Aggregation Operations from Quantum Computing

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***Abstract*—Computer systems based on fuzzy logic are capable of generating a reliable output even when handling inaccurate input data by applying a rule based system. The main contribution of this paper is to show that quantum computing can be used to extend the class of fuzzy sets. The central idea associates the states of a quantum register to membership functions (mFs) of fuzzy subsets, and the rules for the processes of fuzzyfication are performed by unitary quantum transformations. Thus, this paper describes multidimensional quantum registers, associated to mFs on the unitary interval U, in order to introduce a novel interpretation of aggregations found in fuzzy set theory. In particular, t-norms and t-conorms based on quantum gates allows the modeling and interpretation of union, intersection, difference and implication among fuzzy sets, also including an expression for the class of fuzzy S-implications. Furthermore, an interpretation of the symmetric sum was achieved by considering the quantum register sum operator.**

***Index Terms*—Aggregation Operations, Fuzzy Logic, Quantum Operators, Quantum Computing.**

I. Introduction

The similarities between Fuzzy Logic (FL) and Quantum Computing (QC) motivate researches towards a better understanding of their relationship, as can be seen in [1], [2], [3], [4] and [5]. Since both FL and QC describe types of uncertainties, is important to investigate possible contributions from one area to another. Such study is relevant to understand how one can explore the phenomena of quantum mechanics to improve the efficiency of algorithms employed in the design of expert systems.

In this context, the logical structure describing the uncertainty associated with the fuzzy set theory can be modeled by quantum transformations (qTs) and quantum states (qSs). Thus, it is possible to model quantum algorithms which represent operations on fuzzy sets (union, intersection, difference, implication), and the mFs are encoded by qSs, whether in superposition or not.

Our main contribution is the modeling of quantum algorithms for specifying basic fuzzy operations as union, intersection, difference and implication functions. Extending a previous work [6], this paper focuses on the interpretation of aggregation functions. In particular, the symmetric sum is obtained by the summation operator in terms of quantum registers together with its geometric interpretation.

The operations were also studied and simulated in the visual programming environment VPE-qGM (Visual Programming Environment for the Quantum Geometric Machine Model), described in [7] and [8]. The VPE-qGM aims to support the modeling and the sequential or parallel simulation of quantum algorithms through a set of graphical interfaces. The simulations were performed for the union, intersection, difference and implication, but it also can be extended to the symmetric sum and other fuzzy operators.

This paper is organized as follows: Section II presents the foundations on FL. Section III brings the main concepts of QC. In Section IV, the study includes the modeling of fuzzy sets using QC, including some classical concepts such as quantum fuzzy set. Section V presents the operations on fuzzy sets modeled from qTs. Finally, conclusions and further work are discussed in Section VI.

II. Preliminaries on Fuzzy Logic

The non well-defined borders sets called fuzzy sets (FS) were introduced in order to overcome the limitations related to the use of classical sets for dealing with problems where the transitions from one class to another happen smoothly. The definition, properties and operations of FSs are obtained from the generalization of classical set theory, which can be seen as a particular case of fuzzy set theory.

The classical set theory is based on the characteristic function defined from a subset A of $\mathcal{X} \neq \emptyset$ to the Boolean set $\{0, 1\}$, i.e., it assigns to each $x \in \mathcal{X}$ an element of a discrete set $\{0, 1\}$ according to the expression:

$$\lambda_A(x) = \begin{cases} 1, & if\ x \in A, \\ 0, & if\ x \notin A; \end{cases} \quad (1)$$

The fuzzy set theory is based on a generalization of the characteristic function for the interval $U = [0, 1]$. For the membership $f_A(x) : \mathcal{X} \rightarrow U$, the element $x \in \mathcal{X}$ belongs to the subset A with a membership degree given by $f_A(x)$, such that $0 \leq f_A(x) \leq 1$.

Definition 1. *A **fuzzy set A** related to a set $\mathcal{X} \neq \emptyset$ is given by the expression:*

$$A = \{(x, f_A(x)) : x \in \mathcal{X}\}. \quad (2)$$

A. Fuzzy connectives

Definition 2. *A function $N : U \rightarrow U$ is a **fuzzy negation** (FN) when it verifies the following conditions:*

N1 $N(0) = 1$ *and* $N(1) = 0$*;*
N2 *If* $x \leq y$ *then* $N(x) \geq N(y)$*, for all* $x, y \in U$.

FNs verifying the involutive property:

N3 $N(N(x)) = x$, for all $x \in U$,

are called strong FNs. See, e.g., the standard negation:

$$N_S(x) = 1 - x. \quad (3)$$

Let N be a FN. For all $\vec{x} = (x_1, \ldots, x_n) \in U^n$, the N-dual function of $f\colon U^n \to U$ is given by the expression:

$$f_N(\vec{x}) = N(f(N(\vec{x}))), \quad (4)$$

where $N(\vec{x}) = (N(x_1), \ldots, N(x_n)) \in U^n$. Moreover, when $f_N(\vec{x}) = f(\vec{x})$, then f is a self-dual function.

Based on [9], [10], [11], [12] and [13], the general meaning of an aggregation function in FL is to assign a single real number on U to any n-tuple of real numbers belonging to U^n, that is, it is a non-decreasing and idempotent (i.e., it is the identity when an n-tuple is unary) function satisfying boundary conditions.

Among several definitions we will use the following one.

Definition 3. *[14, Definition 2], An* ***aggregation function*** *(AG) $A\colon U^n \to U$ demands, for all $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n) \in U^n$, the following conditions:*

A1: $A(\vec{0}) = A(0, 0, \ldots, 0) = 0$; $A(\vec{1}) = A(1, 1, \ldots 1) = 1$;

A2: *If* $\vec{x} \leq \vec{y}$ *then* $A(\vec{x}) \leq A(\vec{y})$;

A3: $A(\vec{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(x_1, x_2, \ldots, x_n) = A(\vec{x})$.

Extra properties for AGs are reported below:

A4: $A(x, x, \ldots, x) = x$, for all $x \in U$ (idempotency);

A5: $A(\vec{x}) = A_{N_S}(\vec{x})$ (self-duality).

Let $\vee, \wedge\colon U^2 \to U$ be the binary idempotent AGs defined as $\vee(x, y) = \max(x, y)$ and $\wedge(x, y) = \min(x, y)$. So, when A verifies $A4$, for all $x, y \in U$, then

$$\wedge(x, y) \leq A(x, y) \leq \vee(x, y), \quad (5)$$

and A is said to be compensatory in the unit interval.

Among the most often used AGs, frequently classified as compensatory and weighted operators, this paper considers the **symmetric sum**, which is, for all $\vec{x} \in U^n$, a continuous (with respect to each of its variables) and self-dual aggregation (verifying A5). In [15], a binary symmetric sum is expressed as

$$A_G(x, y) = \frac{G(\vec{x})}{G(\vec{x}) + G(N_S(\vec{x}))} \quad (6)$$

whenever $G : U^2 \to U$ is a continuous, increasing and positive function satisfying $G(0, 0) = 0$. It is worth noticing that there is not a unique function G characterizing each symmetric sum. Additionally, symmetric sums are in general not symmetric or commutative.

Proposition 1. *Let $a, b \in U$ and $G : U^2 \to U$ such that $\underline{G}(x, y) = \left(a\sqrt{x} + b\sqrt{y}\right)^2$. A function $A_{\underline{G}} : U^2 \to U$ given as:*

$$A_{\underline{G}}(x, y) = \frac{a\sqrt{x} + b\sqrt{y}}{\left(a\sqrt{x} + b\sqrt{y}\right)^2 + \left(a\sqrt{1-x} + b\sqrt{1-y}\right)^2}, \quad (7)$$

is defined as a symmetric sum as expressed in (6).

Proof. *The continuity of $A_{\underline{G}}$ follows from the composition over continuous functions on U. And, for all $(x, y) \in U^2$,*

$$A_{\underline{G}}(x, y) = \frac{a\sqrt{x} + b\sqrt{y}}{\left(a\sqrt{1-x} + b\sqrt{1-y}\right)^2 + \left(a\sqrt{x} + b\sqrt{y}\right)^2} = 1 - \frac{a\sqrt{1-x} + b\sqrt{1-y}}{\left(a\sqrt{1-x} + b\sqrt{1-y}\right)^2 + \left(a\sqrt{x} + b\sqrt{y}\right)^2}.$$

So, $A_{\underline{G}}(x, y) = A_{N_S}(x, y) = N_S A(N_S(x), N_S(y))$, implying that $A_{\underline{G}}$ is a self N_S-dual function. Moreover, it is immediate that $\underline{G}$ is a continuous, increasing and positive function satisfying $\underline{G}(0, 0) = 0$.

Now, conjunctive and disjunctive AGs are reported.

A **triangular (co)norm** (t-(co)norm) is a binary AG $(S)T\colon U^2 \to U$ satisfying the boundary condition, which is, respectively, given by the expressions:

T1: $T(x, 1) = x$; S1: $S(x, 0) = x$,

and the associativity property, respectively expressed as:

T2: $T(x, T(y, z)) = T(T(x, y), z)$; S2: $S(x, S(y, z)) = S(S(x, y), z)$.

There are many references reporting different definitions of t-norms and t-conorms [16]. Herein, for all $x, y \in U$, we consider the respective t-norm and t-conorm:

- *Algebraic product and algebraic sum*:

$$T_P(x, y) = x \cdot y; \text{ and } S_P(x, y) = x + y - x \cdot y. \quad (8)$$

A binary function $I : U^2 \to U$ is an implication operator (implicator) if the following conditions are satisfied:

I0: $I(1, 1) = I(0, 1) = I(0, 0) = 1$ and $I(1, 0) = 0$.

In [17] and [18], additional properties are considered to define a fuzzy implication obtained by an implicator:

Definition 4. *A* ***fuzzy implication*** *$I : U^2 \to U$ is an implicator verifying, for all $x, y, z \in U$, the conditions:*

I1: *Antitonicity in the first argument:*
if $x \leq z$ then $I(x, y) \geq I(z, y)$;

I2: *Isotonicity in the second argument:*
if $y \leq z$ then $I(x, y) \leq I(x, z)$;

I3: *Falsity dominance in the antecedent: $I(0, y) = 1$;*

I4: *Truth dominance in the consequent: $I(x, 1) = 1$.*

Among the implication classes with explicit representation by fuzzy connectives (negations and AGs) this work considers the class of (S, N)-implication, extending the classical equivalence $p \to q \Leftrightarrow \neg p \vee q$.

Let S be a t-conorm and N be a fuzzy negation. A (S, N)**-implication** is a fuzzy implication $I_{(S,N)} : U^2 \to U$ defined by:

$$I_{(S,N)}(x, y) = S(N(x), y), \forall x, y \in U. \quad (9)$$

If N is an involutive function, an S**-implication** is defined as in (9) [19]. The Reichenbach implication given as:

$$I_{RB}(x, y) = 1 - x + x \cdot y, \forall x, y \in U, \quad (10)$$

is an S-implication, obained by a fuzzy negation $N_S(x) = 1 - x$ and a t-conorm $S_P(x, y) = x + y - x \cdot y$, previously presented in Eqs. (3) and (8b), respectively.

B. Operations over fuzzy sets

Consider in the following definitions and examples of operations defined over the fuzzy sets $A, B \subseteq \mathcal{X}$.

Let $T, S : U^2 \to U$ be a t-(co)norm. [20].

The **complement of** A is a fuzzy set $A' = \{(x, f_{A'}) : x \in \mathcal{X}\}$, with $f_{A'} : \mathcal{X} \to U$ is given by:

$$f_{A'}(x) = N_S(f_A(x)) = 1 - f_A(x), \quad \forall x \in \mathcal{X}. \quad (11)$$

Let $A_{\underline{G}} : U^2 \to U$ be the symmetric sum, according to (7). The **symmetric sum between the fuzzy sets** A **and** B, is the fuzzy set $A \oplus B = \{(x, f_{A\oplus B}(x)) : x \in \mathcal{X}\}$, with $f_{A\oplus B}(x) : \mathcal{X} \to U$ given by:

$$f_{A\oplus B}(x) = A_{\underline{G}}(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \quad (12)$$

The **intersection between the fuzzy sets** A **and** B results in a fuzzy set $A \cap B = \{(x, f_{A\cap B}(x)) : x \in \mathcal{X}\}$, with $f_{A\cap B}(x) : \mathcal{X} \to U$ given by:

$$f_{A\cap B}(x) = T(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \quad (13)$$

An important characterization of the mF related to an intersection $A \cap B$ is obtained by applying the algebraic product to the fuzzy sets A and B, given by the following equation:

$$f_{A\cap B}(x) = f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (14)$$

Let $S : U^2 \to U$ be a t-conorm. A **union operation between fuzzy sets** A **and** B results in a fuzzy set $A \cup B = \{(x, f_{A\cup B}(x)) : x \in \mathcal{X}\}$, whose membership $f_{A\cup B}(x) : \mathcal{X} \to U$ is given by:

$$f_{A\cup B}(x) = S(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \quad (15)$$

A characterization of the fuzzy union $A \cup B$ is obtained by applying the algebraic product defined by

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (16)$$

Let $S : U^2 \to U$ be a t-conorm. An **implication operation between fuzzy sets** A **and** B results in a fuzzy set $A \triangleright B = \{(x, f_{A\triangleright B}(x)) : x \in \mathcal{X}\}$, whose mF $f_{A\triangleright B}(x) : \mathcal{X} \to U$ is given by:

$$f_{A\triangleright B}(x) = S(N(f_A(x)), f_B(x)), \forall x \in \mathcal{X}. \quad (17)$$

Extending the classical equivalence $\neg(p \to q) \Leftrightarrow p \wedge \neg q$, we obtain the difference operator.

Let S be t-conorm, N be a strong FN and I be an S-implication, A and B be FSs. A **difference between** A **and** B results in a FS $A \cup B = \{(x, f_{A-B}(x)) : x \in \mathcal{X}\}$, with$f_{A-B} : \mathcal{X} \to U$ given by:

$$f_{A-B}(x) = N(I_S(f_A(x), f_B(x))) \forall x \in \mathcal{X}. \quad (18)$$

By the composition of N_S and I_{RB} in Eqs.(10) and (3), respectively, see a fuzzy set $A \cup B$ obtained by the mF:

$$\begin{aligned} f_{A-B}(x) &= N_S(S_P(N_S(f_A(x)), f_B(x)) \\ &= f_A(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (19) \end{aligned}$$

III. Foundations on Quantum Computing

The *QC* considers the development of quantum computers, exploring the phenomena predicted by the *QM* (superposition of states, quantum parallelism, interference, entanglement) for better performance when they are compared to the analogous classical approach [21]. These quantum algorithms are modeled considering mathematical foundations which describe the phenomenae of *QM*.

A. Quantum state spaces

In *QC*, the *qubit* is the basic unit of information, being the simplest quantum system, defined by a state vector, unitary and bi-dimensional, generally described, in the notation of Dirac [21], by the expression

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \quad (20)$$

The coefficients α and β are complex numbers corresponding to the amplitudes of the respective states of the computational basis of one-dimensional quantum state space, verifying the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ and ensuring the unitary of the state vector of the quantum system, represented by $(\alpha, \beta)^t$.

The state space of a quantum system with multiple *qubits* is generated (span) by the tensor product of the state space of its subsystems. Considering a two *qubits* quantum system, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$, its tensor product $|\psi\rangle \otimes |\varphi\rangle$ is described by

$$|\psi\varphi\rangle = \alpha \cdot \gamma|00\rangle + \alpha \cdot \delta|01\rangle + \beta \cdot \gamma|10\rangle + \beta \cdot \delta|11\rangle. \quad (21)$$

B. Quantum transformations

The state transition of a quantum system is performed by unitary qTs associated with orthonormalized matrices of order 2^N, with N being the amount of *qubits* within the system. For instance, the definition of the *Pauly X* transformation and its application over a one-dimensional quantum system is described by

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \quad (22)$$

A bi-dimensional construction related to the tensor product of two *Pauly X* qTs is described in Eq (23):

$$X^{\bigotimes 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigotimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (23)$$

Controlled transformations also modify the state of one or more *qubits* considering the current state of another *qubit*. The *Toffoli* transformation [21] is a controlled operation performed over 3 *qubits*, which is obtained by a qT that applies the *NOT* operator (*Pauly X*) over the *qubit* $|\sigma\rangle$ when the current states of the first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both assigned as $|1\rangle$.

C. Measurement operations

The reading of the current state of a quantum system is performed by a measurement operator, which is defined based on a set of linear operators M_m, also called projections, acting on the state spaces. The index m refers to the possible measurement results. If the state of a quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurrence is given by [21]:

$$p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad (24)$$

The measurement operators satisfy the completeness relation $\sum_m M_m^\dagger M_m = I$[1]. For one-dimensional quantum systems, there exist the Hermitian (and thus, normal) matrix representation of these operators, described by

$$M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad and \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Measurement operators are self-adjoint non-reversible operators, satisfying the completeness relation

$$M_0^2{=}M_0^2,\ M_1^2{=}M_1^2 \text{ and } M_0^\dagger M_0 + M_1^\dagger M_1{=}I_2{=}M_0 + M_1.$$

When a *qubit* $|\psi\rangle$, with $\alpha, \beta \neq 0$, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively, given by:

- $p(|0\rangle) = \langle\phi|M_0^\dagger M_0|\phi\rangle = \langle\phi|M_0|\phi\rangle = |\alpha|^2$; and
- $p(|1\rangle) = \langle\phi|M_1^\dagger M_1|\phi\rangle = \langle\phi|M_1|\phi\rangle = |\beta|^2$.

Therefore, after the measure, the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the classical state $|0\rangle$; and $|\beta|^2$ as the probability to be in the other one, the state $|1\rangle$.

IV. Fuzzy Sets from Quantum Computing

The description of FS A from the quantum computing viewpoint considers $f_A(x)$, as state in (2).

Without losing generality, let $\mathcal{X}$ be a finite subset with cardinality N ($|\mathcal{X}| = N$). Thus, the definitions can be extended to infinite sets, considering in this case, a quantum computer with an infinite quantum register [21].

A. Classical fuzzy states - CFS

Definition 5. *[22, Definition 1] Consider $\mathcal{X} \neq \emptyset, |\mathcal{X}| = N$, $i \in \mathbb{N}_N = \{1, 2, ..., N\}$ and a function, $f : \mathcal{X} \to U$. The state of a N-dimensional quantum register, given as:*

$$|s_f\rangle = \bigotimes_{1\leq i\leq N} [\sqrt{1 - f_A(x_i)}|0\rangle + \sqrt{f_A(x_i)}|1\rangle] \quad (25)$$

is called ***classical fuzzy state*** *of N-qubits (CFS). In addition, [CFS] denotes the set of all CFS_S.*

Remark 1. ***Interpreting fuzzy set operations***
Let $\mathcal{A} = \{A_j\}$ be a finite collection of fuzzy sets related to an arbitrary set $\mathcal{X}$, x be an element of $\mathcal{X}$ and $|s_{f_{A_j}}\rangle$ be an one-dimensional classical fuzzy state defined as:

$$|s_{f_{A_j}}\rangle = \sqrt{1 - f_{A_j}(x)}|0\rangle + \sqrt{f_{A_j}(x)}|1\rangle.$$

[1] $M_m^\dagger$ denotes the transpose complex conjugate of the matrix M_m.

Then an interpretation to an N-ary fuzzy operator performed over all the collection $\mathcal{A}$ can be obtained by the following expression:

$$|x\rangle = \bigotimes_{1\leq j\leq N} |s_{f_{A_j}}(x)\rangle$$

In particular, such interpretation extends the notion of union and intersection to the collection $\mathcal{A}$.

Remark 2. ***Interpreting type-2 fuzzy sets***
Under the same conditions stated in Definition 5 , let $\mathbb{N}_N = \{1, 2, ..., N\}$ be a set of independent measurement sources and $|s_{f_A}(x_i)\rangle$ be a one-dimensional CFS defined as:

$$|s_{f_A}(x_i)\rangle = \sqrt{1 - f_A(x_i)}|0\rangle + \sqrt{f_A(x_i)}|1\rangle.$$

Then, the following expression

$$|z\rangle = \bigotimes_{1\leq i\leq N} |s_{f_A}(x_i)\rangle$$

provides an interpretation for the fuzzy set whose values are different membership degrees of an element $x \in \mathcal{X}$ and related to the same fuzzy set A. Such membership degrees can possibly be obtained by different measurement sources. More specifically, $s_{f_A}(x_i)$ indicates the membership degree of the element x to the set A measured by the source i.

Henceforth, this paper considers the interpretation of Remark 1. Generalizing, a state $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is reported as the following definition:

Definition 6. *[22, Section 3] The CFS of $N-$qubits, $|s_f\rangle \in [CFS]$, can be expanded in $\mathcal{C}^{2^N}$ by:*

$$\begin{aligned} |s_f\rangle \ = \ & (1 - f(1))^{\frac{1}{2}}(1 - f(2))^{\frac{1}{2}} \dots (1 - f(n))^{\frac{1}{2}}|00\dots00\rangle + \\ & f(1)^{\frac{1}{2}}(1 - f(2))^{\frac{1}{2}} \dots (1 - f(n))^{\frac{1}{2}}|10\dots00\rangle + \\ & f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}} \dots (1 - f(2))^{\frac{1}{2}}f(n)^{\frac{1}{2}}|11\dots01\rangle + \dots \\ & f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}} \dots f(n)^{\frac{1}{2}}|11\dots11\rangle. \end{aligned} \quad (26)$$

Concluding this section, from the perspective of QC, a fuzzy set consists on a superposition of crisp sets. Each $|s_f\rangle \in [CFS]$ is a representation of a quantum register described as a superposition of crisp sets and generated by the tensor product of non-entangled quantum registers [21].

B. Quantum Fuzzy Sets (QFS)

According to [22], it appears that the fuzzy sets are obtained by overlapping qSs from a conventional fuzzy quantum register. Moreover, from the set of mFs representing the fuzzy classical states, we obtain a linear combination, formalizing the notion of a fuzzy quantum register. In this context, it may be characterized:

- quantum fuzzy sets as quantum superposition of fuzzy subsets, which have different shapes, simultaneously.
- quantum fuzzy sets that are subsets of entangled superpositions of crisp subsets (or classical fuzzy sets).

Proposition 2. *[22, Theorem 1] Consider $N = |\mathcal{X}|$, A as a fuzzy subset. A quantum fuzzy subset related to a fuzzy set A is a point in the quantum states space C^{2^N}.*

Proposition 3. *[22, Theorem 2] Let $f, g : X \to U$ be mFs with respect to $\mathcal{X}$. The classical fuzzy sets $|s_f\rangle$ and $|s_g\rangle$ are mutually orthonormal CFSs if and only if there exists $x \in \mathcal{X}$ such that either $f(x) = 0$ and $g(x) = 1$ or the converse, $f(x) = 1$ and $g(x) = 0$.*

By Proposition 3, a pair of $|s_f\rangle$ and $|s_g\rangle$ in $[CFS]$ are mutually orthogonal CFSs if and only if there exists $x \in X$ such that $f(x) \cdot g(x) = 0$. In Eq (26), a qS $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is characterized, when all vectors are two by two orthonormal elements of a base in $\mathcal{C}^{2^N}$. For further specifications, see [21], [23] and [24].

Definition 7. *Consider $f_i : X \to U$, $i \in \{1, ..., k\}$, as a collection of mF generating FSs A_i and $\{|s_{f_1}\rangle, \ldots, |s_{f_k}\rangle\} \subseteq [CFS]$, such that their components are two by two orthonormal vectors. Let $\{c_1, \ldots, c_k\} \subseteq \mathcal{C}$. A* ***quantum fuzzy set*** *(QFS) $|s\rangle$ is a linear combination given by:*

$$|s\rangle = c_1|s_{f_1}\rangle + \ldots + c_k|s_{f_k}\rangle. \tag{27}$$

[CFQ] denotes the set of all $CFQs$.

From Def. 7, a fuzzy qS of a N-dimensional quantum register, as described in (27), can be entangled or not, depending on the family of classical fuzzy states $|s_{f_i}\rangle$ and the set C_i of chosen amplitudes.

Notice that, in Def. 7, non-entangled fuzzy states can be transformed into classical fuzzy states, by image of rotations on the Bloch's sphere axis (such as rotations of the meridian to achieve a zero phase), see details in [23].

V. Modeling Fuzzy Set Operations from Quantum Transformations

According to [22], fuzzy sets can be obtained by quantum superposition of classical fuzzy states associated with a quantum register. Thus, interpretations related to the fuzzy operations as complement and intersection are obtained from the NOT and AND qTs. Extending this approach, other operations are introduced, such as union, difference and fuzzy implication, which may be derived from interpretations of OR, DIV and IMP quantum operators.

For model, implement and validate these constructions from fuzzy quantum registers we make use of the visual programming environment VPE-qGM. It provides interpretations of the quantum memory, quantum processes and computations related to transition quantum states obtained from the simulation of related qSs and qTs.

For that, let $f_A, f_B : \mathcal{X} \to U$ be mFs obtained according to (25) and by a pair $(|s_{f_A}\rangle, |s_{f_B}\rangle)$ of CFS, given as:

$$|s_{f_A}\rangle \; = \; \sqrt{f_A(x_i)}|1\rangle + \sqrt{1 - f_A(x_i)}|0\rangle, \tag{28}$$

$$|s_{f_B}\rangle \; = \; \sqrt{f_B(x_i)}|1\rangle + \sqrt{1 - f_B(x_i)}|0\rangle, \forall x_i \in \mathcal{X}. \tag{29}$$

In the next sections, in order to simplify the notation, the membership degree defined by $f_A(x_i)$, which is related to an element $x_i \in \mathcal{X}$ in the fuzzy set A, will be denoted by f_A, once only one element will be considered to achieve interpretations for the main fuzzy set operations.

A. Fuzzy Complement

In the interpretation of the complement of a fuzzy set, the standard negation is obtained by the NOT operator related to a multi-dimensional quantum system. The action of the NOT operator is given by the expression:

$$NOT(|s_{f_A}\rangle) \; = \; \sqrt{1 - f_A}|1\rangle + \sqrt{f_A}|0\rangle \tag{30}$$

The complement operator can be applied to the state $|s_{f_A}\rangle$, resulting in an N-dimensional quantum superposition of 1-*qubit* states, described as $\mathcal{C}^{2^N}$ in the computational basis, according to (31):

$$NOT^N(|s_{f_A}\rangle) = NOT(\otimes_{1 \le i \le N}(f_A(i)^{\frac{1}{2}}|1\rangle(1 - f_A(i))^{\frac{1}{2}}|0\rangle)) = \otimes_{1 \le i \le N}((1 - f_A(i))^{\frac{1}{2}}|1\rangle + f_A(i)^{\frac{1}{2}}|0\rangle) \tag{31}$$

Now, Eqs. (32) and (33) describe other applications related to the NOT transformation acting on the 2nd and 3rd-*qubits* of a quantum system, respectively:

$$NOT_2(|s_{f_1}\rangle|s_{f_2}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle; \tag{32}$$

$$NOT_{2,3}(|s_{f_1}\rangle|s_{f_2}\rangle|s_{f_3}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \otimes NOT|s_{f_3}\rangle \tag{33}$$

In the next sections, these equations will describe other fuzzy operations, such as implications and differences.

B. Symmetric Sum

In the interpretation of AGs between the fuzzy sets A and B, related to the mFs $f_A, f_B : \mathcal{X} \to U$, respectively, the symmetric sum is obtained by the summation operator between two one-dimensional quantum registers. The action of such operator interpreting the binary symmetric sum, as stated in (6), is given as a linear combination $|\phi\rangle = a|s_{f_A}\rangle + b|s_{f_B}\rangle$ performed over the registers $|s_{f_A}\rangle$ and $|s_{f_B}\rangle$, by considering scalars $a, b \in U$:

$$|\phi\rangle = (a\sqrt{f_A} + b\sqrt{f_B})|1\rangle + (a\sqrt{1-f_A} + b\sqrt{1-f_B})|0\rangle).$$

Thus, we obtain the following quantum register by applying the normalization operator:

$$\frac{|\phi\rangle}{||\phi\rangle|} = \frac{(a\sqrt{f_A} + b\sqrt{f_B})|1\rangle + (a\sqrt{1-f_A}) + b\sqrt{1-f_B})|0\rangle}{\sqrt{(a\sqrt{f_A} + b\sqrt{f_B})^2 + (a\sqrt{1-f_A}) + b\sqrt{1-f_B})^2}}.$$

And, one of the following situations is obtained by a measurement performed over the above normalized state:

(1) an output (classic state) $|\phi'_1\rangle = |1\rangle$, with probability

$$p_1 = \frac{(a\sqrt{f_A} + b\sqrt{f_B})^2}{(a\sqrt{f_A} + b\sqrt{f_B})^2 + (a\sqrt{1-f_A}) + b\sqrt{1-f_B})^2}.$$

Then, p_1 indicates the membership degree of an element in the fuzzy set $A \oplus B$, as given by Eqs. (13) and (7).

(2) an output $|\phi_2'\rangle = |0\rangle$ with probability

$$p_0 = \frac{\left(a\sqrt{1-f_A} + b\sqrt{1-f_B}\right)^2}{\left(a\sqrt{f_A}+b\sqrt{f_B}\right)^2 + \left(a\sqrt{1-f_A}+b\sqrt{1-f_B}\right)^2}.$$

In this case, an expression of the complement of the symmetric sum between fuzzy sets A and B is given by $p_0 = 1 - p_1$. This probability also indicates the non-membership degree of an element in the fuzzy set $A \oplus B$.

Proposition 4. *For all $x \in \mathcal{X}$, let $0 \leq \frac{\alpha+\beta}{2} \leq \frac{\Pi}{2}$ such that $f_A = \sin^2 \alpha$ and $f_B = \sin^2 \beta$. Then it holds that:*

$$f_{A\oplus B} = \sin(\tfrac{\alpha+\beta}{2})^2.$$

Proof: If $f_A(x) = \sin^2 \alpha$ and $f_B(x) = \sin^2 \beta$, we have:

$$\begin{aligned}
&\sin(\frac{\alpha+\beta}{2})^2 = \frac{1}{2}(1-\cos(\alpha+\beta)) \\
&= \frac{1}{2}(1 + \sqrt{f_A f_B} - \sqrt{(1-f_A)(1-f_B)} \\
&= \frac{(\sqrt{f_A} - \sqrt{f_B})^2(1 + \sqrt{f_A f_B} - \sqrt{(1-f_A)(1-f_B)}}{2(f_A - f_B)^2} \\
&= \frac{(\sqrt{f_A} + \sqrt{f_B})^2(1 + \sqrt{f_A f_B} - \sqrt{(1-f_A)(1-f_B)}}{2(1+\sqrt{f_A f_B})^2 - (1-f_A)(1-f_B)} \\
&= \frac{(\sqrt{f_A} + \sqrt{f_B})^2}{(\sqrt{f_A} + \sqrt{f_B})^2 + (\sqrt{1-f_A)} + \sqrt{1-f_B})^2}
\end{aligned}$$

So, by Eqs.(7) and (12), if $a = b = 1$, $\sin(\frac{\alpha+\beta}{2})^2 = f_{A\oplus B}$. ∎

A geometric representation of results obtained in Proposition 4 is decribed in Figure 1. Moreover, the QFS described by Eqs.(28) and (29) are also quantum registers given as $|s_{f_A}\rangle = \sin\alpha^2|1\rangle + \cos\alpha^2|0\rangle$ and $|s_{f_B}\rangle = \sin\beta^2|1\rangle + \cos\beta^2|0\rangle$, respectively.

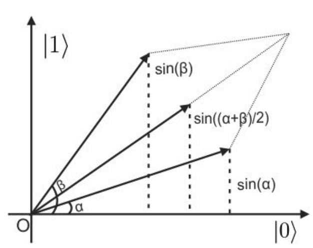

Figure 1. Interpreting symmetric sum from quantum registers.

C. Fuzzy Intersection

Let $|s_{f_A}\rangle$ and $|s_{f_B}\rangle$ be quantum registers given by Eqs. (28) and (29), with mFs $f_A, f_B : \mathcal{X} \to U$ related to an element $x_i \in \mathcal{X}$ respectively; and T be a Toffoli gate, which is an 3-*qubits* qT. An ***AND* operator** models a fuzzy intersection according to the expression:

$$\begin{aligned}
AND(|s_{f_i}\rangle, |s_{g_i}\rangle) &= T(|s_{f_i}\rangle, |s_{g_i}\rangle, |0\rangle) \qquad (34)\\
&= \left(\sqrt{f_A}|1\rangle + \sqrt{1-f_A}|0\rangle\right) \otimes \left(\sqrt{f_B}|1\rangle + \sqrt{1-f_B}|0\rangle\right) \\
&\quad \otimes \left(\sqrt{f_A f_B}|1\rangle + \sqrt{(1-f_A)f_B}|0\rangle\right).
\end{aligned}$$

So, by the distributivity of tensor product related to sum in (34), the next expression is held:

$$\begin{aligned}
AND(|s_{f_i}\rangle, |s_{g_i}\rangle) &= \sqrt{f_A f_B}|111\rangle| + \sqrt{f_A(1-f_B)}|100\rangle + \\
&(\sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle. \qquad (35)
\end{aligned}$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the qS expressed by (35), provides an output $|S_1'\rangle = |111\rangle$, with probability $p = f_A \cdot f_B$. Then, for all $i \in X$, f_A and f_B indicate the probability of $x_i \in \mathcal{X}$ is in the FS defined by $f_A(x) : \mathcal{X} \to U$ and $g_A(x) : \mathcal{X} \to U$, respectively. And then, $f_A \cdot f_B$ indicates the probability of x_i is in the intersection of such FSs. Analogously, a measurement of third *qubit* ($|0\rangle$) in the qS given by (35), returns an output state defined by:

$$\begin{aligned}
|S_2'\rangle = \frac{1}{\sqrt{(1-f_A)f_B}}&(\sqrt{f_A(1-f_B)}|100\rangle + \\
&\sqrt{(1-f_A)f_B}|010\rangle + \sqrt{(1-f_A)(1-f_B)}|000\rangle)
\end{aligned}$$

with probability $p_0 = 1 - f_A \cdot f_B$. In this case, an expression of the complement of the intersection between fuzzy sets A and B is given by $1 - p_0 = f_A \cdot f_B$. This probability indicates the non-membership degree of x is in the fuzzy set $A \cap B$. We also conclude that, by (35), it corresponds to the the standard negation of algebraic product as described in (8) [16].

Consider now, the initial qS resulting the tensor product $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |0\rangle$, according to (36):

$$|S\rangle = \frac{\sqrt{12}}{6}|000\rangle + \frac{\sqrt{6}}{6}|010\rangle + \frac{\sqrt{12}}{6}|100\rangle + \frac{\sqrt{6}}{6}|110\rangle \quad (36)$$

A simulation of the algorithm, considering the qS $|S\rangle$ in (36), was performed in the *VPE-qGM*, as illustrated in Figure 2. The final state is in accordance with the specification of the intersection operation described in (34). After a measurement, two possible situations are held:

- $|S_1'\rangle = |111\rangle$, with probability $p = 17\%$;
- $|S_2'\rangle = \frac{\sqrt{72}}{6\sqrt{5}}|000\rangle + \frac{\sqrt{36}}{6\sqrt{5}}|010\rangle + \frac{\sqrt{72}}{6\sqrt{5}}|100\rangle$, and $p = 83\%$.

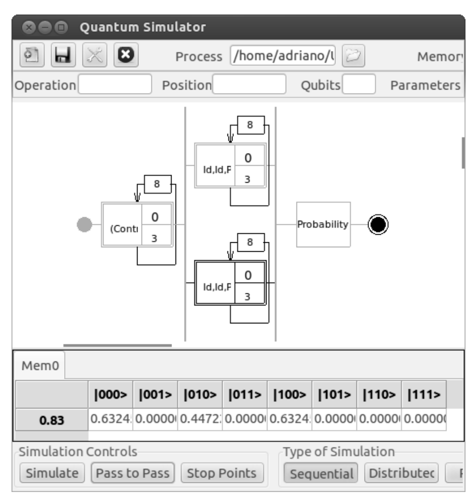

Figure 2. Modeling and simulation in the VPE-qGM of a quantum register interpreting a intersection between fuzzy sets.

D. Fuzzy Union

Let $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ be qSs given by Eqs. (28) and (29), respectively. The union of fuzzy sets is modeled by the ***OR* operator**, based on the expression:

$$\begin{aligned}OR(|s_{f_i}\rangle,|s_{g_i}\rangle)&=NOT^3(AND(NOT|s_{f_i}\rangle,NOT|s_{g_i}\rangle))\\ &=NOT^3(T(NOT|s_{f_i}\rangle,NOT|s_{g_i}\rangle,|0\rangle))\\ &=NOT^3(T(\sqrt{f_Af_B}|000\rangle+\sqrt{f_A(1-f_B)}|010\rangle+\\ &\sqrt{(1-f_A)f_B}|100\rangle+\sqrt{(1-f_A)(1-f_B)}|110\rangle)). \quad (37)\end{aligned}$$

In the sequence, applying the Toffoli transformation and the fuzzy standard negation we have that:

$$\begin{aligned}OR(|s_{f_i}\rangle,|s_{g_i}\rangle)=\sqrt{(1-f_A)(1-f_B)}|000\rangle+ \quad (38)\\ (\sqrt{(1-f_A)f_B}|011\rangle+\sqrt{f_A(1-f_B)}|101\rangle+\sqrt{f_Af_B}|111\rangle).\end{aligned}$$

After a measure performed on third *qubit* of qS:

1) when it is related to $|1\rangle$, we have the qS:

$$\begin{aligned}|S_1'\rangle = \frac{1}{\sqrt{f_B(1-f_A)+f_A)}}(\sqrt{(1-f_A)f_B}|011\rangle+\\ \sqrt{f_A(1-f_B)}|101\rangle+\sqrt{f_Af_B}|111\rangle),\end{aligned}$$

   with corresponding probability $p_1 = f_A+f_B-f_A\cdot f_B$ of $x_i \in \mathcal{X}$ is in both fuzzy sets A e B. See also that union is expressed by (39), which is related to the product t-conorm [16].
2) when it is related to state $|0\rangle$, returns the qS $|S_2'\rangle = |000\rangle$ with $p_0 = (1-f_A)\cdot(1-f_B)$, indicating that $x_i \in \mathcal{X}$ is not in such fuzzy sets (neither A nor B).

The modeling, implementation and simulation on *VPE-qGM* were performed in accordance with the description of union operation in (37) and considering the initial state as defined by (36). Similarly to the intersection operator, an interpretation of the final qS was performed in the VPE-qGM simulator. After the measurement process, one of two states is able to be reached:

- $|S_1'\rangle = \frac{1}{2}|011\rangle + \frac{\sqrt{2}}{2}|101\rangle + \frac{1}{2}|111\rangle$, such that $p = 67\%$;
- $|S_2'\rangle = |000\rangle$, with probability $p = 33\%$.

E. Fuzzy Implications

Fuzzy implications, as many other fuzzy connectives, can be obtained by a composition of quantum operations applied to quantum registers. In the following, this paper introduces the expression of the quantum operator denoted by *IMP*, over which an interpretation of Reichenbach implication is obtained.

For that, consider again the pair $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ of qSs given by Eqs. (28) and (29), respectively. The ***IMP* operator** is defined by:

$$\begin{aligned}IMP(|s_{f_i}\rangle,|s_{g_i}\rangle)&=NOT_2(AND(|s_{f_i}\rangle,NOT|s_{g_i}\rangle)) \quad (39)\\ &=NOT_2(T(|s_{f_i}\rangle,NOT|s_{g_i}\rangle,|0\rangle))\\ &=NOT_2(T(\sqrt{1-f_A)f_B}|000\rangle+\sqrt{1-f_A)(1-f_B)}|010\rangle+\\ &\sqrt{f_A(f_B)}|100\rangle+\sqrt{f_A(1-f_B)}|110\rangle))).\end{aligned}$$

In the following of (40), applying the *Toffoli* and negation quantum transformations, we have that:

$$\begin{aligned}IMP(|s_{f_i}\rangle,|s_{g_i}\rangle)=\sqrt{f_A(1-f_B)}|100\rangle)+ \quad (40)\\ \sqrt{(1-f_A)f_B}|011\rangle+\sqrt{(1-f_A)(1-f_B)}|001\rangle+\sqrt{f_Af_B}|111\rangle.\end{aligned}$$

Applying the same procedure, by a measure performed over the third *qubit* in the state defined by (40) we can get the two following qSs:

1) an output $|S_1'\rangle$, such that

$$\begin{aligned}|S_1'\rangle=\frac{1}{\sqrt{1-f_A+f_Af_B}}(\sqrt{(1-f_A)(1-f_B)}|001\rangle+\\ \sqrt{(1-f_A)f_B}|011\rangle+\sqrt{f_Af_B}|111\rangle), \quad (41)\end{aligned}$$

   with probability $p_1 = 1-f_A+f_A\cdot f_B = f_{A\triangleright B}$. Therefore, p_1 indicates the membership degree of an element in the fuzzy set $A\triangleright B$ (see in (17) related to I_{RB} fuzzy implication [25], as defined in (10).
2) an output $|S_2'\rangle = |100\rangle$ with probability $p_0 = f_A(1-f_B)$. In this case, an expression of the complement of the Reichenbach fuzzy implication related to the fuzzy sets A and B is given by $p_0 = 1-p_1$. This probability also indicates the non-membership degree of an element in the fuzzy set $A\triangleright B$.

Taking $|s_{f_2}\rangle\otimes|s_{f_3}\rangle\otimes|1\rangle$, according to (36). The modeling, implementation and simulation in the *VPE-qGM* based on the operator described in (40) yielded the possible final results as in the following statements:

- $|P_1'\rangle = \frac{\sqrt{2}}{2}|001\rangle + \frac{1}{2}|011\rangle + \frac{1}{2}|111\rangle$, with probability $p = 67\%$;
- $|P_2'\rangle = |100\rangle$, with probability $p = 33\%$.

F. Fuzzy difference

In this section, we introduce the quantum operator denoted by DIF, in order to provide interpretation to the difference between fuzzy sets based on quantum computing. The DIF operator is modeled by a composition of NOT and IMP qTs, previously presented in Sections V-A and V-E, considering the same initial conditions.

The ***DIF* quantum operator** is defined as follow:

$$\begin{aligned}DIF(|s_{f_i}\rangle,|s_{g_i}\rangle)&=NOT_{2,3}(AND(|s_{f_i}\rangle,NOT|s_{g_i}\rangle))\\ &=NOT_{2,3}(T(|s_{f_i}\rangle,NOT|s_{g_i}\rangle,|1\rangle).\\ &=NOT_{2,3}(T(\sqrt{1-f_A)f_B}|000\rangle+\sqrt{1-f_A)(1-f_B)}|010\rangle+\\ &\sqrt{f_A(f_B)}|100\rangle+\sqrt{f_A(1-f_B)}|110\rangle))). \quad (42)\end{aligned}$$

Then, by (33) together with (42) the DIF operator can be expressed as:

$$\begin{aligned}DIF(|\psi\rangle,|\phi\rangle)=\sqrt{(1-f_A)f_B}|01\rangle\otimes|0\rangle+\\ \sqrt{(1-f_A)(1-f_B)}|00\otimes|0\rangle+\sqrt{f_A(f_B)}|11\otimes|0\rangle+\\ \sqrt{f_A(1-f_B)}|10\rangle)\otimes(|1\rangle)). \quad (43)\end{aligned}$$

Thus, also in this last case study, we are able to provide an interpretation. After a measure performed over the third *qubit* of this qS, it returns one of the two the qSs:

1) $|S'_1\rangle = |101\rangle$, with $p_1 = f_A - f_A \cdot f_B = f_{A-B}$ related to the membership degree of an element to the corresponding fuzzy set $A - B$, see in (18); and
2) the superposition quantum state $|S'_2\rangle$, given as:

$$|S'_2\rangle = \frac{1}{\sqrt{(1-f_A)+f_Af_B)}}(\sqrt{(1-f_A(1-f_B)}|00\rangle + \sqrt{(1-f_A)f_B}|01\rangle + \sqrt{f_Af_B}|11\rangle),$$

with $p_0 = 1 - f_A + f_Af_B = 1 - f_{A-B}$ indicating the membership degree of an element in the FS $A - B$.

Preserving the configuration of previous case studies, the initial qS over that the difference operator is implemented and simulated in *VPE-qGM* is given by the tensor product $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |1\rangle$, according to (36).

According to the results presented by the VPE-qGM simulator, both possible qSs are given as:

- $|S'_1\rangle = |101\rangle$, with probability $p = 33\%$;
- $|S'_2\rangle = \frac{\sqrt{2}}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|110\rangle$, with probability $p = 67\%$ obtained by a simulation on VPE-qGM.

VI. Conclusion and Final Remarks

This paper describes fuzzy sets and operations on fuzzy sets through concepts of QC. The mFs are modelled as quantum registers and the operations over fuzzy sets are described by qTs. Hence, this work shows basic constructions in the specification of fuzzy expert systems from QC, in order to obtain new information technologies based on fuzzy approach.

This paper not only analyses the operations of fuzzy complement and fuzzy intersection as described in [22] but also implements and simulates them in the VPE-qGM presenting an extension of such construction to other important fuzzy operations. This extension considers the modeling of the following fuzzy operations obtained from quantum operators: union, difference and implications, focuses on the class of S-implications.

Furthermore, another aggregation operation, now related to the symmetric sum, was defined in terms of quantum registers, expanding the range of possible AGs that can be represented by QC.

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