F-transform in View of Aggregation Functions

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Abstract A relationship between the discrete F-transform and aggregation functions is analyzed. We show that the discrete F-transform (direct or inverse) can be associated with a set of linear aggregation functions that respect a fuzzy partition of a universe. On the other side, we discover conditions that should be added to a set of linear aggregation functions in order to obtain the discrete F-transform. Last but not least, the relationship between two analyzed notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

1 Introduction

In the last ten years, the theory of F-transforms has been intensively developed in many directions and especially in connection with image processing. The following topics have been newly elaborated on the *F-transform* platform: image compression and reconstruction [1, 2, 3], image reduction and sharpening [4], edge detection [5, 6], etc. On the other side, similar applications can be produced with the help of *aggregation functions*, see e.g., [7, 8]. The goal of this contribution is to discover a relationship between both notions.

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To comment the goal, we notice that it is not difficult to show that the discrete F-transform (direct or inverse) can be associated with a set of linear aggregation functions. However, the opposite characterization is not so obvious. In this contribution, we found conditions that should be added to a set of linear aggregation functions of the same number of variables in order to obtain the discrete F-transform. Last but not least, the proposed relationship between two notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

We believe that this investigation contributes to a mutual success of both theories.

2 F-transform on a Generalized Fuzzy Partition

The F-transform technique was introduced in [9]. Below we remind its main principles for the so called *discrete* functions. The latter means that an original function f is known (may be computed) on a finite set $P = \{p_1, \dots, p_l\} \subseteq [a,b]$. The interval $[a,b]$ will be considered as a universe of discourse that is partitioned into $n \geq 3$ fuzzy sets $A_1, \dots, A_n$. We identify fuzzy sets $A_1, \dots, A_n$ with their membership functions that map $[a,b]$ onto $[0,1]$ and call them *basic functions*.

2.1 Generalized Fuzzy Partition

The following is a new definition of a *generalized fuzzy partition* which differs from that in [9] by using a smaller number of axioms.

Definition 1. Let $[a,b]$ be an interval on the real line $\mathbb{R}$, $n > 2$, and let $x_1, \dots, x_n$ be nodes such that $a \leq x_1 < \dots < x_n \leq b$. Let $[a,b]$ be covered by the intervals $[x_k - h'_k, x_k + h''_k] \subseteq [a,b]$, $k = 1, \dots, n$, such that their left and right margins $h'_k, h''_k \geq 0$ fulfill $h'_k + h''_k > 0$.

We say that fuzzy sets $A_1, \dots, A_n : [a,b] \to [0,1]$ constitute a *generalized fuzzy partition* of $[a,b]$ (with nodes $x_1, \dots, x_n$ and margins h'_k, h''_k, $k = 1, \dots, n$), if for every $k = 1, \dots, n$, the following three conditions are fulfilled:

1. (*locality*) — $A_k(x) > 0$ if $x \in (x_k - h'_k, x_k + h''_k)$, and $A_k(x) = 0$ if $x \in [a,b] \setminus (x_k - h'_k, x_k + h''_k)$;
2. (*continuity*) — A_k is continuous on $[x_k - h'_k, x_k + h''_k]$;
3. (*covering*) — for $x \in [a,b]$, $\sum_{k=1}^{n} A_k(x) > 0$.

We say that fuzzy sets $I_1, \dots, I_n : [a,b] \to \{0,1\}$ constitute a *(0-1)-generalized partition* of $[a,b]$ with nodes and margins as above, if for every $k = 1, \dots, n$, I_k fulfills (*locality*) as above, (*continuity*) on $(x_k - h'_k, x_k + h''_k)$ and (*covering*) as above.

If nodes and margins are the same for generalized fuzzy and *(0-1)*-partitions $A_1, \dots, A_n$ and $I_1, \dots, I_n$, respectively, then we say that the latter is a “mask” of the former.

It is worth to remark that given nodes $x_1,\dots,x_n$ and margins h'_k,h''_k, $k=1,\dots,n$, within $[a,b]$, a *(0-1)*-generalized partition $I_1,\dots,I_n$ of $[a,b]$ is uniquely determined.

We say that a generalized fuzzy partition $A_1,\dots,A_n$ of $[a,b]$ with nodes $x_1,\dots,x_n$ and margins h'_k,h''_k, $k=1,\dots,n$, is *centered at nodes* if basic functions are bell-shaped, i.e. for each $k=1,\dots,n$, A_k is monotonically increasing on $[x_k-h'_k,x_k]$ and monotonically decreasing on $[x_k,x_k+h''_k]$.

Further on, the word "generalized" in characterization of fuzzy partitions will be omitted and left only when this fact is essential.

2.2 Discrete F-transform

We assume that a discrete function $f:P\to[0,1]$[1] on a finite domain $P=\{p_1,\dots,p_l\}$, $P\subseteq[a,b]$, is given and that P is *sufficiently dense with respect to a fixed partition* $A_1,\dots,A_n$, of $[a,b]$, i.e.,

$$(\forall k)(\exists j)A_k(p_j)>0.$$

Then, the (discrete) F-transform of f and its inverse are defined as follows.

Definition 2. Let $A_1,\dots,A_n$, for $n>2$, be basic functions that form a generalized fuzzy partition of $[a,b]$, and let function f be defined on the set $P=\{p_1,\dots,p_l\}\subseteq[a,b]$, which is sufficiently dense with respect to the partition. We assume that $n\le l$. The n-tuple of real numbers $F_n[f]=(F_1,\dots,F_n)$ is the *discrete F-transform* of f with respect to $A_1,\dots,A_n$ if

$$F_k=\frac{\sum_{j=1}^{l}f(p_j)A_k(p_j)}{\sum_{j=1}^{l}A_k(p_j)},\ k=1,\dots,n. \tag{1}$$

The *inverse F-transform* $\hat{f}$ of f is a function that is defined on the same set P as above and represented by the following inversion formula:

$$\hat{f}(p_j)=\frac{\sum_{k=1}^{n}F_kA_k(p_j)}{\sum_{k=1}^{n}A_k(p_j)},\quad j=1,\dots,l. \tag{2}$$

Assume that the elements of P are numbered in accordance with their order, i.e., $p_1<\dots<p_l$. Denote $P_k=\{p_j|A_k(p_j)>0\}$, $k=1,\dots,n$. Because P is sufficiently dense with respect to $A_1,\dots,A_n$, each set P_k, $k=1,\dots,n$ is not empty. Moreover, from the property *locality* it follows that for all $k=1,\dots,n$, there exist integers k_1,k_2 such that $1\le k_1\le k_2\le l$ and $P_k=\{p_j\mid k_1\le j\le k_2\}$. We say that P_k is covered by A_k or A_k covers P_k.

Let us identify the function f on P with the l-dimensional vector $(f_1,\dots,f_l)\in[0,1]^l$ of its values such that $f_j=f(p_j)$, $j=1,\dots,l$. Because $A_1,\dots,A_n$ is a fixed partition of $[a,b]$ and f is an arbitrary function on P, the F-transform $F_n[f]$ of f can

[1] The restriction of the range of f to $[0,1]$ is not principal and was assigned due to further correspondence with aggregation functions.

be considered as a result of a linear map $F_n[f] : [0,1]^l \to [0,1]^n$ between linear vector spaces $[0,1]^l$ and $[0,1]^n$. We split this map into n separate maps $F_k : [0,1]^l \to [0,1]$ where $F_k(f_1,\ldots,f_l) = F_k$, $k = 1,\ldots,n$, and consider each map F_k as a real function of l arguments. In the sequel, we will be keeping at this viewpoint.

Let us list basic properties of the map $F_k : [0,1]^l \to [0,1]$, $k = 1,\ldots n$:

P1. (*linearity*) - for all $\mathbf{x},\mathbf{y} \in [0,1]^l$ and $\alpha,\beta \in [0,1]$ such that $\alpha\mathbf{x}+\beta\mathbf{y} \in [0,1]^l$, $F_k(\alpha\mathbf{x}+\beta\mathbf{y}) = \alpha F_k(\mathbf{x})+\beta F_k(\mathbf{y})$;
P2. (*idempotency*) - for all $c \in [0,1]$, $F_k(c,\ldots,c) = c$;
P3. (*non-decreasing*) - if $\mathbf{x},\mathbf{y} \in [0,1]^l$ and $\mathbf{x} \le \mathbf{y}$, then $F_k(\mathbf{x}) \le F_k(\mathbf{y})$;
P4. (*redundancy*) - if basic function A_k covers the set $P_k = \{p_j | k_1 \le j \le k_2\}$, then only those arguments x_j among $x_1,\ldots,x_l$, whose indices are within the interval $k_1 \le j \le k_2$, are essential, i.e. for all $x_1,\ldots,x_l \in [0,1]$, $F_k(x_1,\ldots,x_l) = F_k(0,\ldots,x_{k_1},\ldots,x_{k_2},\ldots,0)$.

It easily follows from properties P1 and P3 that the map F_k, $k = 1,\ldots n$, is monotonously non-decreasing. This fact together with the property P2 proves that the map F_k is an additive and idempotent *aggregation function*[2] (see [10]). Moreover, from property P4 we deduce that the following derived function $F'_k : [0,1]^{l_k} \to [0,1]$ where $l_k = (k_2 - k_1)$ and $F'_k(x_{k_1},\ldots,x_{k_2}) = F_k(0,\ldots,x_{k_1},\ldots,x_{k_2},\ldots,0)$ is an aggregation function as well.

In the following section, we will analyze the inverse problem, i.e., under which conditions n aggregation functions determine the F-transform.

3 Discrete F-transform and Aggregation Functions

The goal of this section is to find conditions that characterize aggregation functions as the F-transform components.

3.1 Aggregation functions and generic fuzzy partition

In this section, we will see that two kinds of properties: functional (additivity, etc.) and spacial (correspondence with a certain partition), should be demanded from a set of aggregation functions if we want them to represent the F-transform components.

Theorem 1. *Let $I_1,\ldots,I_n$, $n > 2$, be a (0-1)-generalized partition of $[a,b]$ with nodes $x_1,\ldots,x_n$ and margins h'_k,h''_k, $k = 1,\ldots,n$, and let finite set $P = \{p_1,\ldots,p_l \subseteq [a,b]\}$ where $l \ge n$ be sufficiently dense with respect to it. Then for any additive, non-decreasing, idempotent aggregation functions $F_1,\ldots,F_n : [0,1]^l \to [0,1]$, that fulfill the property P4 (with respect to $I_1,\ldots,I_n$) there exists a fuzzy partition $A_1,\ldots,A_n$ of*

[2] An aggregation function of l variables in $[0,1]$ is a function which is non-decreasing in each argument and idempotent at boundaries $(0,\ldots,0)$ and $(1,\ldots,1)$.

$[a,b]$ with the mask $I_1,\dots,I_n$, such that for each $k=1,\dots,n$, the k-th F-transform component F_k of a discrete function $f: P \to [0,1]$, identified with $(f_1,\dots,f_l)$, is equal to the value of the corresponding aggregation function F_k at point $(f_1,\dots,f_l)$.

Proof. Let us fix k, $1 \le k \le n$, and prove the assertion for the aggregation function $F_k : [0,1]^l \to [0,1]$. By the assumption, F_k fulfills the properties in the formulation. From the first three, namely additivity, non-decreasing and idempotency, it follows (see, e.g , Proposition 4.21 from [10]) that there exist "weights" $w_{k1},\dots,w_{kl} \in [0,1]$ such that $\sum_{j=1}^{l} w_{kj} = 1$ and

$$F_k(f_1,\dots,f_l) = \sum_{j=1}^{l} w_{kj} f_j, \text{ where } (f_1,\dots,f_l) \in [0,1]^l. \tag{3}$$

Let $P_k = \{p_j \mid k_1 \le j \le k_2\}$ be covered by I_k. By the assumption, $P_k \neq \emptyset$. By the property P4, for all $f_1,\dots,f_l \in \mathbb{R}$, $F_k(f_1,\dots,f_l) = F_k(0,\dots,f_{k_1},\dots,f_{k_2},\dots,0)$. Therefore,

$$F_k(f_1,\dots,f_l) = \sum_{j=1}^{l} w_{kj} f_j = \sum_{j=k_1}^{k_2} w_{kj} f_j.$$

In the above given equality, $(f_1,\dots,f_l)$ is an arbitrary vector in $[0,1]^l$, and this fact implies that coefficients $w_{kj} = 0$, if $j \in \{1,\dots,l\} \setminus \{k_1,\dots,k_2\}$. Let us define the basic function A_k on P as

$$A_k(p_j) = \begin{cases} w_{kj}, & \text{if } k_1 \le j \le k_2, \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

and prove that the k-th F-transform component F_k of $f: P \to [0,1]$ with respect to A_k in (4) is equal to the aggregation $F_k(f_1,\dots,f_l)$ where $f_j = f(p_j)$, $j=1,\dots,l$. Indeed by (1),

$$F_k = \frac{\sum_{j=1}^{l} f(p_j) A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)} = \frac{\sum_{j=1}^{l} f_j w_{kj}}{\sum_{j=1}^{l} w_{kj}} = F_k(f_1,\dots,f_l).$$

To complete the proof it is sufficient to show that A_k can be continuously extended to the whole interval $[a,b]$ with the mask I_k.

By the *locality* of a generalized fuzzy partition, $I_k(x) > 0$ if and only if $x \in (x_k - h'_k, x_k + h''_k)$. By (4), $A_k(p_j) > 0$ if and only if $p_j \in P_k$. Because P_k is covered by I_k, $P_k \subset (x_k - h'_k, x_k + h''_k)$. Therefore, on the first step we construct a continuous extension of A_k to $[x_k - h'_k, x_k + h''_k]$. It can be obtained if we continuously connect the following points on the real plane: $(x_k - h'_k, 0)$, $(p_{k_1}, w_{k,k_1}),\dots,(p_{k_2}, w_{k,k_2})$, $(x_k + h''_k, 0)$. On the second step we put $A_k(x) = 0$ for all $x \in [a,b] \setminus [x_k - h'_k, x_k + h''_k]$, which is a continuous extension of A_k to $[a,b] \setminus [x_k - h'_k, x_k + h''_k]$. It is easy to see that thus extended A_k fulfills all requirements from Definition 1.

In the following corollary, we compose a matrix W so that the vector of F-transform components of f is the product of W by the vector of f.

Corollary 1. *Let the assumptions of Theorem 1 be fulfilled. Then for any additive, non-decreasing, idempotent aggregation functions* $F_1,\dots,F_n : [0,1]^l \to [0,1]$*, that fulfill the property P4, there exists a* $n\times l$ *matrix* W *such that the F-transform* $F_n[f] = (F_1,\dots,F_n)$ *of any discrete function* $f : P \to [0,1]$ *such that* $f(p_j) = f_j$, $j = 1,\dots,l$, *can be computed by the product* $W\mathbf{f}$ *where* $\mathbf{f} = (f_1,\dots,f_l)$*, i.e.*

$$F_n[f] = W\mathbf{f}. \tag{5}$$

Proof. Under the denotation of Theorem 1 and its proof, elements w_{kj} of the matrix W are weights that determine aggregation functions in accordance with (3).

We say that W is an *aggregation matrix that corresponds to the F-transform*.

3.2 Aggregation functions and centered fuzzy partition

This section is focused on fuzzy partitions that are centered at nodes. Our goal is to analyze under which conditions aggregating functions represent the F-transform with respect to this type of partition.

Let us consider aggregation functions of l variables, each one runs over $[0,1]$. We say that the point $\mathbf{y} \in [0,1]^l$ is a result of a *point-spread noise* applied to a point $\mathbf{x} \in [0,1]^l$ if both points differ exactly in one coordinate.

Definition 3. Let $F : [0,1]^l \to [0,1]$ be an aggregation function, $1 \le s \le l$ and $\mathbf{0}_q \in [0,1]^l$ be a point whose coordinates are 0s, except for the q-th one which is equal to 1. We say that aggregation F works as a *"noise damper" centered at s*, if it fulfills the following condition:

$$\text{if } (s \le q_2 < q_1 \le l) \text{ or } (1 \le q_1 < q_2 \le s) \text{ then } F(\mathbf{0}_{q_1}) \le F(\mathbf{0}_{q_2}). \tag{6}$$

Let us explain the above given notions. The value "1" at the q-coordinate in $\mathbf{0}_q$ represents a noise. The "noise damper" centered at s property of F means that the farther is the position of noise "1" from the s-th coordinate, the less is the value of aggregation performed by F.

The following theorem shows that aggregating functions that fulfill conditions of Theorem 1 and work as noise dampers centered at certain nodes represent the F-transform components with respect to a fuzzy partition that is centered at these nodes.

Theorem 2. *Let* $I_1,\dots,I_n$*,* $n > 2$*, be a* $(0-1)$*-generalized partition of* $[a,b]$ *with nodes* $x_1,\dots,x_n$ *and margins* h'_k, h''_k*,* $k = 1,\dots,n$*, and let finite set* $P = \{p_1,\dots,p_l \subseteq [a,b]\}$ *where* $l \ge n$ *be sufficiently dense with respect to it. Assume that* $x_1,\dots,x_n \in P$*, i.e. for all* $1 \le k \le n$*, there exists* $1 \le j_k \le l$ *such that* $x_k = p_{j_k}$*. Let* $F_1,\dots,F_n : [0,1]^l \to [0,1]$ *be additive, non-decreasing, idempotent aggregation functions that fulfill the property P4 (with respect to* $I_1,\dots,I_n$*) and work as noise dampers centered at respective positions* $j_1,\dots,j_n$*. Then there exists a fuzzy partition* $A_1,\dots,A_n$ *of*

$[a,b]$ with the mask $I_1,\dots,I_n$, such that it is centered at nodes $x_1,\dots,x_n$, and for each $k=1,\dots,n$, the k-th F-transform component F_k of any discrete function $f : P \to [0,1]$ is equal to $F_k(f_1,\dots,f_l)$ where $f_j = f(p_j)$, $j=1,\dots,l$.

Proof. Assume that assumptions above are fulfilled. Let us fix k, $1 \le k \le n$, and prove the claim for the aggregation function $F_k : [0,1]^l \to [0,1]$. By Theorem 1, there exist coefficients $w_1,\dots,w_l \in [0,1]$ such that $\sum_{j=1}^{l} w_j = 1$ and

$$F_k(f_1,\dots,f_l) = \sum_{j=1}^{l} w_j f_j, \text{ where } (f_1,\dots,f_l) \in [0,1]^l. \tag{7}$$

Let $P_k = \{p_j \mid k_1 \le j \le k_2\}$ be covered by I_k. By the assumption, $x_k \in P_k$ so that $x_k = p_{j_k}$ for some $k_1 \le j_k \le k_2$. Let us prove that the sequence of coefficients $w_1,\dots,w_l$ non-strictly increases for $i \le j_k$ and non-strictly decreases for $i \ge j_k$, i.e.,

$$w_1 \le \dots \le w_{j_k} \ge w_{j_k+1} \ge \dots \ge w_l. \tag{8}$$

By (6), the aggregation function F_k works as a "noise damper" centered at j_k. Let $1 \le q \le l$, and $\mathbf{0}_q$ be the l-tuple whose elements are 0s, except for the q-th one which is equal to 1. By (7), $F_k(\bar{0}_q) = w_q$. Therefore, by (6),

$$\text{if } (k \le q_2 < q_1 \le l) \text{ or } (1 \le q_1 < q_2 \le l) \text{ then } w_{q_1} \le w_{q_2}.$$

This proves (8). The rest of the proof coincides with the proof of Theorem 1.

4 Inverse F-transform and Aggregation Functions

If we compare expressions (1) and (2) for the direct and inverse F-transform, then we see that they have similar structures. Therefore, the inverse F-transform is expected to be represented by aggregation functions too. The aim of this section is to find a relationship between a set of aggregation functions which determine the direct F-transform and another set of aggregation functions which determine the inverse F-transform.

Assume that the direct F-transform of a discrete function $f : P \to [0,1]$, where the set $P = \{p_1,\dots,p_l\} \subseteq [a,b]$ is sufficiently dense with respect to a certain fuzzy partition $A_1,\dots,A_n$ of $[a,b]$, is determined by a corresponding set of aggregation functions $F_1,\dots,F_n : [0,1]^l \to [0,1]$ such that for every $(f_1,\dots,f_l) \in [0,1]^l$,

$$F_k(f_1,\dots,f_l) = \frac{\sum_{j=1}^{l} f_j A_k(p_j)}{\sum_{j=1}^{l} A_k(p_j)}, k = 1,\dots,n. \tag{9}$$

By this we mean that the k-th F-transform component F_k of the function f is equal to $F_k(f_1,\dots,f_l)$, provided that $f_j = f(p_j)$, $j=1,\dots,l$.

The inverse F-transform $\hat{f}$ of f with respect to the same partition $A_1,\dots,A_n$ is a function on P that is determined by another set of functions $\hat{f}_j : [0,1]^n \to [0,1]$ such that $\hat{f}(p_j) = \hat{f}_j(F_1,\dots,F_n)$, $j = 1,\dots,l$, where $F_1,\dots,F_n$ are the F-transform components of f and

$$\hat{f}_j(F_1,\dots,F_n) = \frac{\sum_{k=1}^n F_k A_k(p_j)}{\sum_{k=1}^n A_k(p_j)}, \quad j = 1,\dots,l. \tag{10}$$

The following reasoning (similar to that in Subsection 2.2) aims at proving that the functions $\hat{f}_j$, $j = 1,\dots,l$, are aggregations. Indeed, the inverse F-transform (10) can be considered as a result of a linear map $\hat{f} : [0,1]^n \to [0,1]^l$ between linear vector spaces $[0,1]^n$ and $[0,1]^l$. We split this map into l separate maps $\hat{f}_j : [0,1]^n \to [0,1]$ so that each one is a real function of n arguments.

The basic properties of $\hat{f}_j : [0,1]^n \to [0,1]$, $j = 1,\dots l$ are the same as they are for the maps $F_k : [0,1]^l \to [0,1]$, $k = 1,\dots n$: linearity, idempotency, non-decreasing and redundancy. The latter differs from the above formulated P4 in interchanging j and k. Let us give the precise formulation.

P5. (*redundancy*) - if a point p_j, $j = 1,\dots,l$, is covered by several basic functions A_k, i.e. $A_k(p_j) > 0$, where $j_1 \le k \le j_2$, then only those arguments x_k among $x_1,\dots,x_n$, whose indices are within the interval $j_1 \le k \le j_2$, are essential, i.e. for all $x_1,\dots,x_n \in [0,1]$, $\hat{f}_j(x_1,\dots,x_n) = \hat{f}_j(0,\dots,x_{j_1},\dots,x_{j_2},\dots,0)$.

Therefore, the maps $\hat{f}_j : [0,1]^n \to [0,1]$, $j = 1,\dots,l$ are linear aggregation functions on $[0,1]^n$ that fulfill the property P5. Conversely, similarly to Theorem 1, any l additive, non-decreasing, idempotent aggregation functions $\hat{f}_j$ on $[0,1]^n$ that fulfill the property P5 can be combined into one function $\hat{f} : P \to [0,1]$ such that $\hat{f}(p_j) = \hat{f}_j(F_1,\dots,F_n)$, $j = 1,\dots,l$.

Our goal is to find conditions on aggregation functions $F_1,\dots,F_n : [0,1]^l \to [0,1]$ and aggregation functions $\hat{f}_j : [0,1]^n \to [0,1]$, $j = 1,\dots,l$, such that they determine the direct and inverse F-transforms with respect to the same partition $A_1,\dots,A_n$. The following theorem gives the solution.

Theorem 3. *Let $I_1,\dots,I_n$, $n > 2$, be a (0-1)-generalized partition of $[a,b]$ with nodes $x_1,\dots,x_n$ and margins h'_k, h''_k, $k = 1,\dots,n$, and let finite set $P = \{p_1,\dots,p_l \subseteq [a,b]\}$ where $l \ge n$ be sufficiently dense with respect to it. Then for any additive, non-decreasing, idempotent aggregation functions $F_1,\dots,F_n : [0,1]^l \to [0,1]$, that fulfill the property P4 there exist additive, non-decreasing, idempotent aggregation functions $\hat{f}_1,\dots,\hat{f}_l : [0,1]^n \to [0,1]$, that fulfill the property P5, both with respect to $I_1,\dots,I_n$, and a fuzzy partition $A_1,\dots,A_n$ of $[a,b]$ with the mask $I_1,\dots,I_n$ such that for any discrete function $f : P \to [0,1]$ such that $f(p_j) = f_j$, $j = 1,\dots,l$,*

(i) the F-transform component F_k, $k = 1,\dots,n$, of f is the value of the corresponding aggregation function F_k at point $(f_1,\dots,f_l)$,
(ii) the inverse F-transform $\hat{f}(p_j)$, $j = 1,\dots,l$, is equal to the corresponding aggregation function $\hat{f}_j$ at point $(F_1,\dots,F_n)$.

In Corollary 1, the aggregation matrix W that corresponds to the F-transform was introduced. A similar result will be established for the inverse F-transform.

Corollary 2. *Let the assumptions of Theorem 1 be fulfilled and $W = (w_{kj})$ be a $n \times l$ matrix that corresponds to the F-transform so that for a function f, (5) holds. Then the related inverse F-transform $\hat{f}$ of f is characterized by the $l \times n$ matrix $\tilde{W} = (\tilde{w}_{jk})$ so that*

$$\hat{f} = \tilde{W} F_n[f]$$

where

$$\tilde{w}_{jk} = \frac{w_{kj}}{\sum_{k=1}^{n} w_{kj}}, \; j = 1, \ldots, l, \; k = 1, \ldots, n.$$

Conclusion

In this contribution, we focused on a relationship between the F-transform and aggregation functions. We showed that the F-transform components can be obtained by linear aggregation functions that respect a fuzzy partition of a universe. On the other side, we discovered conditions that should be added to a set of linear aggregation functions in order to obtain the F-transform components. Similarly, the inverse F-transform can be associated with another set of linear aggregation functions that respect a fuzzy partition of a co-universe. Two sets of linear aggregation functions that are associated with the direct and inverse F-transforms are connected via the so called aggregation matrix. The relationship between two analyzed notions is based on a new (generalized) definition of a fuzzy partition without the Ruspini condition.

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