Why Inverse F-transform? A Compression-Based Explanation

Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
500 W. University
El Paso, TX 679968, USA
vladik@utep.edu

Irina Perfilieva and Vilem Novák
Centre of Excellence IT4Innovations
division of the University of Ostrava
Institute for Research and Applications of Fuzzy Modeling
ul. 30. dubna 22, 701 00 Ostrava 1, Czech Republic
Irina.Perfilieva@osu.cz, Vilem.Novak@osu.cz

***Abstract*—In many practical situations, e.g., in signal processing, image processing, analysis of temporal data, it is very useful to use fuzzy (F-) transforms. In an F-transform, we first replace a function $x(t)$ by a few local averages (this is called *forward* F-transform), and then reconstruct the original function from these averages (this is called *inverse* F-transform). While the formula for the forward F-transform makes perfect intuitive sense, the formula for the inverse F-transform seems, at first glance, somewhat counter-intuitive. On the other hand, its empirical success shows that this formula must have a good justification. In this paper, we provide such a justification – a justification which is based on formulating a reasonable compression-based criterion.**

I. Formulation of the Problem

Main problem. In many real-life situations, fuzzy transform (F-transform, for short) leads to a good quality compression of the original signals, images, etc.

How can we explain this empirical fact? Approximation problems are well-known and well-studied in numerical mathematics, so, at first glance, we should simply use the corresponding well-developed optimization criteria and confirm that F-transform is indeed better according to these criteria. Surprisingly, while F-transform is often *empirically* better, the existing *theoretical* criteria select other approximation methods as much better ones.

This discrepancy between empirical evidence and the existing theoretical criteria shows that these criteria are not fully adequate for comparing real-life compression results.

What we do in this paper. In this paper, we formulate a more adequate theoretical criterion, and we show that – in accordance with the empirical data – that this criterion indeed leads to F-transform.

The structure of the paper. We start with a detailed description of F-transforms in Section 2. This section contains not only the formal definitions, it also contains motivations for these definitions. In Section 3, we explain that with respect to the standard optimization criteria from numerical analysis, the F-transform formulas are not optimal. In Section 4, we present a general definition of the inverse transform, a definition which includes both F-transform and the traditional numerical approximations as particular cases. Finally, in Section 5, we derive the new criterion for selecting compression techniques, and we show that this new criterion leads to F-transform. The results are summarized in the Conclusions section.

II. F-transforms: Reminder

Data compression: one of the problems for which F-transforms were invented. In many practical situations, we need to compress the data. For example, we have records describing how a certain physical characteristic x (e.g., temperature) changes with time. Sometimes, it takes too much space to store all this information; sometimes, it takes too much computation time to process all this information. In all these situations, we need to *compress* the data, i.e., to replace the original values $x(t)$ corresponding to different moments of time t with a few combinations of these values.

What is the best way to compress a 1-D signal? The possibility to compress a signal comes from the fact that usually, the signals slowly change with time, so the values at nearby moments of time are almost identical. As a result, within a short time interval $[\underline{t}, \overline{t}]$, all the values $x(t)$ are approximately equal to each other and so, we can represent all these values $x(t)$ by a single number.

What number should we use? If we knew the exact values of $x(t)$ for $t \in [\underline{t}, \overline{t}]$, then we could use any of these values as the desired single number. In practice, however, the recorded values $x(t)$ come from measurements, and measurements are never absolutely accurate. Due to this inevitable measurement inaccuracy, each recorded value $x(t)$ is, in general, different from the actual value $X(t)$ of the corresponding quantity.

It is well known that we can decrease the measurement error if we repeat a measurement several times and then average the results; see, e.g., [8], [9]. Often, it is better to take a weighted average, so that we give more weight to more accurate measurements and less weight to less accurate ones.

The more measurements we make, the more accurate the resulting average. Thus, instead of a single value $x(t)$, it is better to take the weighted average of all the corresponding values $x(t)$. For example, if we are interested in the value $X(t_m)$ in the midpoint t_m of the interval $[\underline{t}, \overline{t}]$, then:

- the value $x(t_m)$ measured at this same moment of time differs from the desired value $X(t_m)$ only by the measurement error,
- while the values $x(t)$ corresponding to $t \neq t_m$ also differ because the signal somewhat changes between the moments of time t and t_m.

The further away the moment t from t_m, the larger the inaccuracy with which the value $x(t)$ represents $X(t_m)$ and so, the smaller should be the weight with which we take this value. Thus, it is reasonable to use a weighted average

$$\int a(t) \cdot x(t)\, dt$$

to represent all the values from this interval, where the weight $a(t)$ increases until it reaches the midpoint t_m and then decreases.

Mathematical comment. While a weighted average sounds reasonable, a natural question is: why not use some more general (e.g., non-linear) transformation? The answer to this question comes from the fact that both the measurement error $\Delta x(t) \stackrel{\text{def}}{=} x(t) - X(t)$ and the signal $x(t)$ itself come from a large number of different independent factors. According to the Central Limit Theorem (see, e.g., [9]), a joint effect of many such factors is approximately normally distributed. Thus, in a good approximation, we can assume that the joint distribution of all the corresponding variables $X(t)$ and $\Delta x(t)$ is Gaussian.

Under this assumption, we want to minimize, e.g., the mean square difference between the actual (unknown) value $X(t_m)$ and the corresponding estimate $e(x)$, i.e., to find an estimate $e(x)$ depending on all the observed values $x(t)$ for which the expected value $E[(X(t_m) - e(x))^2]$ is the smallest possible. Alternative, instead of reconstructing a single value $X(t_m)$, we can try to reconstruct all the values $X(t)$ corresponding to $t \in [\underline{t}, \overline{t}]$, and thus, minimize the (weighted) average of the corresponding expected values. In all such cases, we want to find an estimate $e(x)$ that minimizes a certain quadratic objective function.

It is known that for Gaussian distribution, the optimal estimate to each such quadratic optimization is a linear function of the inputs [9]. For example, the estimate that minimizes the expected value is simply equal to the conditional expected value $E[X(t_m) \,|\, x(t)]$ under the condition that we know all observed values of $x(t) = X(t) + \Delta x(t)$, and it is known that for the normal distribution, this expected value is a linear function of the variables $x(t)$. Thus, the weighted average is indeed the best possible estimation.

We need weighted averages corresponding to several intervals. To describe the behavior of the observed signal $x(t)$ in different intervals $[\underline{t}_i, \overline{t}_i]$, we need to know the weighted averages $\int a_i(t) \cdot x(t)\, dt$ corresponding to different intervals.

The optimal combination of intervals $[\underline{t}_i, \overline{t}_i]$ and weights $a_i(t)$ depend on the statistical properties of the signal and the noise. In many practical situations, these properties do not change much with time: we use the same measuring instruments, with the same accuracy, and we observe the same physical process. Sometimes, there are abrupt changes that are worthy of our interest, but if we are interested in a local interval in which the signal changes abruptly, then we should not compress the data from this interval, we should take each measurement result into account. The possibility to compress without losing information only comes when we observe a signal that does not change much, and whose statistical characteristics do not change much.

Since the widths of the intervals $[\underline{t}_i, \overline{t}_i]$ and $[\underline{t}_j, \overline{t}_j]$ and the corresponding weights $a_i(t)$ and $a_j(t)$ are determined by the statistical characteristics, and these characteristics do not change when we move to another time interval, it is reasonable to conclude that these widths and weights also do not change from one interval to another. To be more precise:

- all intervals should be of the same width, and
- the values of the weights are the same if we measure them against the midpoints t_{mi} and t_{mj} of the corresponding intervals: $a_i(t) = a(t - t_{mi})$ and $a_j(t) = a(t - t_{mj})$ for the same shape function $a(t)$ that increases until 0 and then decreases.

At first glance, it may seem that it is sufficient to divide the original time interval into subintervals of equal length

$$[t_0, t_0 + H], \quad [t_0 + H, t_0 + 2H], \ldots,$$

and replace the original function $x(t)$ with weighted averages over each of these intervals. The problem with this approach is that this way, we will reconstruct the values at midpoints well, but not the values at the endpoints $t_0 + k \cdot H$ of the corresponding intervals. A natural way to also reconstruct these endpoint values is to supplement the original weighted averages with similar averages centered around these endpoints, i.e., values corresponding to intervals

$$[t_0 + H/2, t_0 + 3H/2], \quad [t_0 + 3H/2, t_0 + 5H/2], \ldots$$

If we denote $h \stackrel{\text{def}}{=} H/2$, we arrive at the following scheme: we select a function $a(t)$ which is:

- defined on an interval $(-h, h)$,
- increasing for $t < 0$, and
- decreasing for $t > 0$,

take the values $t_i = t + i \cdot h$, and replace the original signal $x(t)$ with weighted averages

$$x_i = \int_{t_{i-1}}^{t_{i+1}} a_i(t) \cdot x(t)\, dt, \text{ where } a_i(t) = a(t - t_i).$$

If the signal $x(t)$ does not change with time, then we can compress it into a single value – the value of this signal at any moment of time. It is reasonable to require that for such a constant signal $x(t) = \text{const}$, each compressed value coincides with the same constant, i.e., that

$$\int_{t_{i-1}}^{t_{i+1}} a_i(t)\, dt = \int_{-h}^{h} a(t)\, dt = 1.$$

Because of this property, if we change the step h, we have to also re-scale the weight function $a(t)$. It is therefore

convenient to fix the weight function, e.g., by requiring that its largest value is 1 ($e(0) = 1$), and then, for each h, to *normalize* the function $e(t)$ by dividing it by the corresponding integral $\int e(t)\,dt$:

$$a(t) = \frac{e(t)}{\int e(s)\,ds}.$$

In terms of the non-normalized weight function $e(t)$, the weighted average takes the form

$$x_i = \frac{\int_{t_{i-1}}^{t_{i+1}} e_i(t) \cdot x(t)\,dt}{\int_{t_{i-1}}^{t_{i+1}} e_i(t)\,dt},$$

where $e_i(t) \stackrel{\text{def}}{=} e(t - t_i)$.

An additional simplifying property. All we want from the weighted function $e(t)$ is that it is equal to 1 for $t = 0$, increases for $t < 0$, and decreases for $t > 0$. It make sense to require that $e(-h) = e(h) = 0$, since the width of the interval $[t_i - h, t_i + h]$ is selected by the condition that values outside the interval have changed too much to provide the desired approximation accuracy for the midpoint $x(t_i)$.

The simplest such function is a piece-wise linear function $e(t)$ which linearly increases from 0 to 1 on the the interval $(-h, 0]$ and then linearly decreases from 1 to 0 on the interval $[0, h)$, i.e., the function

$$e(t) = \max\left(1 - \frac{|t|}{h}, 0\right).$$

For this function, we have an additional property that $e_i(t) + e_{i+1}(t) = 1$ for all t. This additional property is often helpful. Thus, we arrive at the following definition; see, e.g., [1], [4], [5], [6], [7]:

Definition 1. *Let t_0 and $h > 0$ be real numbers, let $n > 0$ be an integer, and let $t_i \stackrel{\text{def}}{=} t_0 + i \cdot h$. Let $e(t)$ be a function which:*

- *is non-zero only on the interval $(-h, h)$,*
- *increases from 0 to 1 for negative t and decreases from 1 to 0 for positive h, and*
- *for which, for functions $e_i(t) \stackrel{\text{def}}{=} e(t - t_i)$, we have $e_i(t) + e_{i+1}(t) = 1$ for all integers i and all values $t \in [t_i, t_{i+1}]$.*

Then, for each function $x(t)$, its F-transform *is defined as a sequence of values $x_0, x_2, \ldots, x_n$, where:*

$$x_0 = \frac{\int_{t_0}^{t_1} e_0(t) \cdot x(t)\,dt}{\int_{t_0}^{t_1} e_0(t)\,dt},$$

$$x_i = \frac{\int_{t_{i-1}}^{t_{i+1}} e_i(t) \cdot x(t)\,dt}{\int_{t_{i-1}}^{t_{i+1}} e_i(t)\,dt}, \quad i = 1, 2, \ldots, n-1,$$

$$x_n = \frac{\int_{t_{n-1}}^{t_n} e_n(t) \cdot x(t)\,dt}{\int_{t_{n-1}}^{t_n} e_n(t)\,dt}.$$

Comment. Each function $e_i(t)$ increases from 0 to 1, reaches the value 1 at $t = t_i$, and then decreases from 1 to 0. Thus, it can be interpreted as a membership function describing to what extent t is close to the point t_i; see, e.g., [2], [3], [10]. Since on each interval $[t_i, t_{i+1}]$, only two of the functions $e_j(t)$ are different from 0–the functions $e_i(t)$ and $e_{i+1}(t)$–the condition that $e_i(t) + e_{i+1}(t) = 1$ means that $\sum_{j=0}^{n} e_j(t) = 1$ for all t, i.e., that the membership functions $e_i(t)$ form a *fuzzy partition*.

Inverse F-transform: what it is and how it is usually defined. Once we finished the compression, we replace the original signal by the values $x_0, \ldots, x_n$. As we have mentioned earlier, each of these values x_i if a good approximation to the original value $x(t_i)$. What if we are now interested in estimating the value $x(t)$ for some moment of time which is different from the endpoints t_i, i.e., for some moment of time t which is strictly in between two endpoints t_i and t_{i+1}?

In principle, since we assumed that the value of the signal $x(t)$ does not change much between the moments t_i and t_{i+1}, we can simply take one of the values $x_i \approx x(t_i)$ and $x_{i+1} \approx x(t_{i+1})$ as the desired estimates. But maybe we can do better by appropriately combining several values x_j? Besides, if we simply replace all the values $x(t)$ with one of the values x_i, we transform the original continuous signal with a piece-wise constant discontinuous one, and it is desirable, when reconstructing, to preserve important signal properties like continuity.

As a result, it was proposed to use the following formula:

Definition 2. *Let t_0 and $h > 0$ be real numbers, let $n > 0$ be an integer, and let $t_i \stackrel{\text{def}}{=} t_0 + i \cdot h$. Let $e(t)$ be a function which:*

- *is non-zero only on the interval $(-h, h)$,*
- *increases from 0 to 1 for negative t and decreases from 1 to 0 for positive h, and*
- *for which, for functions $e_i(t) \stackrel{\text{def}}{=} e(t - t_i)$, we have $e_i(t) + e_{i+1}(t) = 1$ for all integers i and all values $t \in [t_i, t_{i+1}]$.*

Let $x_0, \ldots, x_n$ be the F-transform of a function $x(t)$. By an inverse F-transform*, we mean a function*

$$\hat{x}(t) = \sum_{i=1}^{n} x_i \cdot e_i(t).$$

Important remark: applications of F-transform go beyond compression. It should be mentioned that while our motivation for F-transform came from compression, F-transforms are useful beyond compression. For example, when we first perform F-transform and then inverse F-transform, we also get rid of the high-frequency noise, and so, F-transforms are also very useful for de-noising and smoothing; see, e.g., [1], [4], [5], [6], [7].

III. At First Glance, the Formula for the Inverse F-Transform Is Somewhat Counterintuitive

Reconstructing the function $x(t)$: a seemingly reasonable idea. The result of the inverse F-transform is a linear combination of the functions $e_i(t)$: $\hat{x}(t) = \sum_{i=0}^{n} c_i \cdot e_i(t)$ for some

coefficients c_i. As part of our motivations, we wanted to make sure that the mean square approximation error is the smallest possible. It therefore seems to be reasonable to look for the coefficients c_i for which the approximate expression

$$x_{\approx}(t) = \sum_{i=0}^{n} c_i \cdot e_i(t)$$

is the closest to the original function $x(t)$ in the sense that the value

$$\int (x_{\approx}(t) - x(t))^2\, dt = \int \left(\sum_{i=0}^{n} c_i \cdot e_i(t) - x(t) \right)^2$$

is the smallest possible.

The above idea leads to an approximation which is different from the inverse F-transform. The above idea is reasonable and easy to implement, by using the known fact that when a differentiable function attains its minimum, its derivatives are equal to 0. Here, the minimized expression is linear in terms of the unknown coefficients $c_0, c_1, \ldots, c_n$. Thus, if we differentiate the above expression with respect to each unknown c_j, we get the following system of $n+1$ linear equations for $n+1$ unknown c_i:

$$\int 2 \cdot \left(\sum_{i=0}^{n} c_i \cdot e_i(t) - x(t) \right) \cdot e_j(t)\, dt = 0,$$

or, equivalently,

$$\sum_{i=1}^{n} c_i \cdot \int e_i(t) \cdot e_j(t)\, dt = \int x(t) \cdot e_j(t)\, dt.$$

For the functions $e_i(t) = e(t - t_i)$ corresponding to F-transform, this expression can be further simplified if we take into account that when $|i - j| > 1$, the functions $e_i(t)$ and $e_j(t)$ are different from 0 at non-intersecting intervals and thus, $e_i(t) \cdot e_j(t) = 0$ for all t and $\int e_i(t) \cdot e_j(t)\, dt = 0$. Thus, the above system can be simplified as follows:

$$c_0 \cdot \int e_0^2(t)\, dt + c_1 \cdot \int e_0(t) \cdot e_1(t)\, dt =$$

$$\int x(t) \cdot e_0(t)\, dt;$$

$$c_j \cdot \int e_j^2(t)\, dt + c_{j-1} \cdot \int e_j(t) \cdot e_{j-1}(t)\, dt +$$

$$c_{j+1} \cdot \int e_j(t) \cdot e_{j+1}(t)\, dt = \int x(t) \cdot e_j(t)\, dt,$$

$$j = 1, 2, \ldots, n-1;$$

$$c_n \cdot \int e_n^2(t)\, dt + c_{n-1} \cdot \int e_n(t) \cdot e_{n-1}(t)\, dt =$$

$$\int x(t) \cdot e_n(t)\, dt.$$

Each right-hand side $\int x(t) \cdot e_j(t)\, dt$ can be described, in terms of the F-transform x_j, as $x_j \cdot \int e_j(t)\, dt$; thus, by dividing both sides of each equation by the integral $\int e_j(t)\, dt$, we get

$$c_0 \cdot \frac{\int e_0^2(t)\, dt}{\int e_0(t)\, dt} + c_1 \cdot \frac{\int e_0(t) \cdot e_1(t)\, dt}{\int e_0(t)\, dt} = x_0;$$

$$c_j \cdot \frac{\int e_j^2(t)\, dt}{\int e_j(t)\, dt} + c_{j-1} \cdot \frac{\int e_j(t) \cdot e_{j-1}(t)\, dt}{\int e_j(t)\, dt} +$$

$$c_{j+1} \cdot \frac{\int e_j(t) \cdot e_{j+1}(t)\, dt}{\int e_j(t)\, dt} = x_j,$$

$$j = 1, 2, \ldots, n-1;$$

$$c_n \cdot \frac{\int e_n^2(t)\, dt}{\int e_n(t)\, dt} + c_{n-1} \cdot \frac{\int e_n(t) \cdot e_{n-1}(t)\, dt}{\int e_n(t)\, dt} = x_n.$$

Since the functions $e_i(t)$ are obtained from each other by shift, the integrals

$$\int e_j^2(t)\, dt, \quad \int e_j(t)\, dt \text{ and } \int e_j(t) \cdot e_{j-1}(t)\, dt$$

are the same for all $j = 1, 2, \ldots, n$. We will denote these common values by

$$I_2 = \int e_j^2(t)\, dt, \quad I_1 = \int e_j(t)\, dt, \text{ and}$$

$$I_3 = \int e_j(t) \cdot e_{j-1}(t)\, dt.$$

For example, for the above-mentioned triangular function $e(t) = 1 - \frac{|t|}{h}$, we have

$$I_2 = \frac{2}{3} \cdot h^2, \quad I_1 = h, \quad I_3 = \frac{1}{6} \cdot h^2.$$

In these terms, the above system takes the form

$$c_0 \cdot k_0 + c_1 \cdot k_1 = x_0;$$

$$c_j \cdot p_0 + c_{j-1} \cdot p_1 + c_{j+1} \cdot p_1 = x_j, \quad j = 1, \ldots, n-1;$$

$$c_n \cdot k_n + c_{n-1} \cdot k_{n-1} = x_n,$$

where

$$k_0 \stackrel{\text{def}}{=} \frac{\int e_0^2(t)\, dt}{\int e_0(t)\, dt}; \quad k_1 \stackrel{\text{def}}{=} \frac{I_3}{\int e_0(t)\, dt};$$

$$p_0 \stackrel{\text{def}}{=} \frac{I_2}{I_1}; \quad p_1 \stackrel{\text{def}}{=} \frac{I_3}{I_1};$$

$$k_n \stackrel{\text{def}}{=} \frac{\int e_n^2(t)\, dt}{\int e_n(t)\, dt}; \quad k_{n-1} \stackrel{\text{def}}{=} \frac{I_3}{\int e_n(t)\, dt}.$$

In particular, for the triangular function $e(t)$, we have

$$\int e_0^2(t)\, dt = \int e_n^2(t)\, dt = \frac{1}{3} \cdot h^2,$$

$$\int e_0(t)\, dt = \int e_n(t)\, dt = \frac{1}{2} \cdot h,$$

hence

$$k_0 = k_n = p_0 = \frac{2}{3} \cdot h, \quad k_1 = k_{n-1} = \frac{1}{3} \cdot h, \quad p_1 = \frac{1}{6} \cdot h.$$

So, the above system takes the form

$$\frac{2}{3}\cdot h\cdot c_0+\frac{1}{3}\cdot h\cdot c_1=x_0;$$

$$\frac{2}{3}\cdot h\cdot c_j+\frac{1}{6}\cdot h\cdot c_{j-1}+\frac{1}{6}\cdot h\cdot c_{j+1}=x_j;\quad j=1,\ldots,n;$$

$$\frac{2}{3}\cdot h\cdot c_n+\frac{1}{3}\cdot h\cdot c_{n-1}=x_n.$$

Clearly, the solution $c_i=x_i$ corresponding to the inverse F-transform does not satisfy these equations. Thus, with respect to the above optimization criterion, the inverse F-transform is *not* optimal.

Discussion.

- On the one hand, the inverse F-transform does not minimize the mean square approximation error.
- On the other hand, in many practical problems, the inverse F-transform works very well.

This means that this practical success corresponds not to minimizing the mean square approximation error, but to some other reasonable criterion.

What we do in this paper. In this paper, we describe an alternative reasonable criterion, and we show that for this criterion, inverse F-transform is indeed optimal.

IV. A General Definition of an Inverse Transform

Discussion. We want to go from the F-transform values $x_0,\ldots,x_n$ of a function $x(t)$ to an approximation $\sum\limits_{i=0}^{n} c_i\cdot e_i(t)$ to the original function $x(t)$. For fixed functions $e_i(t)$, this means that we need a mapping that maps a vector $(x_0,\ldots,x_n)$ into a vector $(c_0,c_1,\ldots,c_n)$.

Since the transformation from $x(t)$ to x_i is linear, it is reasonable to restrict ourselves to linear inverse transformations, i.e., to the transformations of the type $c_i=\sum\limits_{j=1}^{n} k_{i,j}\cdot x_j$ for some coefficients $k_{i,j}$. The question is then to define appropriate coefficients $k_{i,j}$.

The inverse F-transform corresponds to the unit matrix I for which $k_{i,i}=1$ and $k_{i,j}=0$ for $i\neq j$.

Definition 3. *Let t_0 and $h>0$ be real numbers, let $n>0$ be an integer, and let $t_i\stackrel{\text{def}}{=}t_0+i\cdot h$. Let $e(t)$ be a function which:*

- *is non-zero only on the interval $(-h,h)$,*
- *increases from 0 to 1 for negative t and decreases from 1 to 0 for positive h, and*
- *for which, for functions $e_i(t)\stackrel{\text{def}}{=}e(t-t_i)$, we have $e_i(t)+e_{i+1}(t)=1$ for all integers i and all values $t\in[t_i,t_{i+1}]$.*

By an inverse transform*, we mean a matrix K with elements $k_{i,j}$, $0\le i,j\le n$. Let $x_0,\ldots,x_n$ be the F-transform of a function $x(t)$. For each matrix K, by a K-*inverse transform*, we mean a function*

$$\hat{x}_K(t)=\sum_{i=0}^{n} c_i\cdot e_i(t),$$

where $c_i=\sum\limits_{j=0}^{n} k_{i,j}\cdot x_j$.

V. A New Intuitively Reasonable Criterion Justifies the Inverse F-Transform

A reasonable property: local consistency. The main purpose of the F-transform compression is that x_i should describe a behavior of the signal at an interval $[t_i,t_{i+1}]$. It is reasonable to require that the inverse transform follows the same idea: for example, if the original $x(t)$ signal was constant in a neighborhood of this interval, then the reconstructed function $\hat{x}_K(t)$ should also be equal to the same constant for all values t from this interval.

Let us describe this property in precise terms.

Definition 4. *We say that a matrix K is* locally consistent *if for every $i=0,\ldots,n-1$ and for every function $x(t)$ which is equal to a constant c on the interval $[t_i-h,t_{i+1}+h]$, the reconstructed function $\hat{x}_K(t)$ is equal to the same constant c for all $t\in[t_i,t_{i+1}]$.*

Comment. One can easily check that the inverse F-transform satisfies this property. The F-transform example explains why we require that the reconstructed function $\hat{x}_K(t)$ is equal to the constant c only on a subinterval $[t_i,t_{i+1}]$ of the original interval $[t_i-h,t_{i+1}+h]$: for example, if we take $x(t)=1$ for all $t\in[t_i-h,t_{i+1}+h]$ and $x(t)=0$ for all other t, then the function $\hat{x}_K(t)$ reconstructed by using F-transform is only equal to 1 for $t\in[t_i,t_{i+1}]$; for all other t, we have $\hat{x}_K(t)<1$.

It turns out that the inverse F-transform is the only K-inverse transform that satisfies this local consistency property.

Proposition. *A matrix K is locally consistent if and only if it coincides with the unit matrix.*

Discussion. In other words, the only locally consistent K-inverse transform is the inverse F-transform (which corresponds to the unit matrix $K=I$). So, *the above result provides the desired justification of the inverse F-transform.*

Proof of the Proposition. We will describe the proof for the case when $i=1,2,\ldots,n-2$. The proof for the remaining values $i=0$ and $i=n-1$ is similar.

The local consistency property should hold for every constant c, in particular, it should hold for the constant $c=1$. Let $x(t)$ be a function for which $x(t)=1$ for all t from the interval $[t_i-h,t_{i+1}+h]$. For this function, the local consistency property implies that $\hat{x}_K(t)=1$ for all t from the interval $[t_i,t_{i+1}]$.

For values t from this interval $[t_i,t_{i+1}]$, only two of the functions $e_j(t)$ are different from 0: the function $e_i(t)$ and the function $e_{i+1}(t)$. Thus, for these t, the general formula $\hat{x}_K(t)=\sum\limits_{j=0}^{n} c_j\cdot e_j(t)$ turns into

$$\hat{x}_K(t)=c_i\cdot e_i(t)+c_{i+1}\cdot e_{i+1}(t);$$

so, the local consistency requirement means that

$$c_i\cdot e_i(t)+c_{i+1}\cdot e_{i+1}(t)=1$$

for all $t \in [t_i, t_{i+1}]$.

In particular, for $t = t_i$, by definition of the functions $e_i(t) = e(t - t_i)$ and $e_{i+1}(t) = e(t - t_{i+1})$ and by the properties of the function $e(t)$, we have $e_i(t_i) = 1$ and $e_{i+1}(t_i) = 0$. Thus, for $t = t_i$, the displayed equality takes the form $c_i \cdot 1 + c_{i+1} \cdot 0 = 1$, i.e., the form $c_i = 1$.

Similarly, for $t = t_{i+1}$, we have $e_i(t_{i+1}) = 0$ and $e_{i+1}(t_{i+1}) = 1$ and thus, the displayed equality leads to $c_{i+1} = 1$.

Thus, for each such function $x(t)$, we must have $c_i = c_{i+1} = 1$, i.e.,

$$\sum_{j=0}^{n} k_{i,j} \cdot x_j = 1 \text{ and } \sum_{j=0}^{n} k_{i+1,j} \cdot x_j = 1.$$

Substituting the expression for the forward F-transform x_j into the first of two equalities, we conclude that

$$\sum_{j=0}^{n} k_{i,j} \cdot \int a_j(t) \cdot x(t)\, dt = 1,$$

where

$$a_j(t) = \frac{e_j(t)}{\int e_j(s)\, ds},$$

i.e., that

$$\int w(t) \cdot x(t)\, dt = 1,$$

where $w(t) \stackrel{\text{def}}{=} \sum_{j=0}^{n} k_{i,j} \cdot a_j(t)$.

We know that $x(t) = 1$ for $t \in [t_i - h, t_{i+1} + h]$; for all other values t, we can have arbitrary values $x(t)$. Let us therefore separate the above integral into two parts: an integral over the interval $[t_i - h, t_{i+1} + h]$ for which we know the values of $x(t)$ and the integral over all other (unknown) values $x(t)$:

$$1 = \int w(t) \cdot x(t)\, dt = \int_{t \in [t_i - h, t_{i+1} + h]} w(t) \cdot x(t)\, dt +$$

$$\int_{t \notin [t_i - h, t_{i+1} + h]} w(t) \cdot x(t)\, dt.$$

Substituting $x(t) = 1$ into the first integral, we conclude that

$$1 = \int_{t \in [t_i - h, t_{i+1} + h]} w(t)\, dt + \int_{t \notin [t_i - h, t_{i+1} + h]} w(t) \cdot x(t)\, dt.$$

Both the left-hand side and the right-hand sides are linear functions of the unknowns $x(t)$ corresponding to

$$t \notin [t_i - h, t_{i+1} + h].$$

These two functions must coincide for all possible values of these unknowns. It is well known that the values of two linear functions always coincide if and only they have the same free terms and the same coefficients at all the variables. In other words, we must have

$$1 = \int_{t \in [t_i - h, t_{i+1} + h]} w(t)\, dt$$

and

$$w(t) = \sum_{j=1}^{n} k_{i,j} \cdot a_j(t) = 0$$

for all $t \notin [t_i - h, t_{i+1} + h]$.

For each integer $\ell \le i - 2$, values t from the interval $[t_\ell, t_{\ell+1}]$ are outside the interval $[t_i - h, t_{i+1} + h]$, hence we have

$$\sum_{j=1}^{n} k_{i,j} \cdot a_j(t) = 0$$

for all such t. On this interval, only two functions $a_j(t)$ are different from 0:

- the function $a_\ell(t)$ and
- the function $a_{\ell+1}(t)$.

Thus, the equality $\sum_{j=1}^{n} k_{i,j} \cdot a_j(t) = 0$ takes the form

$$k_{i\ell} \cdot a_\ell(t) + k_{i,\ell+1} \cdot a_{\ell+1}(t) = 0.$$

In particular, for $t = t_\ell$, we have $a_\ell(t_\ell) \ne 0$ and $a_{\ell+1}(t_\ell) = 0$, so $k_{i,\ell} \cdot a_\ell(t_\ell) = 0$ and thus, $k_{i,\ell} = 0$.

Similarly, for $t = t_{\ell+1}$, we have $a_\ell(t_{\ell+1}) = 0$ and $a_{\ell+1}(t_{\ell+1}) \ne 0$, so

$$k_{i,\ell+1} \cdot a_\ell(t_{\ell+1}) = 0$$

and thus, $k_{i,\ell+1} = 0$.

For every integer $\ell \le i - 2$, we thus have $k_{i,\ell} = 0$ and $k_{i,\ell+1} = 0$. So, we have

$$k_{i,0} = k_{i,1} = \ldots = k_{i,i-1} = 0.$$

By considering integers $\ell \ge i + 2$, we can similarly get

$$k_{i,i+2} = \ldots = k_{i,n} = 0.$$

Thus, from the fact that $c_i = 1$, we conclude that the only possibly non-zero elements $k_{i,j}$ are $k_{i,i}$ and $k_{i,i+1}$. This is true for every i, in particular, $k_{i,i-1} = 0$ for every i.

Similarly, from the fact that $c_{i+1} = 1$, we conclude that the only possibly non-zero elements $k_{i+1,j}$ are $k_{i+1,i}$ and $k_{i+1,i+1}$. We cannot have $k_{i+1,i} \ne 0$ since we have already proven that we have $k_{i',i'-1} = 0$ for all i', in particular, for $i' = i + 1$. Thus, the only non-zero element $k_{i+1,j}$ is the diagonal element $k_{i+1,i+1}$.

This is true for every $i' = i + 1$, so we conclude that the matrix $k_{i,j}$ is diagonal, its only non-zero elements are elements of the type $k_{i,i}$. Thus, we have $c_i = k_{i,i} \cdot x_i$.

To complete the proof, let us show that all the diagonal elements $k_{i,i}$ of the matrix $k_{i,j}$ are equal to 1. Indeed, the function $e(t)$ is non-zero for values t from the interval $(-h, h)$ and equal to 0 for all other values. Thus, each function $e_j(t) = e(t - t_j)$ is positive for $t \in (t_j - h, t_j + h)$ and is equal to 0 for all other t. In particular, the function $e_i(t)$ differs from 0 only for values t from the interval $(t_i - h, t_i + h)$. On this interval, the function $x(t)$ coincides with the constant $c = 1$ and thus,

$$x_i = \frac{\int_{t_{i-1}}^{t_{i+1}} e_i(t) \cdot x(t)\, dt}{\int_{t_{i-1}}^{t_{i+1}} e_i(t)\, dt} = \frac{\int_{t_{i-1}}^{t_{i+1}} e_i(t)\, dt}{\int_{t_{i-1}}^{t_{i+1}} e_i(t)\, dt} = 1.$$

Thus, the requirement that $c_i = k_{i,i} \cdot x_i = 1$ leads to

$$k_{i,i} = 1.$$

Since this is true for each element of the diagonal matrix $k_{i,j}$, we thus conclude that this matrix is the unit matrix. The proposition is proven.

VI. Conclusion

Empirically, fuzzy transform (F-transform, for short) often provides a good quality compression of signals and images. However, somewhat surprisingly, with respect to standard criteria used in numerical mathematics to gauge the compression quality, more traditional compression techniques are much better than F-transform. This discrepancy between the empirical efficiency of F-transform and the standard theoretical quality criteria shows that these criteria are not fully adequate to describe the compression quality.

In this paper, we propose a new criterion of *local* consistency between the original and the reconstructed signals, and we show that F-transform is the only scheme that satisfies this criterion. Thus, we provide a theoretical justification of the empirical success of F-transform as a compression technique.

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