Journal of Uncertain Systems
Vol.7, No.x, pp.xx-xx, 2013
Online at: www.jus.org.uk

Possible and Necessary Orders, Equivalences, etc.: From Modal Logic to Modal Mathematics

Francisco Zapata and Olga Kosheleva
University of Texas at El Paso
El Paso, Texas 79968, USA
fazg74@gmail.com, olgak@utep.edu

Received 2 April 2012; Revised 25 August 2012

Abstract

In practice, we are often interested in order relations (e.g., when we describe preferences) or equivalence relations (e.g., when we describe clustering). Often, we do not have a complete information about the corresponding relation; as a result, we have several relations consistent with our knowledge. In such situations, it is desirable to know which elements a and b are *possibly* connected by the relation and which are *necessarily* connected by this relation. In this paper, we provide a full description of all such possible and necessary orders and equivalence relations. For example, possible orders are exactly reflexive relations, while necessary orders are exactly order relations.

1 Introduction

Relations are ubiquitous. In many practical situations, we are interested in a relation $r \subseteq U \times U$ on a given set U:

- In many practical situations, we are interested in an *order* relation that describes preferences.
- In many other practical situations, we are interested in an *equivalence relation* that describes clustering of objects into groups of similar ones.

Need to consider possible and necessary relations. Often, we do not have the full information about the desired relation, so several different relations are consistent with our knowledge. In other words, the class C of all the relations which are consistent with our knowledge has at least two different elements.

If we knew the exact relation r, then, for every two elements $a, b \in U$, we could be able to check whether $a\,r\,b$, i.e., whether these elements are in relation r. For example, we could be able to check whether a is preferable to b, whether a is equivalent to b, etc. Since we do have the full information about r, we cannot always check whether $a\,r\,b$. Instead:

- we can check whether it is *possible* that $a\,r\,b$, i.e., whether $a\,r\,b$ for some $r \in C$, and
- we can check whether it is *necessary* that $a\,r\,b$, i.e., whether $a\,r\,b$ for all $r \in C$.

In modal logic (see, e.g., [1, 2, 3]), *possible* is denoted by $\Diamond$, and necessary by $\Box$. Thus, the corresponding possible and necessary relations can be described as $\Diamond(a\,r\,b)$ and $\Box(a\,r\,b)$

Comment. Possible orders also appear in *cooperative game theory*; see, e.g., [4]. In this theory, we consider games between n players. For each possible coalition $S \subseteq \{1, 2, \ldots, n\}$, we can form an ordering relation $a \leq_S b$ meaning that this coalition can force the outcome to go from the original a to b, increasing their incomes, the outcome b is preferable to the outcome a. Then we define a *dominance* relation: a is dominated by b if $a \leq_S b$ for some coalition S. The *von Neumann-Morgenstern solution* is then defined as a set $\mathcal{C}$ of outcomes for which the following two properties are satisfied:

- no two outcomes from the set $\mathcal{C}$ dominate each other, and
- every outcomes which does not belong to the class $\mathcal{C}$ is dominated by an outcome from this set $\mathcal{C}$.

The meaning of the set $\mathcal{S}$ is that it represents a stable set of socially acceptable outcomes:

- for every outcome which is not in this set, at least one of the coalitions can force this outcome into a socially acceptable one;
- once we agreed to a socially acceptable outcome, no coalition can force us to move to another socially acceptable outcome.

Formulation of the problem. For a given class of relations – e.g., orders, equivalent relations, etc. – how can we describe the corresponding possible and ordinary relations?

What was known. Possible and necessary relations are described, e.g., [5]; see also references therein.

A similar idea of possible and necessary relations was considered in our previous paper [6]. In that paper, however, we consider a different situation, when the relation r is fixed, but the elements a and b are known with uncertainty.

What we do in this paper. This paper provides a description of possible and necessary orders and equivalent relations.

2 Possible and Necessary Orders

Reminder. A relation r is called *reflexive* if $a\,r\,a$ for all a, *antisymmetric* if $a\,r\,b$ and $b\,r\,a$ imply $a = b$, *transitive* if $a\,r\,b$ and $b\,r\,c$ imply $a\,r\,c$, and *order* (or *partial order*) if it is reflexive, antisymmetric, and transitive.

Definition 1. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible order *if there exists a non-empty class C of ordering relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 2. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary order *if there exists a non-empty class C of ordering relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 1. *A relation R is a possible order if and only if it is reflexive.*

Proposition 2. *A relation R is a necessary order if and only if it is an order.*

Proof of Proposition 1.

1°. Let us first prove that every possible order is reflexive.

Indeed, let R be a possible order corresponding to a class C of orders. By definition of an order relation, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ for some $r \in C$ and therefore, $a\,R\,a$.

2°. Vice versa, let us assume that R is a reflexive relation. Let us prove that R is a possible order.

Indeed, for each pair $(x, y) \in R$, we can consider the ordering relation $\le_{u,v}$ that consists of this pair and all pairs (u, u) $(u \in U)$. In this relation, $a \le b$ if and only if either $a = x$ and $b = y$, or $a = b$. One can easily see that this relation $\le_{u,v}$ is indeed an order. So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a \le_{u,v} b$ for some $(u, v) \in R$, i.e., if and only if either $a = u$ and $b = v$ for some pair $(u, v) \in R$ (i.e., equivalently, if $(u, v) \in R$) or $a = b$ – in which case also $(a, b) = (a, a) \in R$. Thus, the corresponding possible order is indeed the original relation R.

The proposition is proven.

Proof of Proposition 2.

1°. One can easily prove that each order $\le$ is a necessary order: it is sufficient to consider the class $C = \{\le\}$ that consists of only this order relation.

2°. Vice versa, let us assume that a relation R is a necessary order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all orders r from some class C. Let us prove that this relation R is an order, i.e., that it is reflexive, antisymmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is indeed reflexive.

2.2°. Let us now prove that the relation R is antisymmetric.

Indeed, if $a\,R\,b$ and $b\,R\,a$, this means that $a\,r\,b$ and $b\,r\,a$ for all $r \in C$. Since each relation r is antisymmetric, this implies $a = b$. We have thus proved that $a\,R\,b$ and $b\,R\,a$ imply $a = b$, i.e., that the necessary order relation is also antisymmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

3 Possible and Necessary Strict Orders

Case of strict orders. Sometimes, it makes sense to consider *strict orders*, i.e., transitive relations which are *strictly antisymmetric*, i.e., for which $a < b$ implies that $b \not< a$. All such relations are *anti-reflexive*: $a \not< a$. For strict order relations, similar proofs lead to the following similar results:

Definition 3. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible strict order *if there exists a non-empty class C of strict orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 4. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary strict order *if there exists a non-empty class C of strict orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 3. *A relation R is a possible strict order if and only if it is anti-reflexive.*

Proposition 4. *A relation R is a necessary strict order if and only if it is a strict order.*

Proof of Proposition 3.

1°. Let us first prove that every possible strict order is anti-reflexive.

Indeed, let R be a possible order corresponding to a class C of strict orders. By definition of a strict order relation, we have $\neg(a\,r\,a)$ for all $r \in C$. Thus, $\neg(a\,R\,a)$.

2°. Vice versa, let us assume that R is an anti-reflexive relation. Let us prove that R is a possible strict order.

Indeed, for each pair $(x, y) \in R$, we can consider the strict ordering relation $<_{u,v}$ that consists only of this pair. In this relation, $a < b$ if and only if $a = x$ and $b = y$. One can easily see that this relation $<_{u,v}$ is indeed an order. So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a <_{u,v} b$ for some $(u, v) \in R$, i.e., if and only if $a = u$ and $b = y$ for some pair $(u, v) \in R$ (i.e., equivalently, if $(u, v) \in R$). Thus, the corresponding possible strict order is indeed the original relation R.

The proposition is proven.

Proof of Proposition 4.

1°. One can easily prove that each strict order $<$ is a necessary strict order: it is sufficient to consider the class $C = \{<\}$ that consists of only this strict order relation.

2°. Vice versa, let us assume that a relation R is a necessary strict order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all strict orders r from some class C. Let us prove that this relation R is asymmetric, and transitive.

2.1°. Let us first prove that the relation R is strictly antisymmetric.

Indeed, let us assume that $a\,R\,b$. This means that $a\,r\,b$ for all $r \in C$. Since all relations $r \in C$ are strict orders, we thus conclude that $\neg(b\,r\,a)$ for all $r \in C$. Thus, $\neg(b\,R\,a)$. Thus, the relation R is indeed strictly antisymmetric.

2.2°. Let us now prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary strict order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

4 Possible and Necessary Linear Orders

Idea. If instead of general (partial) orders, we can consider *linear* (*total*) orders, for which for every a and b, we have $a \le b$ or $b \le a$.

Definition 5. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible linear order *if there exists a non-empty class C of linear orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 6. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary linear order *if there exists a non-empty class C of orders on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 5. *A relation R is a necessary linear order if and only if it is an order.*

Comment. It is not clear how to easily describe *possible* linear orders.

Proof of Proposition 5.

1°. Let us prove that each order $\le$ is a necessary linear order, i.e., that for each (partial) order, $\le$, there exists a family C of linear orders for which, for every a and b, $a \le b \Leftrightarrow \forall r \in C\,(a\,r\,b)$.

This proof is based on the known result that every linear order can be extended to a linear order. This result, in its turn, is based on a following auxiliary result:

1.1°. Let U be an ordered set with an order $\le$ which is not a linear order. The fact that $\le$ is not a linear order means that there exists elements a and b for which $a \not\le b$ and $b \not\le a$. Let us pick two such elements a and b. Then, we can extend the original order $\le$ to a new order $\le^*$ in which $b \le^* a$.

Indeed, we can define the new relation $\le^*$ as follows:

$$p \le^* q \Leftrightarrow (p \le q \vee (p \le b\,\&\,a \le q)).$$

This relation clearly extends the original order $\le$. To prove the above statement, we thus need to prove that this new relation is an order, i.e., that it reflexive, antisymmetric, and transitive.

1.1.1°. Let us first prove that the relation $\le^*$ is reflexive.

Indeed, since $\leq$ is an order, we have $p \leq p$ for every p and thus, $p \leq^* p$ for every p.

1.1.2°. Let us now prove that the relation $\leq^*$ is antisymmetric, i.e., that $p \leq^* q$ and $q \leq^* p$ imply that $p = q$.

The condition $p \leq^* q$ means that either $p \leq q$ or $(p \leq b \,\&\, a \leq q)$. Similarly, the condition $q \leq^* p$ means that either $q \leq p$ or $(q \leq b \,\&\, a \leq p)$. To complete our proof, let us consider all $2 \cdot 2 = 4$ combinations of these conditions.

If $p \leq q$ and $q \leq p$, then $p = q$ since $\leq$ is an order and is, thus, antisymmetric.

If $p \leq b$, $a \leq q$, and $q \leq p$, then, by transitivity, we get $a \leq b$, which contradicts to our original assumption that $a \not\leq b$ and $b \not\leq a$.

Similarly, if $p \leq q$, $q \leq b$, and $a \leq p$, then we also get $a \leq b$ and thus, a contradiction.

Finally, if $p \leq b$, $a \leq q$, $q \leq b$, $a \leq p$, then we get $a \leq q \leq b$ and $a \leq b$, which is also impossible.

Antisymmetry is proven.

1.1.3°. Finally, let us prove that the relation $\leq^*$ is transitive, i.e., that $p \leq^* q$ and $q \leq^* r$ imply that $p \leq^* r$.

The condition $p \leq^* q$ means that either $p \leq q$ or $(p \leq b \,\&\, a \leq q)$. Similarly, the condition $q \leq^* r$ means that either $q \leq r$ or $(q \leq b \,\&\, a \leq r)$. To complete our proof, let us consider all $2 \cdot 2 = 4$ combinations of these conditions.

If $p \leq q$ and $q \leq r$, then $p \leq r$ since $\leq$ is an order and is, thus, transitive.

If $p \leq q$, $q \leq b$, and $a \leq r$, then, by transitivity, we get $p \leq b$ and $a \leq r$, i.e., $p \leq^* r$.

Similarly, if $p \leq b$, $a \leq q$, and $q \leq r$, then by transitivity, we get $p \leq b$ and $a \leq r$, i.e., also $p \leq^* r$.

Finally, if $p \leq b$, $a \leq q$, $q \leq b$, $a \leq r$, then by transitivity, from $a \leq q \leq b$, we conclude that $a \leq b$, which contradicts to our assumption that $a \not\leq b$ and $b \not\leq a$.

Transitivity is proven, so $\leq^*$ is indeed an order.

1.2°. For a finite set U, we can consistently add pairs and thus, eventually get a linear order that extends our original order $\leq$. For an infinite set, we can do the same by using transfinite induction or, equivalently, Zorn's Lemma.

1.3°. We can now prove that a given partial order $\leq$ is a necessary linear order, i.e., that there exists a family C of linear orders for which $a \leq b \Leftrightarrow \forall r \in C\,(a\,r\,b)$.

As this family C, we take all linear orders that extend the original order $\leq$. Let us prove that this family has the desired property, by considering three possible cases: the case when $a \leq b$, the case when $b \leq a$ and $b \neq a$, and the case when $a \not\leq b$ and $b \not\leq a$. We will prove that in all these three case:

- when the condition $a \leq b$ is true, then the condition $\forall r \in C\,(a\,r\,b)$ is also true, and
- when the condition $a \leq b$ is false, then the condition $\forall r \in C\,(a\,r\,b)$ is also false.

1.3.1°. If $a \leq b$, then, since all orders $r \in C$ extend $\leq$, we have $a\,r\,b$ for all $r \in C$. So, the condition $\forall r \in C\,(a\,r\,b)$ is also true.

1.3.2°. If $b \leq a$ and $b \neq a$, then we have $a \not\leq b$, so the condition $a \leq b$ is false. Since all orders $r \in C$ extend $\leq$, we have $b\,r\,a$ for all $r \in C$. Since $b \neq a$ and each r is an order, this implies that $\neg(a\,r\,b)$ for all $r \in C$. Thus, the condition $\forall r \in C\,(a\,r\,b)$ is also false.

1.3.3°. Finally, let us consider the case when $a \not\leq b$ and $b \not\leq a$. Here, the condition $a \leq b$ is false. In this case, as we have proven earlier, we can:

- extend the original order $\leq$ to a new order $\leq^*$ in which $b \leq^* a$, and then
- extend this new order $\leq^*$ to a linear order r.

In this linear order $r \in C$, we have $b\,r\,a$ hence $\neg(a\,r\,b)$, so the condition $\forall r \in C\,(a\,r\,b)$ is also false.

The statement is proven.

2°. Vice versa, let us assume that a relation R is a necessary order linear order, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all linear orders r from some class C. Let us prove that this relation R is an order, i.e., that is is reflexive, antisymmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is indeed reflexive.

2.2°. Let us now prove that the relation R is antisymmetric.

Indeed, if $a\,R\,b$ and $b\,R\,a$, this means that $a\,r\,b$ and $b\,r\,a$ for all $r \in C$. Since each relation r is antisymmetric, this implies $a = b$. We have thus proved that $a\,R\,b$ and $b\,R\,a$ imply $a = b$, i.e., that the necessary order relation is also antisymmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every order $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an order and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary order, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

5 Possible and Necessary Equivalence Relations

Reminder. A relation r is called *symmetric* if $a\,r\,b$ implies $b\,r\,a$, and an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 7. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possible equivalence relation *if there exists a non-empty class C of equivalence relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 8. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessary equivalence relation *if there exists a non-empty class C of equivalence relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 6. *A relation R is a possible equivalence relation if and only if it is reflexive and symmetric.*

Proposition 7. *A relation R is a necessary equivalence relation if and only if it is an equivalence relation.*

Proof of Proposition 6.

1°. Let us first assume that R is a possible equivalence relation corresponding to a class C of equivalent relations. Let us prove that this relation R is reflexive and symmetric.

1.1°. Let us first prove that the relation R is reflexive.

Indeed, by definition of an equivalence relation, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ for some $r \in C$ and therefore, $a\,R\,a$, i.e., R is indeed reflexive.

1.2°. Let us now prove that the relation R is symmetric.

Indeed, if aRb, this means that $a\,r\,b$ for some $r \in C$. This relation r is an equivalence relation, so we have $b\,r\,a$ for this $r \in C$. Thus, we conclude that $b\,R\,a$, i.e., the relation R is indeed symmetric.

2°. Vice versa, let us assume that R is a reflexive and symmetric relation. Let us prove that R is a possible equivalence relation.

Indeed, for each pair $(x, y) \in R$, we can consider the equivalence relation $\equiv_{u,v}$ that consists of this pair (u, v), the "dual" pair (v, u), and all pairs (u, u) $(u \in U)$. In this relation, $a \equiv_{u,v} b$ if and only if either the pair (a, b) coincides with (u, v) or (v, u), or $a = b$. One can easily see that this relation $\equiv_{u,v}$ is indeed an equivalence relation: in this relation, elements u and v are grouped in one equivalence class, while all other equivalence classes consists of a single element.

So, if we take the class of all such relation as C, then $a\,r\,b$ for some $r \in C$ if and only if $a \equiv_{u,v} b$ for some $(u,v) \in R$, i.e., if and only if either $(a,b) = (u,v)$ or $(a,b) = (v,u)$ for some pair $(u,v) \in R$ (since R is symmetric, this is equivalent to $(u,v) \in R$) or $a = b$ – in which case also $(a,b) = (a,a) \in R$. Thus, the corresponding possible equivalence relation is indeed the original relation R.

The proposition is proven.

Proof of Proposition 7.

1°. One can easily prove that each equivalence relation $\equiv$ is a necessary equivalence relation: it is sufficient to consider the class $C = \{\equiv\}$ that consists of only this equivalence relation.

2°. Vice versa, let us assume that R is a necessary equivalence relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all equivalence relations r from some class C. Let us prove that R is an equivalence relation, i.e., that it is reflexive, symmetric, and transitive.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have ara, so we conclude that $a\,R\,a$. Thus, R is reflexive.

2.2°. Let us now prove that the relation R is symmetric.

Indeed, if $a\,R\,b$, this means that $a\,r\,b$ for all $r \in C$. Since each relation r is symmetric, this implies $b\,r\,a$. Since this is true for all $r \in R$, we thus have $b\,R\,a$. We have thus proved that $a\,R\,b$ implies $b\,R\,a$, i.e., that the necessary equivalence relation is also symmetric.

2.3°. Finally, let us prove that the relation R is transitive.

Indeed, if $a\,R\,b$ and $b\,R\,c$, this means that for every equivalence relation $r \in R$, we have $a\,r\,b$ and $b\,r\,c$. Since each r is an equivalence relation and hence, transitive, we conclude that $a\,r\,c$ for all c. By definition of a necessary equivalence relation, this means that $a\,R\,c$. Thus, the relation R is indeed transitive.

The proposition is proven.

6 Auxiliary Results: Possibly and Necessarily Reflexive Relations

Formulation of the question. We have shown that possible orders are exactly reflexive relations, and necessary order relations are orders. It is natural to ask: what are possibly and necessarily reflexive relations?

Definition 9. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly reflexive relation *if there exists a non-empty class C of reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 10. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily reflexive relation *if there exists a non-empty class C of reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 8. *A relation R is a possibly reflexive relation if and only if it is reflexive.*

Proposition 9. *A relation R is a necessarily reflexive relation if and only if it is a reflexive relation.*

Proof of Proposition 8.

1°. Let us first prove that every possibly reflexive relation is reflexive.

Indeed, let R be a possible reflexive relation corresponding to a class C of reflexive relations. For every element a, by definition of a reflexive relation, we have ara for all $r \in C$. Thus, $a\,r\,a$ holds at least for *some* relations $r \in C$ and therefore, $a\,R\,a$. So, the relation R is indeed reflexive.

2°. Vice versa, it is easy to show that every reflexive relation R is possibly reflexive: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 9.

1°. One can easily prove that each reflexive relation r is a necessarily reflexive relation: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily reflexive relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all reflexive relations r from some class C. Let us prove that this relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is reflexive.

The proposition is proven.

7 Auxiliary Results: Possible and Necessary Anti-Reflexive Relations

Formulation of the question. We have shown that possible strict orders are exactly anti-reflexive relations, and necessary strict orders are strict orders. It is natural to ask: what are possibly and necessarily anti-reflexive relations?

Definition 11. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly anti-reflexive relation *if there exists a non-empty class C of anti-reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 12. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily anti-reflexive relation *if there exists a non-empty class C of anti-reflexive relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 10. *A relation R is possibly anti-reflexive if and only if it is anti-reflexive.*

Proposition 11. *A relation R is necessarily anti-reflexive if and only if it is anti-reflexive.*

Proof of Proposition 10.

1°. Let R be a possibly anti-reflexive relation corresponding to a class C of anti-reflexive relations. Let us prove that this relation R is anti-reflexive.

Indeed, since all the relations $r \in C$ are anti-reflexive, we have $\neg(a\,r\,a)$ for all $r \in C$. Thus, we cannot have $a\,r\,a$ for any $r \in C$ and therefore, we have $\neg(a\,R\,a)$.

2°. Vice versa, it is easy to show that every anti-reflexive relation R is possibly anti-reflexive: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 11.

1°. One can easily prove that each anti-reflexive relation r is necessarily anti-reflexive: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily anti-reflexive relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all anti-reflexive relations r from some class C. Let us prove that this relation R is anti-reflexive.

Indeed, for every a, and for every $r \in C$, we have $\neg(a\,r\,a)$, so we conclude that $\neg(a\,R\,a)$. Thus, R is indeed anti-reflexive.

The proposition is proven.

8 Auxiliary Results: Possible and Necessary Reflexive-and-Symmetric Relations

Formulation of the question. We have shown that possible equivalence relations are exactly reflexive and symmetric relations, and necessary equivalent relations are equivalent relations. It is natural to ask: what are possibly and necessarily reflexive-and-symmetric relations?

Definition 13. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* possibly reflexive-and-symmetric relation *if there exists a non-empty class C of reflexive-and-symmetric relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for some $r \in R$:*

$$a\,R\,b \Leftrightarrow \exists r \in C\,(a\,r\,b).$$

Definition 14. *Let U be a set. We say that a relation $R \subseteq U \times U$ is a* necessarily reflexive-and-symmetric relation *if there exists a non-empty class C of reflexive-and-symmetric relations on U for which $a\,R\,b$ if and only if $a\,r\,b$ for all $r \in R$:*

$$a\,R\,b \Leftrightarrow \forall r \in C\,(a\,r\,b).$$

Proposition 12. *A relation R is a possibly reflexive-and-symmetric relation if and only if it is reflexive and symmetric.*

Proposition 13. *A relation R is a necessarily reflexive-and-symmetric relation if and only if it is a reflexive-and-symmetric relation.*

Proof of Proposition 12.

1°. Let R be a possibly reflexive-and-symmetric relation corresponding to a class C of reflexive-and-symmetric relations. Let us prove that this relation R is reflexive and symmetric.

1.1°. Let us prove that the relation R is reflexive.

Indeed, since all the relations $r \in C$ are reflexive, we have $a\,r\,a$ for all $r \in C$. Thus, $a\,r\,a$ at least for some $r \in C$ and therefore, $a\,R\,a$.

1.2°. Let us now prove that the relation R is symmetric, Indeed, suppose that $a\,R\,b$. By definition, this means that $a\,r\,b$ for some $r \in C$. Since every relation $r \in C$ is symmetric, we conclude that $b\,r\,a$, which implies that $b\,R\,a$. Thus, the relation R is indeed symmetric.

2°. Vice versa, it is easy to show that every reflexive-and-symmetric relation R is possibly reflexive-and-symmetric: indeed, we can take a set $S = \{R\}$ consisting of only this relation.

The proposition is proven.

Proof of Proposition 13.

1°. One can easily prove that each reflexive-and-symmetric relation r is necessarily reflexive-and-symmetric: it is sufficient to consider the class $C = \{r\}$ that consists of only this relation.

2°. Vice versa, let us assume that R is a necessarily reflexive-and-symmetric relation, i.e., $a\,R\,b$ if and only if $a\,r\,b$ for all reflexive-and-symmetric relations r from some class C. Let us prove that this relation R is reflexive and symmetric.

2.1°. Let us first prove that the relation R is reflexive.

Indeed, for every a, and for every $r \in C$, we have $a\,r\,a$, so we conclude that $a\,R\,a$. Thus, R is reflexive.

2.2°. Let us now prove that the relation R is symmetric.

Indeed, if $a\,R\,b$, this means that $a\,r\,b$ for all $r \in C$. Since each relation r is symmetric, this implies $b\,r\,a$. So, we have $b\,r\,a$ for all $r \in C$, and thus, we have $b\,R\,a$. We have thus proved that $a\,R\,b$ implies $b\,R\,a$, i.e., that the necessary reflexive-and-symmetric relation is indeed symmetric.

The proposition is proven.

9 Graphical Representation of the Results

Graphical description. By using the symbols $\Diamond$ and $\Box$, we can describe our results in the following graphical form:

$$\begin{array}{ccccc} \leq\overset{\Box}{\Leftrightarrow} & ; & <\overset{\Box}{\Leftrightarrow} & ; & \equiv\overset{\Box}{\Leftrightarrow} \\ \Downarrow\Diamond & ; & \Downarrow\Diamond & ; & \Downarrow\Diamond \\ \text{refl.}\overset{\Diamond,\Box}{\Leftrightarrow} & ; & \text{anti-refl.}\overset{\Diamond,\Box}{\Leftrightarrow} & ; & \text{refl.-sym.}\overset{\Diamond,\Box}{\Leftrightarrow} \end{array}$$

Left diagram. The left diagram means that if we start with an order $\leq$, then:

- the necessary modality $\Box$ leads again to an order – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while
- the possible modality $\Diamond$ leads to reflexive relations – which we denoted by refl.

For reflexive relations, both modalities $\Box$ and $\Diamond$ lead again to reflexive relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

Middle diagram. The middle diagram means that if we start with a strict order $<$, then:

- the necessary modality $\Box$ leads again to a strict order – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while
- the possible modality $\Diamond$ leads to anti-reflexive relations – which we denoted by anti-refl.

For anti-reflexive relations, both modalities $\Box$ and $\Diamond$ lead again to anti-reflexive relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

Right diagram. The right diagram means that if we start with an equivalence relation $\equiv$, then:

- the necessary modality $\Box$ leads again to an equivalence relation – we denoted this by $\overset{\Box}{\Leftrightarrow}$, while
- the possible modality $\Diamond$ leads to reflexive and symmetric relation – which we denoted by refl.-sym.

For reflexive and symmetric relations, both modalities $\Box$ and $\Diamond$ lead again to reflexive and symmetric relations – we denoted this by $\overset{\Diamond,\Box}{\Leftrightarrow}$.

Acknowledgements.

This work was partly supported by a CONACyT scholarship.

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