j, \exists\rangle$.

Such pairs are called *modal intervals*.

In these terms, the condition

$$\forall x_1 \in X_1 \ldots \forall x_n \in X_n$$

$$\exists u_1 \in U_1 \ldots \exists u_m \in U_m\, \exists z \in Z\, (z = f(x,u))$$

can be reformulated as

$$Q_1 x_1 \in X_1 \ldots Q_n x_n \in X_n$$

$$Q'_1 u_1 \in U_1 \ldots Q'_m u_m \in U_m\, \exists z \in Z\, (z = f(x,u)),$$

where Q_i and Q'_j are the quantifiers attached to the corresponding intervals. For the case when all the intervals are traditional (non-controlled), we get the usual expression for the range. Because of this example, we can treat the resulting interval Z as the range defined over modal intervals:

$$Z = f(\langle X_1, \forall\rangle, \ldots, \langle X_n, \forall\rangle, \langle U_1, \exists\rangle, \ldots, \langle U_m, \exists\rangle).$$

The difference between the cases $z^- \le z^+$ and $z^- > z^+$ translates, as we have seen, into the difference between $\exists z \in Z$ and $\forall z \in Z$ in the corresponding formulas. So, the authors of [1] say that when $z^- \le z^+$, the range is the usual interval $\langle Z, \forall\rangle$, while for $z^- > z^+$, the range is the interval $\langle Z, \exists\rangle$.

Relation to Kaucher intervals. The above example shows that the difference between the two types of intervals can also be represented as the difference between the usual intervals, for which $z^- \le z^+$, and the "new" intervals for which $z^- > z^+$. It is therefore reasonable to represent these "new intervals" as $[z^-, z^+]$.

For example, the interval $Z = \langle [2,4], \forall\rangle$ is represented as a usual interval $[2,4]$, while an interval $\langle [2,4], \exists\rangle$ is represented as $[4,2]$. Such intervals have been previously introduced by Kaucher.

This connection with Kaucher intervals is not accidental: indeed, for arithmetic operations $f(x,u)$, the f^*-extensions coincide with the operations of Kaucher arithmetic.

III. Modal Intervals Explain Lattice Order

Main idea. As we have mentioned, when we apply Zadeh's extension principle – i.e., the usual range estimation formula – to the function $\le (a,b)$, we get the relation $\overline{a} \le \underline{b}$ that corresponds to the logical formula

$$\forall a \in \mathbf{a}\, \forall b \in \mathbf{b}\, (a \le b).$$

Our main idea is to consider situations when, instead of one the original intervals $[\underline{a}, \overline{a}]$ and $[\underline{b}, \overline{b}]$, we consider the "dual" intervals $[\overline{a}, \underline{a}]$ and $[\overline{b}, \underline{b}]$.

As we have mentioned earlier, replacing an interval by a dual one means that we replace the corresponding universal quantifier with an existential one. Thus, we get the following two formulas:

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \le b)$$

and

$$\forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \le b).$$

Let us consider these formulas one by one.

First formula. For each a, the existence of $b \in [\underline{b}, \overline{b}]$ for which a is smaller than or equal to b is equivalent to a being smaller than or equal to the largest possible element $\overline{b}$ of the b-interval. Indeed:

- if $a \le b$ for some b for which $\underline{b} \le b \le \overline{b}$ then, by transitivity, we get $a \le \overline{b}$;
- vice versa, if $a \le \overline{b}$, then $a \le b$ for some $b \in [\underline{b}, \overline{b}]$: namely, for $b = \overline{b}$.

Now, the first formula can be equivalently formulated as follows: every value a from the interval $[\underline{a}, \overline{a}]$ is smaller than or equal to $\overline{b}$. Similarly to the previous paragraph, it is sufficient to check this property for the largest possible value $\overline{a}$ of the quantity a. Indeed:

- if $\overline{a} \le \overline{b}$, this implies that for every value $a \le \overline{a}$, we have $a \le \overline{b}$;
- vice versa, if every number a from the interval $[\underline{a}, \overline{a}]$ satisfies the inequality $a \le \overline{b}$, then, in particular, this inequality holds for the value $\overline{a} \in [\underline{a}, \overline{a}]$, i.e., we have

$$\overline{a} \le \overline{b}.$$

Thus, the first formula is equivalent to $\overline{a} \le \overline{b}$.

Second formula. For each b, the existence of $a \in [\underline{a}, \overline{a}]$ for which a is smaller than or equal to b is equivalent to b being larger than or equal to the smallest possible element $\overline{a}$ of the a-interval. Indeed:

- if $a \le b$ for some a for which $\underline{a} \le a \le \overline{a}$ then, by transitivity, we get $\underline{a} \le b$;
- vice versa, if $\underline{a} \le b$, then $a \le b$ for some $a \in [\underline{a}, \overline{a}]$: namely, for $a = \overline{a}$.

Now, the first formula can be equivalently formulated as follows: every value b from the interval $[\underline{b}, \overline{b}]$ is larger than or equal to $\underline{a}$. Similarly to the previous paragraph, it is sufficient to check this property for the smallest possible value $\underline{b}$ of the quantity b. Indeed:

- if $\underline{a} \le \underline{b}$, this implies that for every value $b \ge \underline{b}$, we have $\underline{a} \le b$;
- vice versa, if every number b from the interval $[\underline{b}, \overline{b}]$ satisfies the inequality $\underline{a} \le b$, then, in particular, this inequality holds for the value $\underline{b} \in [\underline{b}, \overline{b}]$, i.e., we have

$$\underline{a} \le \underline{b}.$$

Thus, the second formula is equivalent to $\underline{a} \le \underline{b}$.

Combining the two formulas: the resulting logical interpretation. The first formula is equivalent to $\overline{a} \le \overline{b}$, the second formula is equivalent to $\underline{a} \le \underline{b}$. Thus, the two formulas together are equivalent to lattice order. So, we get the desired logical interpretation of the lattice order.

This interpretation can be described – as with modal logic – in control-type terms. Namely, the order $\underline{a} \le \overline{b}$ means that every element $a \in \mathbf{a}$ is smaller than or equal to every element $b \in \mathbf{b}$. In contrast, the lattice order is equivalent to the following two statements:

- no matter what the actual value $a \in \mathbf{a}$ is, once we know this value, we can always select $b \in \mathbf{b}$ for which $a \le b$;
- vice versa, no matter what the actual value $b \in \mathbf{b}$ is, once we know this value, we can always select $a \in \mathbf{a}$ for which $a \le b$.

Comment: possible generalizations of this interpretation. In the above text, we considered intervals from the real line. In this case, the relation

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow (\underline{a} \le \underline{b} \,\&\, \overline{a} \le \overline{b})$$

forms a *lattice* – in the sense that for every two intervals, there is the least upper bound and the greatest lower bound. A similar definition can be formulated for a more general case, when we consider intervals

$$[a, b] \stackrel{\text{def}}{=} \{x : a \le x \le b\}$$

over an arbitrary partially ordered set. In this case, the above relation is not longer a lattice, but we can still prove that it is equivalent to

$$\forall a \in \mathbf{a}\, \exists b \in \mathbf{b}\, (a \le b) \text{ and } \forall b \in \mathbf{b}\, \exists a \in \mathbf{a}\, (a \le b).$$

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REFERENCES

[1] E. Gardeñes et al., "Modal intervals", *Reliable Computing*, 2001, Vol. 7, pp. 77–111.

[2] L. Jaulin, M. Kieffer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics*, Springer-Verlag, London, 2001.

[3] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic: Theory and Applications*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[4] V. Kreinovich, *Modal intervals: brief description*, Manuscript, 2012.

[5] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, Upper Saddle River, New Jersey, 2001.

[6] J. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, Wiley-IEEE Press, Piscataway and Hoboken, New Jersey, 2010.

[7] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM Press, Philadelphia, Pennsylvania, 2009.

[8] H. T. Nguyen and E. A. Walker, *First Course in Fuzzy Logic* CRC Press, Boca Raton, Florida, 2006.