

F-transform in View of Aggregation Functions

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Abstract A relationship between the discrete F-transform and aggregation functions is analyzed. We show that the discrete F-transform (direct or inverse) can be associated with a set of linear aggregation functions that respect a fuzzy partition of a universe. On the other side, we discover conditions that should be added to a set of linear aggregation functions in order to obtain the discrete F-transform. Last but not least, the relationship between two analyzed notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

1 Introduction

In the last ten years, the theory of F-transforms has been intensively developed in many directions and especially in connection with image processing. The following topics have been newly elaborated on the *F-transform* platform: image compression and reconstruction [1, 2, 3], image reduction and sharpening [4, 5], edge detection [6, 7], etc. On the other side, similar applications can be produced with the help of *aggregation functions*, see e.g., [8, 9]. The goal of this contribution is to discover a relationship between both notions.

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To comment the goal, we notice that it is not difficult to show that the discrete F-transform (direct or inverse) can be associated with a set of linear aggregation functions. However, the opposite characterization is not so obvious. In this contribution, we found conditions that should be added to a set of linear aggregation functions of the same number of variables in order to obtain the discrete F-transform. Last but not least, the proposed relationship between two notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

We believe that this investigation contributes to a mutual success of both theories.

2 F-transform on a Generalized Fuzzy Partition

The F-transform technique was introduced in [10]. Below we remind its main principles for the so called *discrete* functions. The latter means that an original function f is known (may be computed) on a finite set $P = \{p_1, \dots, p_l\} \subseteq [a, b]$. The interval $[a, b]$ will be considered as a universe of discourse that is partitioned into $n \leq 3$ fuzzy sets $A_1, \dots, A_n$. We identify fuzzy sets $A_1, \dots, A_n$ with their membership functions that map $[a, b]$ onto $[0, 1]$ and call them *basic functions*.

2.1 Generalized Fuzzy Partition

The following is a new definition of a *generalized fuzzy partition* which differs from that in [10] by using a smaller number of axioms.

Definition 1. Let $[a, b]$ be an interval on the real line $\mathbb{R}$, $n > 2$, and let $x_1, \dots, x_n$ be nodes such that $a \leq x_1 < \dots < x_n \leq b$. Let $[a, b]$ be covered by the intervals $[x_k - h'_k, x_k + h''_k] \subseteq [a, b]$, $k = 1, \dots, n$, such that their left and right margins $h'_k, h''_k \geq 0$ fulfill $h'_k + h''_k > 0$.

We say that fuzzy sets $A_1, \dots, A_n : [a, b] \rightarrow [0, 1]$ constitute a *generalized fuzzy partition* of $[a, b]$ (with nodes $x_1, \dots, x_n$ and margins h'_k, h''_k , $k = 1, \dots, n$), if for every $k = 1, \dots, n$, the following three conditions are fulfilled:

1. (*locality*) — $A_k(x) > 0$ if $x \in (x_k - h'_k, x_k + h''_k)$, and $A_k(x) = 0$ if $x \in [a, b] \setminus (x_k - h'_k, x_k + h''_k)$;
2. (*continuity*) — A_k is continuous on $[x_k - h'_k, x_k + h''_k]$;
3. (*covering*) — for $x \in [a, b]$, $\sum_{k=1}^n A_k(x) > 0$.

We say that fuzzy sets $I_1, \dots, I_n : [a, b] \rightarrow [0, 1]$ constitute a *(0-1)-generalized partition* of $[a, b]$ with nodes and margins as above, if for every $k = 1, \dots, n$, I_k fulfills (*locality*) as above, (*continuity*) on $(x_k - h'_k, x_k + h''_k)$ and (*covering*) as above.

If nodes and margins are the same for generalized fuzzy and *(0-1)*-partitions $A_1, \dots, A_n$ and $I_1, \dots, I_n$, respectively, then we say that the latter is a “mask” of the former.

It is worth to remark that given nodes $x_1, \dots, x_n$ and margins $h'_k, h''_k, k = 1, \dots, n$, within $[a, b]$, a $(0-1)$ -generalized partition $I_1, \dots, I_n$ of $[a, b]$ is uniquely determined.

We say that a generalized fuzzy partition $A_1, \dots, A_n$ of $[a, b]$ with nodes $x_1, \dots, x_n$ and margins $h'_k, h''_k, k = 1, \dots, n$, is *centered at nodes* if basic functions are bell-shaped, i.e. for each $k = 1, \dots, n$, A_k is monotonically increasing on $[x_k - h'_k, x_k]$ and monotonically decreasing on $[x_k, x_k + h''_k]$.

Further on, the word “generalized” in characterization of fuzzy partitions will be omitted and left only when this fact is essential.

2.2 Discrete F-transform

We assume that a discrete function $f : P \rightarrow [0, 1]^1$ on a finite domain $P = \{p_1, \dots, p_l\}$, $P \subseteq [a, b]$, is given and that P is *sufficiently dense with respect to a fixed partition* $A_1, \dots, A_n$, of $[a, b]$, i.e.,

$$(\forall k)(\exists j)A_k(p_j) > 0.$$

Then, the (discrete) F-transform of f and its inverse are defined as follows.

Definition 2. Let $A_1, \dots, A_n$, for $n > 2$, be basic functions that form a generalized fuzzy partition of $[a, b]$, and let function f be defined on the set $P = \{p_1, \dots, p_l\} \subseteq [a, b]$, which is sufficiently dense with respect to the partition. We assume that $n \leq l$. The n -tuple of real numbers $F_n[f] = (F_1, \dots, F_n)$ is the *discrete F-transform* of f with respect to $A_1, \dots, A_n$ if

$$F_k = \frac{\sum_{j=1}^l f(p_j)A_k(p_j)}{\sum_{j=1}^l A_k(p_j)}, k = 1, \dots, n. \quad (1)$$

The *inverse F-transform* $\hat{f}$ of f is a function that is defined on the same set P as above and represented by the following inversion formula:

$$\hat{f}(p_j) = \frac{\sum_{k=1}^n F_k A_k(p_j)}{\sum_{k=1}^n A_k(p_j)}, \quad j = 1, \dots, l. \quad (2)$$

Assume that the elements of P are numbered in accordance with their order, i.e., $p_1 < \dots < p_l$. Denote $P_k = \{p_j | A_k(p_j) > 0\}$, $k = 1, \dots, n$. Because P is sufficiently dense with respect to $A_1, \dots, A_n$, each set P_k , $k = 1, \dots, n$ is not empty. Moreover, from the property *locality* it follows that for all $k = 1, \dots, n$, there exist integers k_1, k_2 such that $1 \leq k_1 \leq k_2 \leq l$ and $P_k = \{p_j | k_1 \leq j \leq k_2\}$. We say that P_k is covered by A_k or A_k covers P_k .

Let us identify the function f on P with the l -dimensional vector $(f_1, \dots, f_l) \in [0, 1]^l$ of its values such that $f_j = f(p_j)$, $j = 1, \dots, l$. Because $A_1, \dots, A_n$ is a fixed partition of $[a, b]$ and f is an arbitrary function on P , the F-transform $F_n[f]$ of f can

¹ The restriction of the range of f to $[0, 1]$ is not principal and was assigned due to further correspondence with aggregation functions.

be considered as a result of a linear map $F_n[f] : [0, 1]^l \rightarrow [0, 1]^n$ between linear vector spaces $[0, 1]^l$ and $[0, 1]^n$. We split this map into n separate maps $F_k : [0, 1]^l \rightarrow [0, 1]$ where $F_k(f_1, \dots, f_l) = F_k$, $k = 1, \dots, n$, and consider each map F_k as a real function of l arguments. In the sequel, we will be keeping at this viewpoint.

Let us list basic properties of the map $F_k : [0, 1]^l \rightarrow [0, 1]$, $k = 1, \dots, n$:

- P1. (*linearity*) - for all $\mathbf{x}, \mathbf{y} \in [0, 1]^l$ and $\alpha, \beta \in [0, 1]$ such that $\alpha\mathbf{x} + \beta\mathbf{y} \in [0, 1]^l$, $F_k(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha F_k(\mathbf{x}) + \beta F_k(\mathbf{y})$;
- P2. (*idempotency*) - for all $c \in [0, 1]$, $F_k(c, \dots, c) = c$;
- P3. (*non-decreasing*) - if $\mathbf{x}, \mathbf{y} \in [0, 1]^l$ and $\mathbf{x} \leq \mathbf{y}$, then $F_k(\mathbf{x}) \leq F_k(\mathbf{y})$;
- P4. (*redundancy*) - if basic function A_k covers the set $P_k = \{p_j | k_1 \leq j \leq k_2\}$, then only those arguments x_j among $x_1, \dots, x_l$, whose indices are within the interval $k_1 \leq j \leq k_2$, are essential, i.e. for all $x_1, \dots, x_l \in [0, 1]$, $F_k(x_1, \dots, x_l) = F_k(0, \dots, x_{k_1}, \dots, x_{k_2}, \dots, 0)$.

It easily follows from properties P1 and P3 that the map F_k , $k = 1, \dots, n$, is monotonously non-decreasing. This fact together with the property P2 proves that the map F_k is an additive and idempotent *aggregation function*² (see [11]). Moreover, from property P4 we deduce that the following derived function $F'_k(x_{k_1}, \dots, x_{k_2}) = F_k(0, \dots, x_{k_1}, \dots, x_{k_2}, \dots, 0)$ with the reduced number $l_k = (k_2 - k_1)$ of arguments is an aggregation function as well.

In the following section, we will analyze the inverse problem, i.e., under which conditions n aggregation functions determine the F-transform.

3 Discrete F-transform and Aggregation Functions

The goal of this section is to find conditions that characterize aggregation functions as the F-transform components.

3.1 Aggregation functions and generic fuzzy partition

In this section, we will see that two kinds of properties: functional (additivity, etc.) and spacial (correspondence with a certain partition), should be demanded from a set of aggregation functions if we want them to represent the F-transform components.

Theorem 1. *Let $I_1, \dots, I_n$, $n > 2$, be a (0-1)-generalized partition of $[a, b]$ with nodes $x_1, \dots, x_n$ and margins h'_k, h''_k , $k = 1, \dots, n$, and let finite set $P = \{p_1, \dots, p_l \subseteq [a, b]\}$ where $l \geq n$ be sufficiently dense with respect to it. Then for any additive, non-decreasing, idempotent aggregation functions $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$,*

² An aggregation function of l variables in $[0, 1]$ is a function which is non-decreasing in each argument and idempotent at boundaries $(0, \dots, 0)$ and $(1, \dots, 1)$.

that fulfill the property P4 (with respect to $I_1, \dots, I_n$) there exists a fuzzy partition $A_1, \dots, A_n$ of $[a, b]$ with the mask $I_1, \dots, I_n$ such that for each $k = 1, \dots, n$, the k -th F-transform component F_k of any discrete function $f : P \rightarrow [0, 1]$ such that $f(p_j) = f_j$, $j = 1, \dots, l$, is the value of the corresponding aggregation function F_k at point $(f_1, \dots, f_l)$.

Proof. Let us fix k , $1 \leq k \leq n$, and prove the assertion for the aggregation function $F_k : [0, 1]^l \rightarrow [0, 1]$. By the assumption, F_k fulfills the properties in the formulation. From the first three, namely additivity, non-decreasing and idempotency, it follows (see, e.g., Proposition 4.21 from [11]) that there exist “weights” $w_{k1}, \dots, w_{kl} \in [0, 1]$ such that $\sum_{j=1}^l w_{kj} = 1$ and

$$F_k(f_1, \dots, f_l) = \sum_{j=1}^l w_{kj} f_j, \text{ where } (f_1, \dots, f_l) \in [0, 1]^l. \quad (3)$$

Let $P_k = \{p_j \mid k_1 \leq j \leq k_2\}$ be covered by I_k . By the assumption, $P_k \neq \emptyset$. By the property P4, for all $f_1, \dots, f_l \in \mathbb{R}$, $F_k(f_1, \dots, f_l) = F_k(0, \dots, f_{k_1}, \dots, f_{k_2}, \dots, 0)$. Therefore,

$$F_k(f_1, \dots, f_l) = \sum_{j=1}^l w_{kj} f_j = \sum_{j=k_1}^{k_2} w_{kj} f_j.$$

In the above given equality, $(f_1, \dots, f_l)$ is an arbitrary vector in $[0, 1]^l$, and this fact implies that coefficients $w_{kj} = 0$, if $j \in \{1, \dots, l\} \setminus \{k_1, \dots, k_2\}$. Let us define the basic function A_k on P as

$$A_k(p_j) = \begin{cases} w_{kj}, & \text{if } k_1 \leq j \leq k_2, \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

and prove that the k -th F-transform component F_k of arbitrary function $f : P \rightarrow [0, 1]$, that is computed on the basis of (1) with respect to the above given A_k , equals aggregation $F_k(f_1, \dots, f_l)$ where $f_j = f(p_j)$, $j = 1, \dots, l$. Indeed,

$$F_k = \frac{\sum_{j=1}^l f(p_j) A_k(p_j)}{\sum_{j=1}^l A_k(p_j)} = \frac{\sum_{j=1}^l f_j w_{kj}}{\sum_{j=1}^l w_{kj}} = F_k(f_1, \dots, f_l).$$

To complete the proof it is sufficient to show that A_k can be continuously extended to the whole interval $[a, b]$ with the mask I_k .

By the *locality* of a generalized fuzzy partition, $I_k(x) > 0$ if and only if $x \in (x_k - h'_k, x_k + h''_k)$. By (4), $A_k(p_j) > 0$ if and only if $p_j \in P_k$. Because P_k is covered by I_k , $P_k \subset (x_k - h'_k, x_k + h''_k)$. Therefore, on the first step we construct a continuous extension of A_k to $[x_k - h'_k, x_k + h''_k]$. It can be obtained if we continuously connect the following points on the real plane: $(x_k - h'_k, 0)$, (p_{k_1}, w_{k,k_1}) , $\dots$, (p_{k_2}, w_{k,k_2}) , $(x_k + h''_k, 0)$. On the second step we put $A_k(x) = 0$ for all $x \in [a, b] \setminus [x_k - h'_k, x_k + h''_k]$, which is a continuous extension of A_k to $[a, b] \setminus [x_k - h'_k, x_k + h''_k]$. It is easy to see that thus extended A_k fulfills all requirements from Definition 1.

In the following corollary we unify aggregation functions into one matrix W so that the vector of F-transform components of f is the product of W by the vector of f .

Corollary 1. *Let the assumptions of Theorem 1 be fulfilled. Then for any additive, non-decreasing, idempotent aggregation functions $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$, that fulfill the property P4, there exists a $n \times l$ matrix W such that the F-transform $F_n[f] = (F_1, \dots, F_n)$ of any discrete function $f : P \rightarrow [0, 1]$ such that $f(p_j) = f_j$, $j = 1, \dots, l$, can be computed by the product $W\mathbf{f}$ where $\mathbf{f} = (f_1, \dots, f_l)$, i.e.*

$$F_n[f] = W\mathbf{f}. \quad (5)$$

Proof. Under the denotation of Theorem 1 and its proof, elements w_{kj} of the matrix W are weights that determine aggregation functions in accordance with (3).

We say that W is an *aggregation matrix that corresponds to the F-transform*.

3.2 Aggregation functions and centered fuzzy partition

This section is focused on fuzzy partitions that are centered at nodes. Our goal is to analyze under which conditions aggregating functions represent the F-transform with respect to this type of partition.

Let us consider aggregation functions of l variables, each one runs over $[0, 1]$. We say that the point $\mathbf{y} \in [0, 1]^l$ is a result of a *point-spread noise* applied to a point $\mathbf{x} \in [0, 1]^l$ if both points differ exactly in one coordinate.

Definition 3. Let $F : [0, 1]^l \rightarrow [0, 1]$ be an aggregation function, $1 \leq s \leq l$ and $\mathbf{0}_q \in [0, 1]^l$ be a point whose coordinates are 0s, except for the q -th one which is equal to 1. We say that aggregation F works as a “noise damper” centered at s , if it fulfills the following condition:

$$\text{if } (s \leq q_2 < q_1 \leq l) \text{ or } (1 \leq q_1 < q_2 \leq s) \text{ then } F(\mathbf{0}_{q_1}) \leq F(\mathbf{0}_{q_2}). \quad (6)$$

Let us explain the above given notions. In Definition 3, a point-spread noise in $\mathbf{0}_q \in [0, 1]^l$ is represented by the value “1”. The “noise damper” centered at s means that the farther “1” is from the s -th coordinate, the less is the value of aggregation.

The following theorem shows that aggregating functions that fulfill conditions of Theorem 1 and work as noise dampers centered at certain nodes represent the F-transform components with respect to a fuzzy partition that is centered at these nodes.

Theorem 2. *Let $I_1, \dots, I_n$, $n > 2$, be a $(0-1)$ -generalized partition of $[a, b]$ with nodes $x_1, \dots, x_n$ and margins h'_k, h''_k , $k = 1, \dots, n$, and let finite set $P = \{p_1, \dots, p_l \subseteq [a, b]\}$ where $l \geq n$ be sufficiently dense with respect to it. Assume that $x_1, \dots, x_n \in P$ so that for all $1 \leq k \leq n$, there exists $1 \leq j_k \leq l$ such that $x_k = p_{j_k}$. Let $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$ be additive, non-decreasing, idempotent aggregation functions that*

fulfill the property P4 (with respect to $I_1, \dots, I_n$) and work as noise dampers centered at respective positions $j_1, \dots, j_n$. Then there exists a fuzzy partition $A_1, \dots, A_n$ of $[a, b]$ with the mask $I_1, \dots, I_n$, such that it is centered at nodes $x_1, \dots, x_n$, and for each $k = 1, \dots, n$, the k -th F-transform component F_k of any discrete function $f : P \rightarrow [0, 1]$ is equal to $F_k(f_1, \dots, f_l)$ where $f_j = f(p_j)$, $j = 1, \dots, l$.

Proof. Assume that assumptions above are fulfilled. Let us fix k , $1 \leq k \leq n$, and prove the claim for the aggregation function $F_k : [0, 1]^l \rightarrow [0, 1]$. By Theorem 1, there exist coefficients $w_1, \dots, w_l \in [0, 1]$ such that $\sum_{j=1}^l w_j = 1$ and

$$F_k(f_1, \dots, f_l) = \sum_{j=1}^l w_j f_j, \text{ where } (f_1, \dots, f_l) \in [0, 1]^l. \quad (7)$$

Let $P_k = \{p_j \mid k_1 \leq j \leq k_2\}$ be covered by I_k . By the assumption, $x_k \in P_k$ so that $x_k = p_{j_k}$ for some $k_1 \leq j_k \leq k_2$. Let us prove that the sequence of coefficients $w_1, \dots, w_l$ non-strictly increases for $i \leq j_k$ and non-strictly decreases for $i \geq j_k$, i.e.,

$$w_1 \leq \dots \leq w_{j_k} \geq w_{j_k+1} \geq \dots \geq w_l. \quad (8)$$

By (6), the aggregation function F_k works as a “noise damper” centered at j_k . Let $1 \leq q \leq l$, and $\mathbf{0}_q$ be the l -tuple whose elements are 0s, except for the q -th one which is equal to 1. By (7), $F_k(\mathbf{0}_q) = w_q$. Therefore, by (6),

$$\text{if } (k \leq q_2 < q_1 \leq l) \text{ or } (1 \leq q_1 < q_2 \leq l) \text{ then } w_{q_1} \leq w_{q_2}.$$

This proves (8). The rest of the proof coincides with the proof of Theorem 1.

4 Inverse F-transform and Aggregation Functions

If we compare expressions (1) and (2) for the direct and inverse F-transform, then we see that they have similar structures. Therefore, the inverse F-transform is expected to be represented by aggregation functions too. The aim of this section is to find a relationship between a set of aggregation functions which determine the direct F-transform and another set of aggregation functions which determine the inverse F-transform.

Assume that the direct F-transform of a discrete function $f : P \rightarrow [0, 1]$, where the set $P = \{p_1, \dots, p_l\} \subseteq [a, b]$ is sufficiently dense with respect to a certain fuzzy partition $A_1, \dots, A_n$ of $[a, b]$, is determined by a corresponding set of aggregation functions $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$ such that for every $(f_1, \dots, f_l) \in [0, 1]^l$,

$$F_k(f_1, \dots, f_l) = \frac{\sum_{j=1}^l f_j A_k(p_j)}{\sum_{j=1}^l A_k(p_j)}, k = 1, \dots, n. \quad (9)$$

By this we mean that the k -th F-transform component F_k of the function f is equal to $F_k(f_1, \dots, f_l)$, provided that $f_j = f(p_j)$, $j = 1, \dots, l$.

The inverse F-transform $\hat{f}$ of f with respect to the same partition $A_1, \dots, A_n$ is a function on P that is determined by another set of functions $\hat{f}_j : [0, 1]^n \rightarrow [0, 1]$ such that $\hat{f}(p_j) = \hat{f}_j(F_1, \dots, F_n)$, $j = 1, \dots, l$, where $F_1, \dots, F_n$ are the F-transform components of f and

$$\hat{f}_j(F_1, \dots, F_n) = \frac{\sum_{k=1}^n F_k A_k(p_j)}{\sum_{k=1}^n A_k(p_j)}, \quad j = 1, \dots, l. \quad (10)$$

The following reasoning (similar to that in Subsection 2.2) aims at proving that the functions $\hat{f}_j$, $j = 1, \dots, l$, are aggregations. Indeed, the inverse F-transform (10) can be considered as a result of a linear map $\hat{f} : [0, 1]^n \rightarrow [0, 1]^l$ between linear vector spaces $[0, 1]^n$ and $[0, 1]^l$. We split this map into l separate maps $\hat{f}_j : [0, 1]^n \rightarrow [0, 1]$ so that each one is a real function of n arguments.

The basic properties of $\hat{f}_j : [0, 1]^n \rightarrow [0, 1]$, $j = 1, \dots, l$ are the same as they are for the maps $F_k : [0, 1]^l \rightarrow [0, 1]$, $k = 1, \dots, n$: linearity, idempotency, non-decreasing and redundancy. The latter differs from the above formulated P4 in interchanging j and k . Let us give the precise formulation.

P5. (redundancy) - if a point p_j , $j = 1, \dots, l$, is covered by several basic functions A_k , i.e. $A_k(p_j) > 0$, where $j_1 \leq k \leq j_2$, then only those arguments x_k among $x_1, \dots, x_n$, whose indices are within the interval $j_1 \leq k \leq j_2$, are essential, i.e. for all $x_1, \dots, x_n \in [0, 1]$, $\hat{f}_j(x_1, \dots, x_n) = \hat{f}_j(0, \dots, x_{j_1}, \dots, x_{j_2}, \dots, 0)$.

Therefore, the maps $\hat{f}_j : [0, 1]^n \rightarrow [0, 1]$, $j = 1, \dots, l$ are linear aggregation functions on $[0, 1]^n$ that fulfill the property P5. Conversely, similarly to Theorem 1, any l additive, non-decreasing, idempotent aggregation functions $\hat{f}_j$ on $[0, 1]^n$ that fulfill the property P5 can be combined into one function $\hat{f} : P \rightarrow [0, 1]$ such that $\hat{f}(p_j) = \hat{f}_j(F_1, \dots, F_n)$, $j = 1, \dots, l$.

Our goal is to find conditions on aggregation functions $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$ and aggregation functions $\hat{f}_j : [0, 1]^n \rightarrow [0, 1]$, $j = 1, \dots, l$, such that they determine the direct and inverse F-transforms with respect to the same partition $A_1, \dots, A_n$. The following theorem gives the solution.

Theorem 3. Let $I_1, \dots, I_n$, $n > 2$, be a $(0-1)$ -generalized partition of $[a, b]$ with nodes $x_1, \dots, x_n$ and margins h'_k, h''_k , $k = 1, \dots, n$, and let finite set $P = \{p_1, \dots, p_l \subseteq [a, b]\}$ where $l \geq n$ be sufficiently dense with respect to it. Then for any additive, non-decreasing, idempotent aggregation functions $F_1, \dots, F_n : [0, 1]^l \rightarrow [0, 1]$, that fulfill the property P4 there exist additive, non-decreasing, idempotent aggregation functions $\hat{f}_1, \dots, \hat{f}_l : [0, 1]^n \rightarrow [0, 1]$, that fulfill the property P5, both with respect to $I_1, \dots, I_n$, and a fuzzy partition $A_1, \dots, A_n$ of $[a, b]$ with the mask $I_1, \dots, I_n$ such that for any discrete function $f : P \rightarrow [0, 1]$ such that $f(p_j) = f_j$, $j = 1, \dots, l$,

- (i) the F-transform component F_k , $k = 1, \dots, n$, of f is the value of the corresponding aggregation function F_k at point $(f_1, \dots, f_l)$,
- (ii) the inverse F-transform $\hat{f}(p_j)$, $j = 1, \dots, l$, is equal to the corresponding aggregation function $\hat{f}_j$ at point $(F_1, \dots, F_n)$.

In Corollary 1, the aggregation matrix W that corresponds to the F-transform was introduced. A similar result will be established for the inverse F-transform.

Corollary 2. *Let the assumptions of Theorem 1 be fulfilled and $W = (w_{kj})$ be a $n \times l$ matrix that corresponds to the F-transform so that for a function f , (5) holds. Then the related inverse F-transform $\hat{f}$ of f is characterized by the $l \times n$ matrix $\tilde{W} = (\tilde{w}_{jk})$ so that*

$$\hat{f} = \tilde{W}F_n[f]$$

where

$$\tilde{w}_{jk} = \frac{w_{kj}}{\sum_{k=1}^n w_{kj}}, j = 1, \dots, l, k = 1, \dots, n.$$

Conclusion

In this contribution, we focused on a relationship between the F-transform and aggregation functions. We showed that the F-transform components can be obtained by linear aggregation functions that respect a fuzzy partition of a universe. On the other side, we discovered conditions that should be added to a set of linear aggregation functions in order to obtain the F-transform components. Similarly, the inverse F-transform can be associated with another set of linear aggregation functions that respect a fuzzy partition of a co-universe. Two sets of linear aggregation functions that are associated with the direct and inverse F-transforms are connected via the so called aggregation matrix. The relationship between two analyzed notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

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