F-transform in View of Trend Extraction*

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Abstract. In the analysis of time series, it is important to decompose the original values into trend, cycle, seasonal component, and noise. In this paper, we provide a theoretical justification of the fact that the F-transform can be used for this purpose. We formulate "natural" requirements on the trend extraction procedure and then show that the inverse F-transform fulfils all of them.

Keywords: time series, trend, F-transform, basic function

1 Introduction

Need for the time series analysis. In application areas such as meteorology, financial analysis, etc., it is desirable to predict the value of a monitoring variable, e.g., temperature, stock price, and so on. To make this prediction, we observe values of the monitoring variable at different time moments, and use the results of these observations to make a prediction. Usually, measurements are performed at regular time intervals, i.e., at moments t_1, $t_2 = t_1 + \Delta, \ldots, t_k = t_1 + (k-1)\Delta, \ldots,$ – e.g., daily, monthly, hourly, etc. The result of each measurement at the corresponding time moment t_k, $k \geq 1$, is a real number, say x_{t_k}. The result of all measurements, i.e. the (ordered) sequence of real numbers is called a *time series*, see e.g., [3].

$$x_t = (x_1, \ldots, x_N) \tag{1}$$

where $N > 0$ is an integer. If $\mathbb{N}$ denotes the set of natural numbers, $[1, N] \subset \mathbb{N}$ and $\mathbb{R}$ denotes the set of reals, then we can say that the time series x_t is a function $x : [1, N] \to \mathbb{R}$ such that $x_t = x(t)$, $t = 1, \ldots, N$. Thus, we will identify the denotation x_t of the whole time series as a sequence with the denotation of its value at the moment t.

Seasonal and other components of a time series. An observed time series x_t can be naturally represented (additively decomposed) as a sum of four time series with different behavior: a long-term trend (slowly changing time series), cycles (medium-term changes), seasonal (short-term changes), and noise; see, e.g., [2]. Namely,

$$x_t = \mathrm{Tr}_t + \mathrm{Cy}_t + \mathrm{Se}_t + \mathrm{No}_t, \qquad t = 1, \ldots, N, \tag{2}$$

where Tr_t is a trend, Cy_t is a cycle, Se_t is a seasonal time series, and No_t is the random error (quite often assumed to be a white noise).

In applications (see, e.g., [2]), it is important to separate seasonal (short-term) changes from long-term ones. In such situation, it is reasonable to combine trend and cycle into a single trend-cycle time series $y_t \stackrel{\text{def}}{=} \mathrm{Tr}_t + \mathrm{Cy}_t$, and to combine seasonal time series and noise into a single seasonal-noise time series $z_t \stackrel{\text{def}}{=} \mathrm{Se}_t + \mathrm{No}_t$. After this combination, the decomposition (2) takes the simplified form

$$x_t = y_t + z_t, \qquad t = 1, \ldots, N. \tag{3}$$

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2 F-transform approach to time series: a background.

In [5, 6, 8], we propose to apply the F-transform to time series analysis and forecast. The F-transform technique itself was introduced in [7]. Before we explain how the F-transform is used in the analysis of time series, let us remind its main principles. In particular, the *discrete* form of the F-transform will be recalled. The definition below is adjusted to the denotation above, so that a time series is identified with a function $x : [1, N] \to \mathbb{R}$ where $[1, N]$ is an interval of $\mathbb{N}$.

The first step in the definition of the F-transform of x is the selection of a *fuzzy partition* of the interval $[1, N]$ by a finite number $n \geq 3$ of fuzzy sets $A_1, \ldots, A_n$. In [7], we used five axioms to characterize a fuzzy partition. In [9], the number of axioms was reduced to three and a fuzzy partition was called *relaxed*. Below, we repeat the last definition and adjust it to the case of $[1, N]$. This means that we will drop the axiom of continuity and leave only two axioms.

Definition 1. *Let $[1, N]$ be an interval of $\mathbb{N}$, $n \geq 3$, and $t_0, t_1, \ldots, t_n, t_{n+1} \in [1, N]$ be nodes*[4] *such that $1 = t_0 \leq t_1 < \ldots < t_n \leq t_{n+1} = N$. We say that the fuzzy sets $A_1, \ldots, A_n : [1, N] \to [0, 1]$, identified with their membership functions, constitute a* fuzzy partition *of $[1, N]$ if the following conditions are satisfied:*

1. *(locality) - for every $k = 1, \ldots, n$, $A_k(t) = 0$ if $t \in [1, N] \setminus (t_{k-1}, t_{k+1})$;*
2. *(density) - $\sum_{t=1}^{N} A_k(t) > 0$, $k = 1, \ldots, n$.*

The membership functions $A_1, \ldots, A_n$ of the respective fuzzy partition are called *basic functions*. A point t is *covered* by A_k if $A_k(t) > 0$. A fuzzy partition is called *uniform* if basic function A_1 is symmetrical (with respect to the axis $t = t_1$) and $A_2, \ldots, A_n$ are shifted copies of A_1, i.e.,

$$A_k(t) = A_1(t_1 + t - t_k), \quad t_{k-1} \leq t \leq t_{k+1}, k = 2, \ldots, n. \tag{4}$$

In the case of uniform partition, basic function A_1 is called *generating*.

The direct F-transform. Once the basic functions $A_1, \ldots, A_n$ are selected, we define (see [7]) the (direct) *F-transform* of the time series $x : [1, N] \longrightarrow \mathbb{R}$ as a vector $F_n(x) = (X_1, \ldots, X_n)$, where the k-th *component* X_k is equal to

$$X_k = \frac{\sum_{t=1}^{N} x(t) \cdot A_k(t)}{\sum_{t=1}^{N} A_k(t)}, \quad k = 1, \ldots, n. \tag{5}$$

The definition is correct due to the property "density". To stress that the *F-transform components* $X_1, \ldots, X_n$ depend on $A_1, \ldots, A_n$, we say that the F-transform is taken with respect to $A_1, \ldots, A_n$.

The following properties characterize the F-transform $F_n(x)$ [7]:

P1. The mapping $F_n : \mathbb{R}^N \to \mathbb{R}^n$ such that $F_n : x \mapsto F_n(x)$ is linear. In the formulation of this property, the time series x is identified with the vector $(x_1, \ldots, x_N)$ of its values on $[1, N]$.

P2. If the time series x is constant, i.e., $x(t) = C$, $t = 1, \ldots, N$, then the components of its F-transform $F_n(x)$ are constants as well; moreover, $F_n(x) = (C, \ldots, C)$.

P3. Components $X_1, \ldots, X_n$ of $F_n(x)$ minimize the following function:

$$\Psi(y_1, \ldots, y_n) = \sum_{k=1}^{n} \sum_{t=1}^{N} (x(t) - y_k)^2 A_k(t), \tag{6}$$

which can be considered to be a *weighted least square mean criterion*. By this property, the F-transform components are weighted least square means of x.

The inverse F-transform. In our Definition 1, we have used the relaxed fuzzy partition of $[1, N]$ (e.g., the so called Ruspini condition was not demanded). One of the consequences of this is the fact that the inverse F-transform has a slightly different form, see below.

The *inverse F-transform* of the time series x is defined on the set $D = \{t : \sum_{k=1}^{n} A_k(t) > 0\}$ where it is given by the inversion formula

$$x_{F,n}(t) = \frac{\sum_{k=1}^{n} X_k A_k(t)}{\sum_{k=1}^{n} A_k(t)}, \quad t = 1, \ldots, N, \tag{7}$$

which represents a function on $[1, N]$.

Similarly to [7], it can be shown that the inverse F-transform $x_{F,n}$ approximates the original time series x on the set D.

[4] The nodes $t_0, t_1, \ldots, t_n, t_{n+1}$ establish a (sparse) grid on the set $[1, N]$.

3 Formalization of Trend

3.1 Our goal

Our goal is to show that given a time series x_t, a trend-cycle of it (below, we will write "trend" instead of "trend-cycle") can be represented by the respective inverse F-transform $x_{F,n}$ where n is the the number of basic functions in the partition $A_1, \dots, A_n$ of $[1, N]$. To realize this goal, we will formalize the notion of trend by listing its properties and then show that the inverse F-transform fulfils all of them. In more details we will proceed as follows.

Let x_t, $t = 1, \dots, N$, be a time series. Based on the observed values x_t, we want to represent this time series in the form (3). It is clear that the solution of this problem is not unique. In practice, every decomposition (3) is an acceptable solution, if y_t is a slowly changing discrete function and z_t is the rest. Therefore, we will formulate reasonable properties of a single trend value y_s where s is a time moment within $[1, N]$, and analyze how y_s depends on values of x_t in a certain neighborhood of x_s. Then we will show that the inverse F-transform interpolates values y_s for a chosen sequence of time moments s and by this, can be considered as a trend of x_t.

Assume that y_s is a trend value of a time series x_t, $t = 1, \dots, N$, at a time moment s, $s = 1, \dots, N$. Being a trend value, y_s characterizes a "behavior" of the time series x_t in a certain neighborhood, say $[x_{s-\ell}, x_{s+u}]$ of its value x_s. To be more precise, we assume that integers $\ell, u > 0$ are such that $1 \le s - \ell < s + u \le N$ and that the value y_s depends on values x_t for $t \in [s - \ell, s + u]$. Moreover, we assume that this dependence is the same for all time moments s, i.e. there exists a function F of $q = \ell + u + 1$ real variables $v_1, \dots, v_q$, such that whenever $v_1 = x_{s-\ell}, \dots, v_q = x_{s+u}$, the value $F(x_{s-\ell}, \dots, x_{s+u})$ is the trend value y_s at the moment s.

Thus, the necessary requirements on a trend will be formulated in the form of necessary properties of the "trend extraction" function $F(v_1, \dots, v_q)$. Once these properties are listed, we will describe the class of functions that satisfy all of them.

3.2 Trend Extraction and Its Properties

Let x_t, $t = 1, \dots, N$, be a time series and $F : \mathbb{R}^q \to R$ be a function. Below, we list properties of F such that for some ℓ, s, u, where $1 \le s - \ell \le u - s \le N$, the value $F(x_{s-\ell}, \dots, x_{s+u})$ can be considered as a trend value of the time series x_t on the interval $[s - \ell, u - s]$ in the sense that the following properties are fulfilled.

Continuity. The values of a time series come from measurements or from expert estimates. Neither measurements nor expert estimates are absolutely accurate. Thus, an actual value x_t^{act} of a time series is slightly different from its observed values x_t. It is therefore desirable that the values $F(x_{s-\ell}^{\text{act}}, \dots, x_{s+u}^{\text{act}})$ and $F(x_{s-\ell}, \dots, x_{s+u})$ are slightly different as well. In other words, we wish the function F to be *continuous*.

Additivity. Assume that a time series x_t is sum of two different time series, i.e. $x_t = x_t^{(1)} + x_t^{(2)}$, and that each of $x_t^{(1)}$ and $x_t^{(2)}$ is decomposed according to (3), so that $x_t^{(1)} = y_t^{(1)} + z_t^{(1)}$ and $x_t^{(2)} = y_t^{(2)} + z_t^{(2)}$. For example, the varying price of the financial portfolio can be represented as a sum of the prices corresponding to different parts of this portfolio: e.g., stocks and bonds. In this case,

$$x_t = x_t^{(1)} + x_t^{(2)} = (y_t^{(1)} + z_t^{(1)}) + (y_t^{(2)} + z_t^{(2)}) = (y_t^{(1)} + y_t^{(2)}) + (z_t^{(1)} + z_t^{(2)}).$$

The sum $y_t^{(1)} + y_t^{(2)}$ contains slowly-changing (trend-cycle) terms, while the sum $z_t^{(1)} + z_t^{(2)}$ contains shortly-changing (seasonal-noise) terms. Thus, the sum $y_t^{(1)} + y_t^{(2)}$ is the trend-cycle of the resulting time series $x_t = x_t^{(1)} + x_t^{(2)}$. Therefore, the trend extraction function F should be *additive*, i.e., for every two q-tuples $(v_1, \dots, v_q)$ and $(u_1, \dots, u_q)$, the following should be valid:

$$F(v_1 + u_1, \dots, v_q + u_q) = F(v_1, \dots, v_q) + F(u_1, \dots, u_q). \tag{8}$$

Non-negativity. In economical and financial analysis, experts compare time series according to behavior of their trends, e.g., they compare share indices, behavior of the Gross Domestic Product (GDP) in successful years, etc. Therefore, it is important to distinguish between increases and decreases of time series and the respective changes of their trends. Formally, we require that the trend extraction function F should be *non-negative*, i.e.

$$\text{if } v_1 \ge 0, \dots, v_q \ge 0 \text{ then } F(v_1, \dots, v_q) \ge 0. \tag{9}$$

It is easy to show that if the trend extraction function F fulfils (9) then it is non-decreasing in the sense that

$$\text{if } v_1 \le u_1, \ldots, v_q \le u_q \text{ then } F(v_1, \ldots, v_q) \le F(u_1, \ldots, u_q).$$

Constant Preserving. If a time series is equal to a constant c such that $x_t = c$ for all $t = 1, \ldots, N$, then the trend of x_t should also be equal to the same constant c. Therefore, it is reasonable to require that the trend extraction function F is a *constant preserving*, i.e.

$$F(c, \ldots, c) = c. \tag{10}$$

Noise damper. The next property is a *noise damper* with respect to a point-spread noise[5]. By this we mean that when a time series $x_t^{\text{act}} = c$ is constant, but the observed time series x_t contains a point-spread noise then it is desirable that the trend extraction function F "recognizes" its presence and diminishes its influence the more the farther it is from a fixed designated point. In order to formalize this property, we notice that by the property of additivity, it is enough to consider the time series x_t whose values are zeroes except for one designated time moment with the value 1.

Let us denote $\bar{0}_k^q$ the q-tuple whose elements are 0s, except for the k-th one which is equal to 1. We say that the trend extraction function F works as a *"noise damper" centered at s*, if it fulfils the following condition:

$$\text{if } (s \le k_2 < k_1 \le q) \text{ or } (1 \le k_1 < k_2 \le s) \text{ then } F(\bar{0}_{k_1}^q) \le F(\bar{0}_{k_2}^q). \tag{11}$$

Remark 1. If a trend extraction function F fulfils (11) with the designated point s, then s should be reflected in its denotation. From now on, the trend extraction function will be denoted by F_s where $1 \le s \le q$ is the designated point to which we referred in (11).

By combining all requirements given above, we arrive at the following definition.

Definition 2. *Let $q \ge 2$ and $1 \le s \le q$. We say that the function $F_s : \mathbb{R}^q \to \mathbb{R}$ is a trend extraction function centered at s if it satisfies five properties given above, namely:* continuity*,* additivity *(8),* non-negativity *(9),* constant-preserving *(10), and* noise-damping centered at s *(11).*

In order to describe all possible trend extractor functions, we will make use of the following notion:

Definition 3. *A function $A : \mathbb{N} \to [0, 1]$ is called an (ℓ, s, u)-fuzzy number on $\mathbb{N}$ if there exist $\ell, s, u \in \mathbb{N}$ such that $\ell < u$, $\ell \le s \le u$, and*

(i) $A(i) = 0$ for all $i \in \mathbb{N} \setminus [\ell, u]$;
(ii) $A(i)$ non-strictly increases if $i \in [\ell, s]$ and non-strictly decreases if $i \in [s, u]$;
(iii) $A(s) = 1$.

Theorem 1. *Let $q \ge 2$ and $1 \le s \le q$. A function $F_s : \mathbb{R}^q \to \mathbb{R}$ is a trend extraction function centered at s if and only if there exists an $(1, s, q)$-fuzzy number A on $\mathbb{N}$ such that for all $x_1, \ldots, x_q \in \mathbb{R}$,*

$$F_s(x_1, \ldots, x_q) = \frac{\sum_{t=1}^{q} A(t) \cdot x_t}{\sum_{t=1}^{q} A(t)}. \tag{12}$$

Proof. One can easily check that for each $(1, s, q)$-fuzzy number A, the function F_s given by (12) is a trend extraction function centered at s. Let us prove that, vice versa, for every trend extraction function F_s centered at s there is an $(1, s, q)$-fuzzy number A for which F_s has the form (12).

Let F_s be a trend extraction function, i.e. it is continuous and fulfils (8) - (11). We will first prove that F_s is linear, i.e., it has the following form

$$F_s(x_1, \ldots, x_q) = \sum_{i=1}^{q} f_i \cdot x_i, \tag{13}$$

where $f_1, \ldots, f_q \in \mathbb{R}$ are some coefficients. By the assumption, F_s is continuous and additive, i.e., it satisfies the property (8). It is known (see, e.g., [1]) that every continuous additive function is a homogeneous linear function, i.e., that it has the form (13).

[5] We say that an observed time series x_t contains a point-spread noise if it differs from an actual time series x_t^{act} at a single time moment.

Let us prove that coefficients $f_1, \dots, f_q$, in (13) fulfil

$$\sum_{i=1}^{q} f_i = 1. \tag{14}$$

Indeed, by the assumption, F_s preserves constants, i.e., if $c = 1$, then $F_s(1, \dots, 1) = 1$. Together with (13) it gives

$$1 = F_s(1, \dots, 1) = \sum_{i=1}^{q} f_i \cdot 1 = \sum_{i=1}^{q} f_i.$$

Let us prove that the coefficients $f_1, \dots, f_q$, in (13) are non-negative. Indeed by (9), the trend extraction function F_s is non-negative. If we fix an arbitrary $i \in [1, q]$, and consider (13) where $x_i = 1$ and $x_j = 0$, $j \neq i$, then (13) takes the form

$$F_s(0, \dots, 0, \overbrace{1}^{\text{i}}, 0, \dots, 0) = f_i.$$

By (9), $f_i \geq 0$. Because it is true for all $i \in [1, N]$ the statement above is proven.

Finally, let us prove that the sequence $f_1, \dots, f_q$ (non-strictly) increases for $i \leq s$ and non-strictly decreases for $i \geq s$, i.e.,

$$f_1 \leq \ldots \leq f_s \geq f_{s+1} \geq \ldots \geq f_q. \tag{15}$$

By (11), the trend extraction function F works as a *"noise damper" centered at* s. Let $1 \leq k \leq q$, and $\bar{0}_k^q$ be the q-tuple whose elements are 0s, except for the k-th one which is equal to 1. By (13), $F_s(\bar{0}_k^q) = f_k$. Therefore, by (11),

$$\text{if } (s \leq k_2 < k_1 \leq q) \text{ or } (1 \leq k_1 < k_2 \leq s) \text{ then } f_{k_1} \leq f_{k_2},$$

which coincides with (15).

Let us now define the $(1, s, q)$-fuzzy number $A : \mathbb{N} \to [0, 1]$ which makes (12) true:

$$A(i) = \begin{cases} \frac{f_i}{f_s}, & \text{if } i \in [1, q], \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

By (13),

$$F_s(x_1, \dots, x_q) = \sum_{i=1}^{q} f_i \cdot x_i = f_s \sum_{i=1}^{q} A(i) \cdot x_i.$$

By (14),

$$1 = \sum_{i=1}^{q} f_i = f_s \sum_{i=1}^{q} A(i).$$

Therefore,

$$F_s(x_1, \dots, x_q) = f_s \sum_{i=1}^{q} A(i) \cdot x_i = \frac{\sum_{i=1}^{q} A(i) \cdot x_i}{\sum_{i=1}^{q} A(i)},$$

which proves (12).

3.3 Trend Extraction and the F-transform

In this subsection, we realize the goal which has been formulated above: a trend y_t of a time series x_t can be represented by its inverse F-transform $x_{F,n}$. We assume that the trend y_t is connected with a certain trend extraction function in such a way that for time moments $t_1, \dots, t_n \in [1, N]$, $y_{t_k} = F_s(x_{t_k - s + 1}, \dots, x_{t_k - s + q})$. Then we prove that the respective inverse F-transform $x_{F,n}$ interpolates values y_t, $t = t_1, \dots, t_n$, and by this, can be considered as a trend of x_t. As a preliminary result, we will show that the values of a trend extraction function applied to x_t are the F-transform components of x_t with respect to a certain fuzzy partition of $[1, N]$.

Theorem 2. *Let x_t, $t = 1, \ldots, N$, be a time series, $q \geq 2$, $1 \leq s \leq q$, and $F_s : \mathbb{R}^q \to \mathbb{R}$ be a trend extraction function centered at s. Then there exists a fuzzy partition $A_1, \ldots, A_n$ of $[1, N]$ with nodes $t_0, t_1, \ldots, t_n, t_{n+1} \in [1, N]$ such that for every $k = 1, \ldots, n$, the F-transform component X_k of the time series x_t is the value of F_s at the respective q-tuple of arguments, i.e.*

$$X_k = F_s(x_{t_k - s+1}, \ldots, x_{t_k - s+q}). \tag{17}$$

Proof. Let the assumptions above be fulfilled. Then by Theorem 1, there exists an $(1, s, q)$-fuzzy number A on $\mathbb{N}$ such that for all $x_1, \ldots, x_q \in \mathbb{R}$, (12) holds. For certainty, we assume that $s \leq q - s + 1$ and that $A(1) = A(q) = 0$. Let us choose the following nodes: $t_0 = 1$, $t_k = k(q - s)$, $k = 1, \ldots, n$, $t_{n+1} = N$, where we denote $n = \lfloor \frac{N}{q-s} \rfloor$[6]. For all $k = 1, \ldots, n$, $t_{k-1} \leq t \leq t_{k+1}$, we define the function A_k as follows:

$$A_k(t) = \begin{cases} 0, & \text{if } t_{k-1} \leq t \leq t_k - s, \\ A(s + t - t_k), & \text{if } t_k - s + 1 \leq t \leq t_k + q - s, \\ 0, & \text{if } t_k + q - s < t \leq t_{k+1}. \end{cases} \tag{18}$$

It is easy to see that $A_1, \ldots, A_n$ are basic functions of a certain fuzzy partition of $[1, N]$ with nodes $t_0, t_1, \ldots, t_n, t_{n+1}$. Moreover, on the interval $[1, t_n + q - s]$, this partition is uniform. The rest of the proof easily follows from (12).

Remark 2. It is clear from the proof of Theorem 2, that a fuzzy partition which guarantees (17) is not unique.

It remains to show that if a time series x_t is decomposed into a trend-cycle y_t and a seasonal-noise component z_t, then there exists a fuzzy partition $A_1, \ldots, A_n$ of $[1, N]$ such that the respective inverse F-transform $x_{F,n}$ interpolates those values y_t that belong to a certain sequence of time moments, and by this, $x_{F,n}$ can be considered as a trend of x_t.

Theorem 3. *Let a time series x_t be decomposed into a trend-cycle y_t and a seasonal-noise component z_t and moreover, there exists a trend extraction function $F_s : \mathbb{R}^q \to R$, $q \geq 2$, $1 \leq s \leq q$, centered at s and such that for time moments $t_1, \ldots, t_n \in [1, N]$,*

$$y_{t_k} = F_s(x_{t_k - s+1}, \ldots, x_{t_k - s+q}). \tag{19}$$

Assume that the distance between any two neighboring time moments is not greater than $(q - 2)$, i.e. for all $k = 1, \ldots, n - 1$, $(t_{k+1} - t_k) \leq q - 2$. Then there exists a fuzzy partition $A_1, \ldots, A_n$ of $[1, N]$ with nodes $1, t_1, \ldots, t_n, N \in [1, N]$ such that for every $k = 1, \ldots, n$,

$$y_{t_k} = x_{F,n}(t_k), \tag{20}$$

where $x_{F,n}$ is the inverse F-transform of x_t which is taken with respect to $A_1, \ldots, A_n$.

Proof. Let all the assumptions above be fulfilled. By Theorem 1, there exists an $(1, s, q)$-fuzzy number A on $\mathbb{N}$ such that for all $x_1, \ldots, x_q \in \mathbb{R}$, (12) holds. For certainty, we assume that $s \leq q - s + 1$, $A(1) = A(q) = 0$ and $A(t) > 0$ for $t = 2, \ldots, q - 1$. Let us define a fuzzy partition $A_1, \ldots, A_n$ of $[1, N]$ with nodes $1, t_1, \ldots, t_n, N \in [1, N]$ in accordance with (18). Then by similar reasoning as in the proof of Theorem 2, we have the F-transform component X_k, $k = 1, \ldots, n$, is a value of F_s at the respective q-tuple of arguments, i.e.

$$X_k = F_s(x_{t_k - s+1}, \ldots, x_{t_k - s+q}).$$

This equality together with the assumption (19) imply that $y_{t_k} = X_k$, $k = 1, \ldots, n$. It remains to prove that the inverse F-transform is an interpolating function on the domain $[t_1, t_n]$ with nodes $t_1, \ldots, t_n$ and the respective values $X_1, \ldots, X_n$.

At first, we verify that the inverse F-transform that is given by (7), is defined on $[t_1, t_n]$. This requires to prove that for all $t \in [t_1, t_n]$,

$$\sum_{k=1}^{n} A_k(t) > 0. \tag{21}$$

Indeed, let $t \in [t_k, t_{k+1}]$, where $k = 1, \ldots, n - 1$. By (18), the basic function A_k has $q - s$ positive values on $[t_k, t_{k+1}]$ including the value at t_k. Similarly, the basic function A_{k+1} has $s - 1$ positive values on

[6] By $\lfloor r \rfloor$ we denote the largest integer such that it is smaller than r.

$[t_k, t_{k+1}]$ including the value at t_{k+1}. If A_k and A_{k+1} have positive values at different points of $[t_k, t_{k+1}]$ then the number of points in $[t_k, t_{k+1}]$ is greater or equal to $(q - s) + (s - 1) = q - 1$. On the other side, the number of points in $[t_k, t_{k+1}]$ is equal to $(t_{k+1} - t_k) + 1 = q - 2 + 1 = q - 1$. Therefore, at each point of $[t_k, t_{k+1}]$, at least one function A_k or A_{k+1} is positive. Thus, (21) is true.

At second, we prove that for all $k = 1, \dots, n$, $x_{F,n}(t_k) = X_k$. Let k be within $[1, n]$. By (18), $A_j(t_k) = 0$, for all $j \neq k$. By (7),

$$x_{F,n}(t_k) = \frac{\sum_{j=1}^{n} X_j A_j(t_k)}{\sum_{j=1}^{n} A_j(t_k)} = \frac{X_k A_k(t_k)}{A_k(t_k)} = X_k.$$

Finally, by the fact that $y_{t_k} = X_k$, we proved (20).

4 Conclusion

We showed that a trend of a time series x_t, can be represented by its respective inverse F-transform $x_{F,n}$ where n is the the number of basic functions in the partition $A_1, \dots, A_n$ of $[1, N]$. For this purpose, we formalized the notion of a trend extraction function by listing its properties and then showed that components of the F-transform fulfil all of them. Finally, we showed that the respective inverse F-transform $x_{F,n}$ interpolates values of the trend extraction function at chosen nodes and by this, can be taken as a trend.

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