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KINEMATIC SPACES AND DE VRIES ALGEBRAS: TOWARDS POSSIBLE PHYSICAL MEANING OF DE VRIES ALGEBRAS

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Traditionally, in physics, space-times are described by (pseudo-)Riemann spaces, i.e., by smooth manifolds with a tensor metric field. However, in several physically interesting situations smoothness is violated: near the Big Bang, at the black holes, and on the microlevel, when we take into account quantum effects. In all these situations, what remains is causality – an ordering relation. To describe such situations, in the 1960s, geometers H. Busemann and R. Pimenov and physicists E. Kronheimer and R. Penrose developed a theory of *kinematic spaces*. Originally, kinematic spaces were formulated as topological ordered spaces, but it turned out that kinematic spaces allow an equivalent purely algebraic description as sets with two related orders: causality and "kinematic" causality (possibility to influence by particles with non-zero mass, particles that travel with speed smaller than the speed of light). In this paper, we analyze the relation between kinematic spaces and *de Vries algebras* – another mathematical object with two similarly related orders.

1. Kinematic Spaces: Brief Introduction

Order relations are needed in describing space-time. Traditionally, in physics, space-times are described by (pseudo-)Riemann spaces, i.e., by smooth manifolds with a tensor metric field $g_{ij}(x)$; see, e.g., [9]. However, in several physically interesting situations smoothness is violated and metric is undefined [9]:

- near the singularity (Big Bang),
- at the black holes, and
- on the microlevel, when we take into account quantum effects.

In all these situations, what remains is causality $\preccurlyeq$ – an ordering relation. To describe such situations, in the 1960s, geometers H. Busemann and R. Pimenov and physicists E. Kronheimer and R. Penrose developed a theory of *kinematic spaces* [6, 8, 10].

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Causality: a brief history. In Newton's physics, signals can potentially travel with an arbitrarily large speed. To describe the corresponding causality relation between events, let us denote an event occurring at the spatial location x at time t by $a = (t, x)$. In these notations, Newton's causality relation is trivial: an event $a = (t, x)$ can influence an event $a' = (t', x')$ if and only if $t \le t'$:

$$(t, x) \preceq (t', x') \Leftrightarrow t \le t'.$$

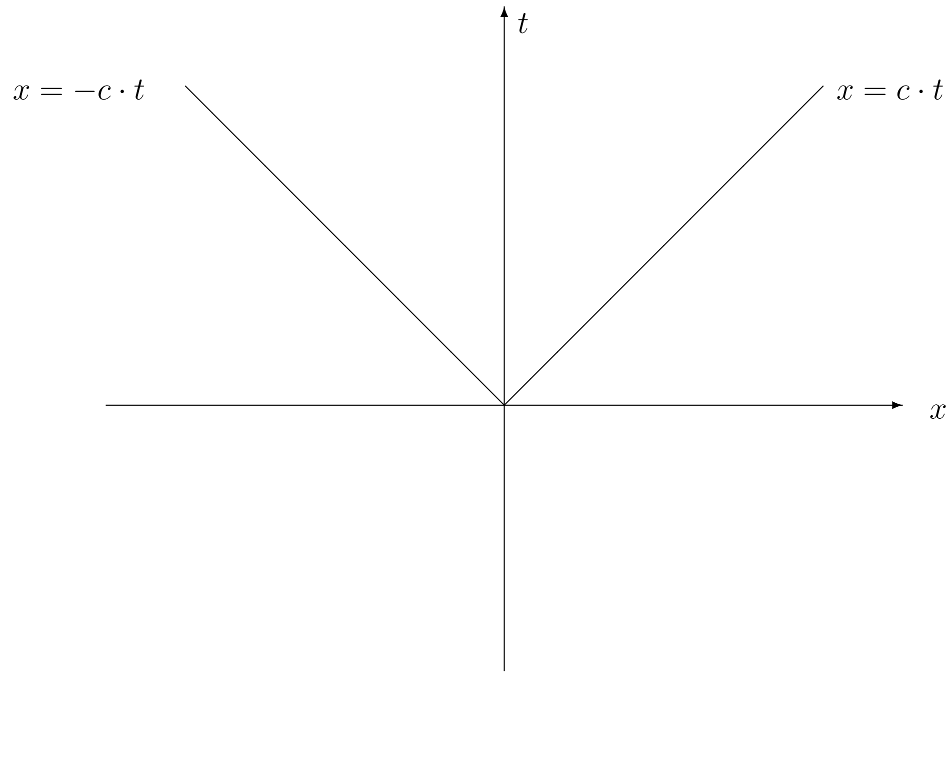
The fundamental role of the non-trivial causality relation emerged with the Special Relativity. In Special Relativity, the speed of all the signals is limited by the speed of light c. As a result, $a = (t, x) \preccurlyeq a' = (t', x')$ if and only if $t' \ge t$ and in time $t' - t$, the speed needed to traverse the distance $d(x, x')$ does not exceed c, i.e., $\dfrac{d(x, x')}{t' - t} \le c$. The resulting causality relation has the form

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge d(x, x'),$$

i.e.,

$$(t, x) \preccurlyeq (t', x') \Leftrightarrow c \cdot (t' - t) \ge \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2}.$$

This relation can be graphically described as follows:



Importance of causality. In the original special relativity theory, causality was just one of the concepts. Its central role was revealed by A. D. Alexandrov [1,2] who showed that in Special Relativity, causality implied Lorenz group. To be more precise, he proved that every order-preserving transforming of the corresponding partial ordered set is linear, and is a composition of:

- spatial rotations,
- Lorentz transformations (describing a transition to a moving reference frame), and
- re-scalings $x \to \lambda \cdot x$ (corresponding to a change of unit for measuring space and time).

This theorem was later generalized by E. Zeeman [11] and is therefore known as the *Alexandrov-Zeeman theorem.*

When is causality experimentally confirmable? Since causality is important, it is desirable to analyze how it can be experimentally detected whether an event a can influence event b.

In many applications, we only observe an event b with some accuracy. For example, in physics, we may want to check what is happening exactly 1 second after a certain reaction. However, in practice, we cannot measure time exactly, so, we observe an event occurring 1 ± 0.001 sec after a.

In general, we can only guarantee that the observed event is within a certain neighborhood U_b of the event b. Because of this uncertainty, the only possibility to experimentally confirm that a can influence b is when a can influence *all* the events from a neighborhood. i.e., when

$$\exists U_b \, \forall \widetilde{b} \in U_b \, \left(a \preccurlyeq \widetilde{b} \right).$$

Let us denote this "experimentally conformable" causality relation by $a \prec b$. In topological terms, $a \prec b$ means that b is in the interior K_a^+ of the *future cone* $C_a^+ \stackrel{\text{def}}{=} \{c : a \preccurlyeq c\}$, i.e., of the set of all the events that can be influenced by the event a.

Kinematic orders. In special relativity, the relation $a \prec b$ correspond to influences with speeds smaller than the speed of light. This relation has a natural physical interpretation if we take into account that in special relativity, there are two types of objects:

- objects with non-zero rest mass can travel with any possible speed $v < c$ but not with the speed c; and
- objects with zero rest mass (e.g., photons) can travel only with the speed c, but not with $v < c$.

In these terms, the relation $\prec$ correspond to causality by traditional (kinematic) objects. Because of this fact, the relation $\prec$ is called *kinematic causality*, and spaces with this relation $\prec$ are called *kinematic spaces.*

Kinematic spaces: towards a description. To describe space-time, we thus need a (pre-)ordering relation $\preccurlyeq$ (causality) and topology (= closeness). There are some natural relation between them.

For example, a natural continuity idea implies that in every neighborhood of an event a, there are events causally following a and causally preceding a. In other words, for every event a and for every neighborhood U_a, there exist a^- and a^+ for which $a^- \prec a$ and $a \prec a^+$.

It is reasonable to assume that if the events $a' \prec a''$ are close to the event a, then every event in between a' and a'' should also be close to a. In precise terms, every neighborhood U_a should contains an entire open interval

$$(a', a'') \stackrel{\text{def}}{=} \{b : a' \prec c \prec a''\}.$$

Another reasonable requirement comes from the fact a motion with speed c is a limit of motions with speeds $v < c$ when $v \to c$. It is therefore reasonable to require that the future cone $C_a^+ = \{b : a \preccurlyeq b\}$ in terms of the original causality relation $\preccurlyeq$ is a closure of the future cone $K_a^+ = \{b : a \prec b\}$ in terms of the kinematic causality: $C_a^+ = \overline{K_a^+}$. A similar property holds for the past cones

$$C_a^- \stackrel{\text{def}}{=} \{b : b \preccurlyeq a\} \text{ and } K_a^- \stackrel{\text{def}}{=} \{b : b \prec a\} :$$

$C_a^- = \overline{K_a^-}$. In other words,

$$a \preccurlyeq b \Leftrightarrow \forall U_b \, \exists \widetilde{b} \left(\widetilde{b} \in U_b \,\&\, a \prec \widetilde{b} \right).$$

In particular, for the neighborhood $U_b = (b', b'')$, for which $b \prec b''$, we get $a \prec \widetilde{b} \prec b''$ and hence $a \prec b''$. Thus,

$$a \preccurlyeq b \Leftrightarrow \forall c (b \prec c \Rightarrow a \prec c).$$

These requirements lead to the following definition of a kinematic space.

Definition 1.

- A set X with a partial order $\prec$ is called a *kinematic space* if is satisfies the following conditions:

$$\forall a \, \exists a_-, a_+ \, (a_- \prec a \prec a_+);$$
$$\forall a, b \, (a \prec b \to \exists c \, (a \prec c \prec b));$$
$$\forall a, b, c \, (a \prec b, c \to \exists d \, (a \prec d \prec b, c));$$
$$\forall a, b, c \, (b, c \prec a \to \exists d \, (b, c \prec d \prec a)).$$

- On a kinematic space, we take a topology generated by *intervals*

$$(a, b) \stackrel{\text{def}}{=} \{c : a \prec c \prec b\}.$$

- A kinematic space is called *normal* if

$$b \in \overline{\{c : a \prec c\}} \Leftrightarrow a \in \overline{\{c : c \prec b\}}.$$

- For a normal kinematic space, we denote $b \in \overline{\{c : a \prec c\}}$ by $a \preccurlyeq b$.

Remark 1. It has been proven that $a \prec b \preccurlyeq c$ or $a \preccurlyeq b \prec c$ imply $a \prec c$ [10].

Symmetry: a fundamental property of the physical world. One of the main objectives of science is prediction. The main basis for prediction is that we have observed similar situations in the past, and thus we expect similar outcomes as in those past situations.

In mathematical terms, similarity corresponds to symmetry, and similarity of outcomes – to invariance. For example, suppose that I dropped the ball, then it fall down. I can then shift my position, and drop the ball again. It is reasonable to expect that the ball will fall, and that its trajectory will be the same as before – to be more precise, obtained by shift from the previous one. Similarly, if I rotate myself 90 degrees and drop the ball again, I will get the same trajectory but rotated by 90 degrees.

The notion of symmetry is very important in modern physics, to the extent that, starting with the quarks, physical theories are usually formulated in terms of symmetries – and not in terms of differential equations as in the past; see, e.g., [7].

In particular, an important symmetry is *T-transformation*, a symmetry with respect to reversal of time $t \to -t$. One important property of this transformation is that if we apply it twice, we get the same point back. Another property is that T-transformation reverses the order of causality. Thus, we arrive at the following definition.

Definition 2. A 1-1 mapping $t : X \to X$ of a kinematic space onto itself is called a *T-transformation* if $t(t(a)) = a$ for all a and $a \preccurlyeq b \Leftrightarrow t(b) \preccurlyeq t(a)$.

2. de Vries Algebras and their Relation to Kinematic Spaces

de Vries algebras. In mathematics, there is another case when we have a set with two orders: the case of so-called *de Vries algebras*. The original example of such an object is the class $\mathcal{R}X$ of all regular open subsets of a compact Hausdorff space X, i.e., open subsets A for which the interior $\text{Int}\left(\overline{A}\right)$ of the closure $\overline{A}$ coincides with the original set A.

On $\mathcal{R}X$, we can define $A \preccurlyeq B \Leftrightarrow A \subseteq B$ and $A \prec B \Leftrightarrow \overline{A} \subseteq B$. One can check that the class $\mathcal{R}X$ with the relation $\preccurlyeq$ is a complete Boolean algebra, with negation $\neg A \stackrel{\text{def}}{=} \text{Int}(X - A)$. In general, a de Vries algebra is defined as a Boolean algebra with an additional relation $\prec$ that satisfies the same properties as the algebra $\mathcal{R}X$. This idea leads to the following definition [3–5]:

Definition 3. A *de Vries algebra* is a pair consisting of a complete Boolean algebra $(B, \preccurlyeq)$ with the relation $\preccurlyeq$ (*precedes*) and a binary relation $\prec$ (*strictly precedes*) for which:

- $1 \prec 1$;
- $a \prec b$ implies $a \preccurlyeq b$;
- $a \preccurlyeq b \prec c \preccurlyeq d$ implies $a \prec d$;

- $a \prec b, c$ implies $a \prec b \wedge c$;
- $a \prec b$ implies $\neg b \prec \neg a$;
- $a \prec b$ implies that there exists a c such that $a \prec c \prec b$;
- $a \neq 0$ implies that there exists a $b \neq 0$ such that $b \prec a$.

Possible relations with kinematic spaces: discussion. In a kinematic space, if we associate with every element a an open cone K_a^+, then we get $a \preccurlyeq b \Leftrightarrow K_b^+ \subseteq K_a^+$ and $a \prec b \Leftrightarrow \overline{K_b^+} \subseteq K_a^+$ [10].

Please note that we need to be cautious about this observation, since standard examples of de Vries algebras come from a compact space X, while a kinematic space is never compact [10]. However, this observation can indeed be transformed into a formal relation between kinematic spaces and de Vries algebras. To describe this formal relation, we need to introduce the following auxiliary definitions.

Definition 4. We say that a de Vries algebra is *connected* if $a \prec a$ implies that $a = 0$ or $a = 1$.

Remark 2. The name comes from the fact that, as it is easy to check, for an algebra $\mathcal{R}X$, this is indeed equivalent to connectedness of the topological space X, i.e., to the fact that the space X cannot be represented as a union of two disjoint open sets (which are different from X and $\emptyset$).

Theorem 1. *For every connected de Vries algebra B:*

- *the set $B - \{0, 1\}$ with a proximity relation $\prec$ is a normal kinematic space, and*
- *the original relation $\preccurlyeq$ coincides with the closure of $\prec$ in the sense of kinematic spaces.*

Proof. 1°. Let us first prove that for every $a \in B - \{0, 1\}$, there exists a point $a^- \in B - \{0, 1\}$ for which $a^- \prec a$.

Indeed, by the definition of de Vries algebra, since $a \neq 0$, there exists an $b \neq 0$ for which $b \prec a$. We will show that this b is the desired a^-. We already know that $b \prec a$ and that $b \neq 0$. So, to complete the proof, it is sufficient to show that $b \neq 1$.

We can prove the inequality $b \neq 1$ by contradiction. Indeed, if $b = 1$, then from $b \prec a$, we would be able to conclude that $1 \prec a$. Since $a \preceq 1$, from $a \preceq 1 \prec a$, we would then conclude that $a \prec a$, which contradicts to our assumption that the de Vries algebra B is connected.

2°. Let us now prove that for every $a \in B - \{0, 1\}$, there exists a point $a^+ \in B - \{0, 1\}$ for which $a \prec a^+$.

Let us use an auxiliary element $b \stackrel{\text{def}}{=} \neg a$. Since $\neg$ is a 1-1 mapping and it maps 0 to 1 and vice versa, we conclude that b is different from 0 and 1, i.e., that $b \in B - \{0, 1\}$. Thus, due to Part 1 of this proof, there exists a value $b^- \in B - \{0, 1\}$ for which $b^- \prec b$.

By definition of the de Vries algebra, this implies $\neg b \prec \neg b^-$, i.e., $a \prec a^+ \stackrel{\rm def}{=} \neg b^-$. Due to $b^- \in B - \{0,1\}$, we get $a^+ = \neg b^- \in B - \{0,1\}$. The statement is proven.

3°. If $a \prec b$, then the existence of a c for which $a \prec c \prec b$ follows directly from the definition of a de Vries algebra.

4°. Let us prove that if $a \prec b$ and $a \prec c$, then there exists a d for which $a \prec d \prec b, c$.

Indeed, due to the properties of a de Vries algebra, $a \prec b, c$ implies that $a \prec b \wedge c$, where $b \wedge c \preccurlyeq b, c$. Due to Part 3 of this proof, there exists a d for which $a \prec d \prec b \wedge c$. From $d \prec b \wedge c \preccurlyeq b, c$, we conclude that $d \prec b, c$. The statement is proven.

5°. The dual statement, that if $b \prec b$ and $c \prec a$, then there exists a d for which $c, d \prec d \prec a$, follows from Part 4 of this proof by considering the values $\neg a$, $\neg b$, and $\neg c$ (just like we reduced Part 2 of this proof to Part 1).

6°. So, the set $B - \{0,1\}$ is indeed a kinematic space. Thus, intervals form a basis of a topology. Let us prove that $\preccurlyeq$ is indeed a closure of $\prec$ in this topology, and that this kinematic space is normal.

7°. Let us prove that in $B - \{0,1\}$, we have

$$a \preccurlyeq b \Leftrightarrow \forall c\, (b \prec c \rightarrow a \prec c).$$

Indeed, if $a \preccurlyeq b$ and $b \prec c$, then, due to the definition of a de Vries algebra, we have $a \prec c$. Vice versa, let us assume that $\forall c\, (b \prec c \rightarrow a \prec c)$, i.e., that a strictly precedes ($\prec$) all the elements that b strictly precedes. It is known that in a de Vries algebra, we have $b = \vee\{c : b \prec c\}$ [3–5]. Since a strictly precedes all elements of the set $\{c : b \prec c\}$, it thus precedes ($\preccurlyeq$) all these elements and thus, precedes their infimum b: $a \prec b$.

8°. By using duality, we can now prove that

$$a \preccurlyeq b \Leftrightarrow \forall c\, (c \prec a \rightarrow c \prec b).$$

9°. From Parts 7 and 8 of this proof, we can now conclude, by using known results about kinematic spaces [10], that for every a:

- the cone $C_a^+ = \{b : a \preccurlyeq b\}$ is equal to the closure of the cone
$$K_a^+ = \{b : a \prec b\}, \text{ and}$$
- the cone $C_a^- = \{b : b \preccurlyeq a\}$ is equal to the closure of the cone
$$K_a^- = \{b : b \prec a\}.$$

The fact that $b \in K^+a \Leftrightarrow a \preccurlyeq b \Leftrightarrow a \in K^-_b$ means that the kinematic space $(B - \{0,1\}, \prec)$ is normal.

The theorem is proven. ■

The inverse is also true, in the following sense.

Theorem 2. *Let S be a normal kinematic space, let t be a T-transformation, and let us assume that when we add the smallest element 0 and the largest element 1 to the corresponding set $(S, \preccurlyeq)$, we get a complete Boolean algebra, with t as negation. In this case, if we extent $\prec$ to $S \cup \{0,1\}$ by taking and that $0 \prec a \prec 1$ for all a, then $(S \cup \{0,1\}, \preccurlyeq, \prec)$ becomes a connected de Vries algebra.*

Proof. Let us prove the properties of de Vries algebra one by one.

1°. The property $1 \prec 1$ follows from our definition of the order $\prec$.

2°. For $a, b \in S$, the desired property – that $a \prec b$ implies $a \preccurlyeq b$ – comes from the known properties of a kinematic space. When in the pair (a, b), at least one of the elements a or b is equal to 0 or 1, this implication follows from our definitions of $\prec$ and $\preccurlyeq$ for such pairs.

3°. For $a, b, c \in S$, the desired property – that $a \preccurlyeq b \prec c \preccurlyeq d$ implies $a \prec d$ – follows from the above-mentioned results about kinematic spaces. When at least one of the elements a, b, c, or d is equal to 0 or 1, this implication follows from the above results and from our definitions of $\prec$ and $\preccurlyeq$ for the pairs containing 0 or 1.

4°. When $a \prec b, c$ for $a, b, c \in S$, then, according to the definition of a kinematic space, there exists a d for which $a \prec d \prec b, c$. Since $\prec$ implies $\preccurlyeq$, we have $d \preccurlyeq b, c$ and thus, $a \preccurlyeq b \wedge c$. From $a \prec d \preccurlyeq b \wedge c$, we conclude that $a \prec b \wedge c$. This inequality is also easy to prove when one of the elements a, b, and c coincides with 0 or 1.

5°. For $a, b \in S$, the desired property – that $a \prec b$ implies $\neg b \prec \neg a$ – follows from the fact that t is a T-transformation. When in the pair (a, b), at least one of the elements a or b is equal to 0 or 1, this implication follows from our definition of $\prec$ for such pairs.

6°. When $a \prec b$ for $a, b \in S$, then, according to the definition of a kinematic space, there exists a c for which $a \prec c \prec b$. This inequality is also easy to prove when one of the elements a or b coincides with 0 or 1: e.g., if $0 \prec a$, then $0 \prec 0 \prec a$, so we can take $c = 0$. If $a \prec 1$, then $a \prec 1 \prec 1$, so we can take $c = 1$.

7°. Let us assume that $a \neq 0$, We need to prove that there exists $b \neq 0$ for which $b \prec a$. To prove this property, let us consider two cases: $a = 1$ and $a \neq 1$.

7.1°. If $a = 1$, we can take $b = 1$. In this case, $1 \neq 0$ and $1 \prec 1$ (due to Part 1 of this proof).

7.2°. If $a \neq 1$, then, since a is also different from 0, the element a belongs to the original set S. Thus, due to the definition of a kinematic space, there exists an element $a^- \in S$ for which $a^- \prec a$. This element is different from 0, so we can take it as the desired element b.

The theorem is proven. ■

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