

Membership Functions or α -Cuts? Algorithmic (Constructivist) Analysis Justifies an Interval Approach

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Abstract

In his pioneering papers, Igor Zaslavsky started an algorithmic (constructivist) analysis of fuzzy logic. In this paper, we extend this analysis to fuzzy mathematics and fuzzy data processing. Specifically, we show that the two mathematically equivalent representations of a fuzzy number – by a membership function and by α -cuts – are *not* algorithmically equivalent, and only the α -cut representation enables us to efficiently process fuzzy data.

1 First Result: Two Representations Are Not Equivalent

Definition 1. By a *c*-membership function, we mean a tuple consisting of two real numbers $\underline{\Delta}$ and $\overline{\Delta}$ and a function $\mu : [\underline{\Delta}, \overline{\Delta}] \rightarrow [0, 1]$ for which $\mu(\underline{\Delta}) = \mu(\overline{\Delta}) = 0$, $\max_x \mu(x) = 1$, and $a \leq b \leq c$ implies that $\mu(b) \geq \min(\mu(a), \mu(c))$.

Definition 2. We say that a *c*-membership function $\langle \underline{\Delta}, \overline{\Delta}, \mu \rangle$ is computable if both real numbers $\underline{\Delta}$ and $\overline{\Delta}$ are computable and the function μ is computable.

Definition 3. By a family of α -cuts (or simply α -cuts, for short) corresponding to a *c*-membership function μ , we mean a pair of mappings $\underline{x} : [0, 1] \rightarrow \mathbb{R}$ and $\overline{x} : [0, 1] \rightarrow \mathbb{R}$ for which, for every $\alpha \in [0, 1]$, we have $\{x : \mu(x) \geq \alpha\} = [\underline{x}(\alpha), \overline{x}(\alpha)]$.

Definition 4. We say that α -cuts are computable if both mapping \underline{x} and \overline{x} are computable.

Proposition 1. There exists a computable *c*-membership function for which the corresponding α -cuts are not computable.

Proposition 2. There exist computable α -cuts for which the corresponding *c*-membership function is not computable.

2 Only α -Cuts Guarantee Algorithmic Fuzzy Data Processing

Since the two representations of fuzzy are not computationally equivalent, it is desirable to analyze which of them leads to an algorithmic fuzzy data processing. Here are the results of this analysis: fuzzy data processing is computable for α -cuts but, in general, not computable for membership functions.

Definition 5. Let μ_1, \dots, μ_n be membership functions, and let $f(x_1, \dots, x_n)$ be a function. By the result of applying f to fuzzy sets μ_1, \dots, μ_n , we mean a membership function defined by the formula $\mu(y) = \max_{x_1, \dots, x_n: y=f(x_1, \dots, x_n)} \min(\mu_1(x_1), \dots, \mu_n(x_n))$.

Proposition 3. There exists a computable c -membership function $\mu_1(x_1)$ and a computable function $f(x_1)$ for which the result μ of applying f to μ_1 is not computable.

Proposition 4. There exists an algorithm that, given n computable families of α -cuts corresponding to the membership functions μ_1, \dots, μ_n and a computable function $f(x_1, \dots, x_n)$, returns computable α -cuts for the result μ of applying f to μ_1, \dots, μ_n .

3 Auxiliary Result: Why min and Not Any Other And-Operation

We want all the property to satisfy the “convexity” condition, that if $a \leq b \leq c$, then $\mu(b) \geq \min(\mu(a), \mu(c))$. Sometimes, we know that the actual value x satisfies *two* properties S' and S'' characterized by membership functions $\mu'(x)$ and $\mu''(x)$; then, the degree $\mu(x)$ to which a real number x is consistent with this information can be described as $\mu(x) = f_{\&}(\mu'(x), \mu''(x))$. It is reasonable to require that this combined property should also be “convex” (in the above sense).

Definition 6. A function $\mu : \mathbb{R} \rightarrow [0, 1]$ is called *f-convex* if $a \leq b \leq c$ implies that $\mu(b) \geq \min(\mu(a), \mu(c))$.

Definition 7. By a generalized and-operation, we mean a function $f : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which satisfies the following two properties:

- for all a, a', b , and b' , if $a \leq a'$ and $b \leq b'$, then $f(a, b) \leq f(a', b')$ (monotonicity);
- for all a , we have $f(a, 1) = f(1, a) = a$.

Proposition 5. Let $f(a, b)$ be a generalized and-operation. Then, the following two conditions are equivalent to each other:

- for every two *f-convex* functions $\mu'(x)$ and $\mu''(x)$, the function $\mu(x) = f(\mu'(x), \mu''(x))$ is also *f-convex*;
- $f(a, b) = \min(a, b)$.

References

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