# Decision Making under Interval and Fuzzy Uncertainty: Towards an Operational Approach

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**Abstract**

Traditional decision theory is based on a simplifying assumption that for each two alternatives, a user can always meaningfully decide which of them is preferable. In reality, often, when the alternatives are close, the user is either completely unable to select one of these alternatives, or selects one of the alternatives only "to some extent". How can we extend the traditional decision theory to such realistic interval and fuzzy cases? In their previous papers, the first two authors proposed a natural generalization of the usual decision theory axioms to interval and fuzzy cases, and described decision coming from this generalization. In this paper, we make the resulting decisions more intuitive by providing commonsense operational explanation.

**Introduction.** Traditional decision theory is based on a simplifying assumption that for each two alternatives, a user can always meaningfully decide which of them is preferable. In reality, often, when the alternatives are close, the user is either completely unable to select one of these alternatives, or selects one of the alternatives only "to some extent". How can we extend the traditional decision theory to such realistic interval and fuzzy cases?

In their papers [1, 2, 3, 4, 5, 6, 7, 8, 9], summarized in [10], the first two authors proposed a natural generalization of the usual decision theory axioms to interval and fuzzy cases, and described decision coming from this generalization.

In this paper, we make the resulting decisions more intuitive by providing commonsense operational explanation. This paper is structured as follows: first, we recall the main ideas and results of the traditional decision theory. We then consider the case when in addition to deciding which of the two alternatives is better, the user can also reply that he/she is unable to decide between the two close alternatives; this leads to interval uncertainty. Finally, we consider the general case when the user makes fuzzy statements about preferences.

**Traditional decision theory: brief reminder.** Following [11, 13, 14], let us describe the main ideas and results of the traditional decision theory.

**Main assumption behind the traditional decision theory.** Let us assume that for every two alternatives $A'$ and $A''$, a user can tell:

- whether the first alternative is better for him/her; we will denote this by $A'' < A'$;

- or the second alternative is better; we will denote this by $A' < A''$;
- or the two given alternatives are of equal value to the user; we will denote this by $A' = A''$.

**The notion of utility.** Under the above assumption, we can form a natural numerical scale for describing attractiveness of different alternatives. Namely, let us select a very bad alternative $A_0$ and a very good alternative $A_1$, so that most other alternatives are better than $A_0$ but worse than $A_1$. Then, for every probability $p \in [0,1]$, we can form a lottery $L(p)$ in which we get $A_1$ with probability $p$ and $A_0$ with the remaining probability $1-p$.

When $p = 0$, this lottery simply coincides with the alternative $A_0$: $L(0) = A_0$. The larger the probability $p$ of the positive outcome increases, the better the result, i.e., $p' < p''$ implies $L(p') < L(p'')$. Finally, for $p = 1$, the lottery coincides with the alternative $A_1$: $L(1) = A_1$. Thus, we have a continuous scale of alternatives $L(p)$ that monotonically goes from $A_0$ to $A_1$.

We have assumed that most alternatives $A$ are better than $A_0$ but worse than $A_1$: $A_0 < A < A_1$. Since $A_0 = L(0)$ and $A_1 = L(1)$, for such alternatives, we thus get $L(0) < A < L(1)$. We assumed that every two alternatives can be compared. Thus, for each such alternative $A$, there can be at most one value $p$ for which $L(p) = A$; for others, we have $L(p) < A$ or $L(p) > A$. Due to monotonicity of $L(p)$ and transitivity of preference, if $L(p) < A$, then $L(p') < A$ for all $p' \le p$; similarly, if $A < L(p)$, then $A < L(p')$ for all $p' > p$. Thus, the supremum (= least upper bound) $u(A)$ of the set of all $p$ for which $L(p) < A$ coincides with the infimum (= greatest lower bound) of the set of all $p$ for which $A < L(p)$. For $p < u(A)$, we have $L(p) < A$, and for for $p > u(A)$, we have $A < L(p)$. This value $u(A)$ is called the *utility* of the alternative $A$.

It may be possible that $A$ is equivalent to $L(u(A))$; however, it is also possible that $A \ne L(u(A))$. However, the difference between $A$ and $L(u(A))$ is extremely small: indeed, no matter how small the value $\varepsilon > 0$, we have $L(u(A)-\varepsilon) < A < L(u(A)+\varepsilon)$. We will describe such (almost) equivalence by $\equiv$, i.e., we write that $A \equiv L(u(A))$.

**How can we actually find utility values.** The above definition of utility is somewhat theoretical, but in reality, utility can be found reasonably fast by the following iterative bisection procedure.

We want to find the probability $u(A)$ for which $L(u(A)) \equiv A$. On each stage of this procedure, we have the values $\underline{u} < \overline{u}$ for which $L(\underline{u}) < A < L(\overline{u})$. In the beginning, we have $\underline{u} = 0$ and $\overline{u} = 1$, with $|\overline{u} - \underline{u}| = 1$.

To find the desired probability $u(A)$, we compute the midpoint $\widetilde{u} = \dfrac{\underline{u} + \overline{u}}{2}$ and compare the alternative $A$ with the corresponding lottery $L(\widetilde{u})$. Based on our assumption, there are three possible results of this comparison:

- if the user concludes that $L(\widetilde{u}) < A$, then we can replace the previous lower bound $\underline{u}$ with the new one $\widetilde{p}$;
- if the user concludes that $A < L(\widetilde{u})$, then we can replace the original upper bound $\overline{u}$ with the new one $\widetilde{u}$;
- finally, if $A = L(\widetilde{u})$, this means that we have found the desired probability $u(A)$.

In this third case, we have found $u(A)$, so the procedure stops. In the first two cases, the new distance between the bounds $\underline{u}$ and $\overline{u}$ is the half of the original distance. By applying this procedure $k$ times, we get values $\underline{u}$ and $\overline{u}$ for which $L(\underline{u}) < A < L(\overline{u})$ and $|\overline{u} - \underline{u}| \le 2^{-k}$. One can easily check that the desired value $u(A)$ is within the interval $[\underline{u}, \overline{u}]$, so the midpoint $\widetilde{u}$ of this interval is an $2^{-(k+1)}$-approximation to the desired utility value $u(A)$.

In other words, for any given accuracy, we can efficiently find the corresponding approximation to the utility $u(A)$ of the alternative $A$.

**How to make a decision based on utility values.** If we know the utilities $u(A')$ and $u(A'')$ of the alternatives $A'$ and $A''$, then which of these alternatives should we choose? By definition of utility, we have $A' \equiv L(u(A'))$ and $A'' \equiv L(u(A''))$. Since $L(p') < L(p'')$ if and only if $p' < p''$, we can thus conclude that $A'$ is preferable to $A''$ if and only if $u(A') > u(A'')$. In other words, we should always select an alternative with the largest possible value of utility.

**How to estimate utility of an action: why expected utility.** To apply the above idea to decision making, we need to be able to compute utility of different actions. For each action, we usually know possible outcomes $S_1, \ldots, S_n$, and we can often estimate the probabilities $p_1, \ldots, p_n$, $\sum\limits_{i=1}^{n} p_i = 1$, of these outcomes. Let $u(S_1), \ldots, u(S_n)$ be utilities of the situations $S_1, \ldots, S_n$. What is then the utility of the action?

By definition of utility, each situation $S_i$ is equivalent (in the sense of the relation $\equiv$) to a lottery $L(u(S_i))$ in which we get $A_1$ with probability $u(S_i)$ and $A_0$ with the remaining probability $1 - u(S_i)$. Thus, the action in which we get $S_i$ with probability $p_i$ is equivalent to complex lottery in which:

- first, we select one of the situations $S_i$ with probability $p_i$: $P(S_i) = p_i$;
- then, depending on the selected situation $S_i$, we get $A_1$ with probability $u(S_i)$ and $A_0$ with probability $1 - u(S_i)$: $P(A_1 \mid S_i) = u(S_i)$ and $P(A_0 \mid S_i) = 1 - u(S_i)$.

In this complex lottery, we end up either with the alternative $A_1$ or with the alternative $A_0$. The probability of getting $A_1$ can be computed by using the complete probability formula:

$$P(A_1) = \sum_{i=1}^{n} P(A_1 \mid S_i) \cdot P(S_i) = \sum_{i=1}^{n} u(S_i) \cdot p_i.$$

Thus, the original action is equivalent to a lottery in which we get $A_1$ with probability $\sum\limits_{i=1}^{n} p_i \cdot u(S_i)$ and $A_0$ with the remaining probability. By definition of utility, this means that the utility of our action is equal to $\sum\limits_{i=1}^{n} p_i \cdot u(S_i)$.

In probability theory, this sum is known as the expected value of utility $u(S_i)$. Thus, we can conclude that the utility of each action is equal to its expected utility; in other words, among several possible actions, we should select the one with the largest value of expected utility.

**Non-uniqueness of utility.** The above definition of utility depends on a selection of two alternatives $A_0$ and $A_1$. What if we select different alternatives $A'_0$ and $A'_1$? How will utility change? In other words, if $A$ is an alternative with utility $u(A)$ in the scale determined by $A_0$ and $A_1$, what is its utility $u'(A)$ in the scale determined by $A'_0$ and $A'_1$?

Let us first consider the case when $A'_0 < A_0 < A_1 < A'_1$. In this case, since $A_0$ is in between $A'_0$ and $A'_1$, for each of them, there exists a probability $u'(A_0)$ for which $A_0$ is equivalent to a lottery $L'(u'(A_0))$ in which we get $A'_1$ with probability $u'(A_0)$ and $A'_0$ with the remaining probability

$1 - u'(A_0)$. Similarly, there exists a probability $u'(A_1)$ for which $A_1$ is equivalent to a lottery $L'(u'(A_1))$ in which we get $A'_1$ with probability $u'(A_1)$ and $A'_0$ with the remaining probability $1 - u'(A_1)$.

By definition of the utility $u(A)$, the original alternative $A$ is equivalent to a lottery in which we get $A_1$ with probability $u(A)$ and $A_0$ with the remaining probability $1-u(A)$. Here, $A_1$ is equivalent to the lottery $L'(u'(A_1))$, and $A_0$ is equivalent to the lottery $L'(u'(A_0))$. Thus, the alternative $A$ is equivalent to a complex lottery, in which:

- first, we select $A_1$ with probability $u(A)$ and $A_0$ with probability $1 - u(A)$;
- then, depending on the selection $A_i$, we get $A'_1$ with probability $u'(A_i)$ and $A'_0$ with the remaining probability $1 - u'(A_i)$.

In this complex lottery, we end up either with the alternative $A'_1$ or with the alternative $A'_0$. The probability $u'(A) = P(A'_1)$ of getting $A'_1$ can be computed by using the complete probability formula:

$$u'(A) = P(A'_1) = P(A'_1 \mid A_1) \cdot P(A_1) + P(A'_1 \mid A_0) \cdot P(A_0) = u'(A_1) \cdot u(A) + u'(A_0) \cdot (1 - u(A)) =$$

$$u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0).$$

Thus, the original alternative $A$ is equivalent to a lottery in which we get $A'_1$ with probability $u'(A) = u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0)$. By definition of utility, this means that the utility $u'(A)$ of the alternative $A$ in the scale determined by the alternatives $A'_0$ and $A'_1$ is equal to $u'(A) = u(A) \cdot (u'(A_1) - u'(A_0)) + u'(A_0)$.

Thus, in the case when $A'_0 < A_0 < A_1 < A'_1$, when we change the alternatives $A_0$ and $A_1$, the new utility values are obtained from the old ones by a linear transformation. In other cases, we can use auxiliary events $A''_0$ and $A''_1$ for which $A''_0 < A_0, A'_0$ and $A_1, A'_1 < A''_1$. In this case, as we have proven, transformation from $u(A)$ to $u''(A)$ is linear and transformation from $u'(A)$ to $u''(A)$ is also linear. Thus, by combining linear transformations $u(A) \to u''(A)$ and $u''(A) \to u'(A)$, we can conclude that the transformation $u(A) \to u'(A)$ is also linear.

So, in general, utility is defined modulo an (increasing) linear transformation $u' = a \cdot u + b$, with $a > 0$.

*Comment.* So far, once we have selected alternatives $A_0$ and $A_1$, we have defined the corresponding utility values $u(A)$ only for alternatives $A$ for which $A_0 < A < A_1$. For such alternatives, the utility value is always a number from the interval $[0, 1]$.

For other alternatives, we can define their utility $u'(A)$ with respect to different pairs $A'_0$ and $A'_1$, and then apply the corresponding linear transformation to re-scale to the original units. The resulting utility value $u(A)$ can now be an arbitrary real number.

**Subjective probabilities.** In our derivation of expected utility, we assumed that we know the probabilities $p_i$ of different outcomes. In practice, we often do not know these probabilities, we have to rely on a subjective evaluation of these probabilities. For each event $E$, a natural way to estimate its subjective probability is to compare the lottery $\ell(E)$ in which we get a fixed prize (e.g., \$1) if the event $E$ occurs and 0 is it does not occur, with a lottery $\ell(p)$ in which we get the same amount with probability $p$. Here, similarly to the utility case, we get a value $ps(E)$ for which $\ell(E)$ is (almost) equivalent to $\ell(ps(E))$ in the sense that $\ell(ps(E) - \varepsilon) < \ell(E) < \ell(ps(E) + \varepsilon)$ for every $\varepsilon > 0$. This value $ps(E)$ is called the *subjective probability* of the event $E$.

From the viewpoint of decision making, each event $E$ is equivalent to an event occurring with the probability $ps(E)$. Thus, if an action has $n$ possible outcomes $S_1, \ldots, S_n$, in which $S_i$ happens if the event $E_i$ occurs, then the utility of this action is equal to $\sum\limits_{i=1}^{n} ps(E_i) \cdot u(S_i)$.

**Beyond traditional decision making: towards a more realistic description.** Instead of assuming that a user can always decide which of the two alternatives $A'$ and $A''$ is better, let us now consider a more realistic situation in which a user is allowed to say that he or she is unable to meaningfully decide between the two alternatives; we will denote this option by $A' \parallel A''$.

In mathematical terms, this means that the preference relation is no longer a *total* (linear) order, it can be a *partial* order.

**From utility to interval-valued utility.** Similarly to the traditional decision making approach, we can select two alternatives $A_0 < A_1$ and compare each alternative $A$ which is better than $A_0$ and worse than $A_1$ with lotteries $L(p)$. The main difference is that here, the supremum $\underline{u}(A)$ of all the values $p$ for which $L(p) < A$ is, in general, smaller than the infimum $\overline{u}(A)$ of all the values $p$ for which $A < L(p)$. Thus, for each alternative $A$, instead of a single value $u(A)$ of the utility, we now have an *interval* $[\underline{u}(A), \overline{u}(A)]$ such that:

- if $p < \underline{u}(A)$, then $L(p) < A$;
- if $p > \overline{u}(A)$, then $A < L(p)$; and
- if $\underline{u}(A) < p < \overline{u}(A)$, then $A \parallel L(p)$.

We will call this interval the *utility* of the alternative $A$.

**How to efficiently find the interval-valued utility.** To elicit the corresponding utility interval from the user, we can use a slightly modified version of the above bisection procedure. At first, the procedure is the same as before: namely, we produce a narrowing interval $[\underline{u}, \overline{u}]$ for which $L(\underline{u}) < A < L(\overline{u})$.

We start with the interval $[\underline{u}, \overline{u}] = [0, 1]$, and we repeatedly compute the midpoint $\widetilde{u} = \dfrac{\underline{u} + \overline{u}}{2}$ and compare $A$ with $L(\widetilde{u})$. If $L(\widetilde{u}) < A$, we replace $\underline{u}$ with $\widetilde{u}$; if $A < L(\widetilde{u})$, we replace $\overline{u}$ with $\widetilde{u}$. If we get $A \parallel L(\widetilde{p})$, then we switch to the new second stage of the iterative algorithm. Namely, now, we have *two* intervals:

- an interval $[\underline{u}_1, \overline{u}_1]$ (which is currently equal to $[\underline{u}, \widetilde{u}]$) for which $L(\underline{u}_1) < A$ and $L(\widetilde{u}_1) \parallel A$, and
- an interval $[\underline{u}_2, \overline{u}_2]$ (which is currently equal to $[\widetilde{u}, \overline{u}]$) for which $L(\underline{u}_2) \parallel A$ and $A < L(\overline{u}_2)$.

Then, we perform bisection of each of these two intervals. For the first interval, we compute the midpoint $\widetilde{u}_1 = \dfrac{\underline{u}_1 + \overline{u}_1}{2}$, and compare the alternative $A$ with the lottery $L(\widetilde{u}_1)$:

- if $L(\widetilde{u}_1) < A$, then we replace $\underline{u}_1$ with $\widetilde{u}_1$;
- if $L(\widetilde{u}_1) \parallel A$, then we replace $\overline{u}_1$ with $\widetilde{u}_1$.

As a result, after $k$ iterations, we get the value $\underline{u}(A)$ with accuracy $2^{-k}$.

Similarly, for the second interval, we compute the midpoint $\widetilde{u}_2 = \dfrac{\underline{u}_2 + \overline{u}_2}{2}$, and compare the alternative $A$ with the lottery $L(\widetilde{u}_2)$:

- if $L(\widetilde{u}_2) \parallel A$, then we replace $\underline{u}_2$ with $\widetilde{u}_2$;
- if $A < L(\widetilde{u}_2)$, then we replace $\overline{u}_2$ with $\widetilde{u}_2$.

As a result, after $k$ iterations, we get the value $\overline{u}(A)$ with accuracy $2^{-k}$.

**Interval-valued subjective probability.** Similarly, when we are trying to estimate the probability of an event $E$, we no longer get a single value $ps(E)$, we get an interval $[\underline{ps}(E), \overline{ps}(E)]$ of possible values of probability.

**Need for decision making under interval uncertainty.** In the traditional approach, for each alternative $A$, we produce a number $u(A)$ – the utility of this alternative. Then, an alternative $A'$ is preferable to the alternative $A''$ if and only if $u(A') > u(A'')$.

How can we make a similar decision in situations when we only know interval-valued probabilities? At first glance, the situation may sound straightforward: if $A' \parallel A''$, it does not matter whether we select $A'$ or $A''$. However, this is *not* a good way to make a decision. For example, let us assume that there is an alternative $A$ about which we know nothing. In this case, we have no reason to prefer $A$ or $L(p)$, so we have $A \parallel L(p)$ for all $p$. By definition of $\underline{u}(A)$ and $\overline{u}(A)$, this means that we have $\underline{u}(A) = 0$ and $\overline{u}(A) = 1$, i.e., the alternative $A$ is characterized by the utility interval $[0, 1]$.

In this case, the alternative $A$ is indistinguishable both from a good lottery $L(0.999)$ (in which the good alternative $A_1$ appears with probability 99.9%) and from a bad lottery $L(0.001)$ (in which the bad alternative $A_0$ appears with probability 99.9%). If we recommend, to the user, that $A$ is equivalent both to to $L(0.999)$ and $L(0.001)$, then this user will feel comfortable exchanging his chance to play in the good lottery with $A$, and then – following the same logic – exchanging $A$ with a chance to play in a bad lottery. As a result, following our recommendations, the user switches from a very good alternative to a very bad one.

This argument does not depend on the fact that we assumed complete ignorance about $A$. Every time we recommend that the alternative $A$ is equivalent to $L(p)$ and $L(p')$ with two different values $p < p'$, we make the user vulnerable to a similar switch from a better alternative $L(p')$ to a worse one $L(p)$. Thus, there should be only a single value $p$ for which $A$ can be reasonably exchanged with $L(p)$.

In precise terms: we start with the utility interval $[\underline{u}(A), \overline{u}(A)]$, and we need to select a single utility value $u$ for which it is reasonable to exchange the alternative $A$ with a lottery $L(u)$. How can we find this value $u$?

**How to make decisions under interval uncertainty: Hurwicz optimism-pessimism criterion.** The problem of decision making under such interval uncertainty was first handled by the future Nobelist L. Hurwicz in [12].

We need to assign, to each interval $[\underline{u}, \overline{u}]$, a utility value $u(\underline{u}, \overline{u})$.

No matter what value $u$ we get from this interval, this value will be larger than or equal to $\underline{u}$ and smaller than or equal to $\overline{u}$. Thus, the equivalent utility value $u(\underline{u}, \overline{u})$ must satisfy the same inequalities: $\underline{u} \le u(\underline{u}, \overline{u}) \le \overline{u}$. In particular, for $\underline{u} = 0$ and $\overline{u} = 1$, we get $0 \le \alpha_H \le 1$, where we denoted $\alpha_H \stackrel{\text{def}}{=} u(0, 1)$.

We have mentioned that the utility is determined modulo a linear transformation $u' = a \cdot u + b$. It is therefore reasonable to require that the equivalent utility does not depend on what scale we use, i.e., that for every $a > 0$ and $b$, we have

$$u(a \cdot \underline{a} + b, a \cdot \overline{u} + b) = a \cdot u(\underline{u}, \overline{u}) + b.$$

In particular, for $\underline{u} = 0$ and $\overline{u} = 1$, we get

$$u(b, a + b) = a \cdot u(0, 1) + b = a \cdot \alpha_H + b.$$

So, for every $\underline{u}$ and $\overline{u}$, we can take $b = \underline{u}$, $a = \overline{u} - \underline{u}$, and get

$$u(\underline{u}, \overline{u}) = \underline{u} + \alpha_H \cdot (\overline{u} - \underline{u}) = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}.$$

This expression is called *Hurwicz optimism-pessimism criterion*, because:

- when $\alpha_H = 1$, we make a decision based on the most optimistic possible values $u = \overline{u}$;
- when $\alpha_H = 0$, we make a decision based on the most pessimistic possible values $u = \underline{u}$;
- for intermediate values $\alpha_H \in (0, 1)$, we take a weighted average of the optimistic and pessimistic values.

So, if we have two alternatives $A'$ and $A''$ with interval-valued utilities $[\underline{u}(A'), \overline{u}(A')]$ and $[\underline{u}(A''), \overline{u}(A'')]$, we recommend an alternative for which the equivalent utility value is the largest. In other words, we recommend to select $A'$ if $\alpha_H \cdot \overline{u}(A') + (1 - \alpha_H) \cdot \underline{u}(A') > \alpha_H \cdot \overline{u}(A'') + (1 - \alpha_H) \cdot \underline{u}(A'')$ and $A''$ otherwise.

**Which value $\alpha_H$ should we choose? An argument in favor of $\alpha_H = 0.5$.** Which value $\alpha_H$ should we choose?

To answer this question, let us take an event $E$ about which we know nothing. For a lottery $L^+$ in which we get $A_1$ if $E$ and $A_0$ otherwise, the utility interval is $[0, 1]$, thus, from a decision making viewpoint, this lottery should be equivalent to an event with utility $\alpha_H \cdot 1 + (1 - \alpha_H) \cdot 0 = \alpha_H$.

Similarly, for a lottery $L^-$ in which we get $A_0$ if $E$ and $A_1$ otherwise, the utility interval is $[0, 1]$, thus, this lottery should also be equivalent to an event with utility $\alpha_H \cdot 1 + (1 - \alpha_H) \cdot 0 = \alpha_H$.

We can now combine these two lotteries into a single complex lottery, in which we select either $L^+$ or $L^-$ with equal probability 0.5. Since $L^+$ is equivalent to a lottery $L(\alpha_H)$ with utility $\alpha_H$ and $L^-$ is also equivalent to a lottery $L(\alpha_H)$ with utility $\alpha_H$, the complex lottery is equivalent to a lottery in which we select either $L(\alpha_H)$ or $L(\alpha_H)$ with equal probability 0.5, i.e., to $L(\alpha_H)$. Thus, the complex lottery has an equivalent utility $\alpha_H$.

On the other hand, no matter what is the event $E$, in the above complex lottery, we get $A_1$ with probability 0.5 and $A_0$ with probability 0.5. Thus, this complex lottery coincides with the lottery $L(0.5)$ and thus, has utility 0.5. Thus, we conclude that $\alpha_H = 0.5$.

**Which action should we choose?** Suppose that an action has $n$ possible outcomes $S_1, \ldots, S_n$, with utilities $[\underline{u}(S_i), \overline{u}(S_i)]$, and probabilities $[\underline{p}_i, \overline{p}_i]$. How do we then estimate the equivalent utility of this action?

We know that each alternative is equivalent to a simple lottery with utility $u_i = \alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)$, and that for each $i$, the $i$-th event is – from the viewpoint of decision making – equivalent to $p_i = \alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i$. Thus, from the viewpoint of decision making, this action is equivalent to a situation in which we get utility $u_i$ with probability $p_i$. We know that the utility of such a situation is equal to $\sum\limits_{i=1}^{n} p_i \cdot u_i$. Thus, the equivalent utility of the original action is equivalent to

$$\sum_{i=1}^{n} p_i \cdot u_i = \sum_{i=1}^{n} (\alpha_H \cdot \overline{p}_i + (1 - \alpha_H) \cdot \underline{p}_i) \cdot (\alpha_H \cdot \overline{u}(S_i) + (1 - \alpha_H) \cdot \underline{u}(S_i)).$$

**Observation: the resulting decision depends on the level of detail.** We make a decision in a situation when we do not know the exact values of the utilities and when we do not know the exact values of the corresponding probabilities. Clearly, if gain new information, the equivalent utility may change. For example, if we know nothing about an alternative $A$, then its utility is $[0, 1]$ and thus, its equivalent utility is $\alpha_H$. Once we narrow down the utility of $A$, e.g., to the interval $[0.5, 0.9]$, we get a different equivalent utility $\alpha_H \cdot 0.9 + (1 - \alpha_H) \cdot 0.5 = 0.5 + 0.4 \cdot \alpha_H$. On this example, the fact that we have different utilities makes perfect sense.

However, there are other examples where the corresponding difference is not as intuitively clear. Let us consider a situation in which, with some probability $p$, we gain a utility $u$, and with the remaining probability $1 - p$, we gain utility 0. If we know the exact values of $u$ and $p$, we can then compute the equivalent utility of this situation as the expected utility value $p \cdot u + (1 - p) \cdot 0 = p \cdot u$.

Suppose now that we only know the interval $[\underline{u}, \overline{u}]$ of possible values of utility and the interval $[\underline{p}, \overline{p}]$ of possible values of probability. Since the expression $p \cdot u$ for the expected utility of this situation is an increasing function of both variables:

- the largest possible utility of this situation is attained when both $p$ and $u$ are the largest possible: $u = \overline{u}$ and $p = \overline{p}$, and
- the smallest possible utility is attained when both $p$ and $u$ are the smallest possible: $u = \underline{u}$ and $p = \underline{p}$.

In other words, the resulting amount of utility ranges from $\underline{p} \cdot \underline{u}$ to $\overline{p} \cdot \overline{u}$.

If we know the structure of the situation, then, according to our derivation, this situation has an equivalent utility $u_k = (\alpha_H \cdot \overline{p} + (1 - \alpha_H) \cdot \underline{p}) \cdot (\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u})$ ($k$ for *k*now). On the other hand, if we do not know the structure, if we only know that the resulting utility is from the interval $[\underline{p} \cdot \underline{u}, \overline{p} \cdot \overline{u}]$, then, according to the Hurwicz criterion, the equivalent utility is equal to $u_d = \alpha_H \cdot \overline{p} \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u}$ ($d$ for *d*on't know). One can check that

$$u_d - u_k = \alpha_H \cdot \overline{p} \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u} - \alpha_H^2 \cdot \overline{p} \cdot \overline{u} - \alpha_H \cdot (1 - \alpha_H) \cdot (\underline{p} \cdot \overline{u} + \overline{p} \cdot \underline{u}) - (1 - \alpha_H)^2 \cdot \underline{p} \cdot \underline{u} =$$

$$\alpha_H \cdot (1 - \alpha_H) \cdot \overline{p} \cdot \overline{u} + \alpha_H \cdot (1 - \alpha_H) \cdot \underline{p} \cdot \underline{u} - \alpha_H \cdot (1 - \alpha_H) \cdot (\underline{p} \cdot \overline{u} + \overline{p} \cdot \underline{u}) =$$

$$\alpha_H \cdot (1 - \alpha_H) \cdot (\overline{p} - \underline{p}) \cdot (\overline{u} - \underline{u}).$$

This difference is always positive, meaning that additional knowledge decreases the utility of the situation. (This is maybe what the Book of Ecclesiastes means by "For with much wisdom comes much sorrow"?)

**From intervals to general sets.** In the ideal case, we know the exact situation $s$ in all the detail, and we can thus determine its utility $u(s)$. Realistically, we have an imprecise knowledge, so instead of a single situation $s$, we only know a *set* $S$ of possible situations $s$. Thus, instead of a single value of the utility, we only know that the actual utility belongs to the set $U = \{u(s) : s \in S\}$. If this set $S$ is an interval $[\underline{u}, \overline{u}]$, then we can use the above arguments to come up with its equivalent utility value $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$.

What is $U$ is a generic set? For example, we can have a 2-point set $U = \{\underline{u}, \overline{u}\}$. What is then the equivalent utility?

Let us first consider the case when the set $U$ contains both its infimum $\underline{u}$ and its supremum $\overline{u}$. The fact that we only know the set of possible values and have no other information means that *any* probability distribution on this set is possible (to be more precise, it is possible to have any probability distribution on the set of possible situations $S$, and this leads to the probability distribution on utilities). In particular, for each probability $p$, it is possible to have a distribution in which we have $\overline{u}$ with probability $p$ and $\underline{u}$ with probability $1 - p$. For this distribution, the expected utility is equal to $p \cdot \overline{u} + (1 - p) \cdot \underline{u}$. When $p$ goes from 0 to 1, these values fill the whole interval $[\underline{u}, \overline{u}]$. Thus, every value from this interval is the possible value of the expected utility. On the other hand, when $u \in [\underline{u}, \overline{u}]$, the expected value of the utility also belongs to this interval – no matter what the probability distribution. Thus, the set of all possible utility values is the whole interval $[\underline{u}, \overline{u}]$ and so, the equivalent utility is equal to $\alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$.

When the infimum and/or supremum are not in the set $S$, then the set $S$ contains points as close to them as possible. Thus, the resulting set of possible values of utility is as close as possible to the interval $[\underline{u}, \overline{u}]$ – and so, it is reasonable to assume that the equivalent utility is as close to $u_0 = \alpha_H \cdot \overline{u} + (1 - \alpha_H) \cdot \underline{u}$ as possible – i.e., coincides with this value $u_0$.

**From sets to fuzzy sets: main idea.** What if utility is a fuzzy number, described by a membership function $\mu(u)$? One of the natural interpretations of a fuzzy set is via its nested intervals $\alpha$-cuts $\mathbf{u}(\alpha) = [\underline{u}(\alpha_H), \overline{u}(\alpha_H)] = \{u : \mu(u) \geq \alpha\}$. For example, when we are talking about a measurement error of a given measuring instrument, then we know the guaranteed upper bound, i.e., the guaranteed interval that contains all possible values of the measurement error. In addition to this guaranteed interval, experts can usually pinpoint a narrower interval that contains the measurement error with some certainty; the narrower the interval, the smaller our certainty. Thus, we are absolutely sure (with certainty 1) that the actual value $u$ belongs to the $\alpha$-cut $\mathbf{u}(0)$; also, with a degree of certainty $1 - \alpha$, we claim that $x \in \mathbf{u}(\alpha)$. Thus, if we select some small value $\Delta\alpha$ and take $\alpha = 0, \Delta\alpha, 2\Delta\alpha, \ldots$, we conclude that:

- with probability $\Delta\alpha$, the set of possible values of $u$ is the interval $[\underline{u}(0), \overline{u}(0)]$;
- with probability $\Delta\alpha$, the set of possible values of $u$ is the interval $[\underline{u}(\Delta\alpha), \overline{u}(\Delta\alpha)]$;
- ...
- with probability $\Delta\alpha$, the set of possible values of $u$ is the interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$;
- ...

For each interval, the equivalent utility value is $\alpha \cdot \overline{u}(\alpha) + (1 - \alpha) \cdot \underline{u}(\alpha)$. The entire situation is a probabilistic combination of such intervals, so the resulting equivalent utility is equal to the expected value of the above utility, i.e., to

$$\alpha \cdot \int_0^1 \overline{u}(\alpha)\, d\alpha + (1 - \alpha) \cdot \int_0^1 \underline{u}(\alpha)\, d\alpha.$$

**Acknowledgments.** This work was supported in part by the National Science Foundation grants HRD-0734825 (Cyber-ShARE Center of Excellence) and DUE-0926721, by Grant 1 T36 GM078000-01 from the National Institutes of Health, and by a grant on F-transforms from the Office of Naval Research.

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