Why Complex-Valued Fuzzy? Why Complex Values in General? A Computational Explanation

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***Abstract*—In the traditional fuzzy logic, as truth values, we take all real numbers from the interval $[0,1]$. In some situations, this set is not fully adequate for describing expert uncertainty, so a more general set is needed. From the mathematical viewpoint, a natural extension of *real* numbers is the set of *complex* numbers. Complex-valued fuzzy sets have indeed been successfully used in applications of fuzzy techniques. This practical success leaves us with a puzzling question: why complex-valued degree of belief, degrees which do not seem to have a direct intuitive meaning, have been so successful? In this paper, we use latest results from theory of computation to explain this puzzle. Namely, we show that the possibility to extend to complex numbers is a necessary condition for fuzzy-related computations to be feasible. This computational result also explains why complex numbers are so efficiently used beyond fuzzy, in physics, in control, etc.**

I. Formulation of the Problem

From the mathematical viewpoint, complex-valued fuzzy sets are natural. In classical (2-valued) logic, every statement is either true or false. In the computer, "true" is usually represented as 1, and "false" as 0. As a result, in the 2-valued logic, the set of possible truth values is a 2-element set $\{0,1\}$.

The traditional 2-valued logic is well equipped to represent:

- situations when we are completely confident that a given statement is true and
- situations when we are completely confident that a given statement is false.

However, the traditional logic cannot adequately represent intermediate situations, when we only have some *degree* of confidence that a statement is true. To describe such intermediate situations, L. Zadeh invented fuzzy logic; see, e.g., [7], [11], [13]. In the original version of fuzzy logic, the set of possible truth values is an interval $[0,1]$; this is still the most widely used set of possible truth values.

Fuzzy logic has been successful in many applications. However, in some applications, the $[0,1]$-based fuzzy logic is itself not fully adequate. For example, the $[0,1]$-based logic assumes that we can describe an expert's intermediate degree of confidence by an *exact* number from the interval $[0,1]$. Real-life experts often cannot meaningfully distinguish between nearby numbers: it is difficult to meaningfully distinguish between degree of confidence 0.78 and 0.79. So, a more adequate description is needed.

One of the main objectives of studying fuzzy knowledge is to use the resulting models in computer-based systems. Because of this objective, researchers are looking for well-developed models, models which use well-established mathematical constructions. Thus, we are looking for well-established mathematical constructions that generalize the set of all real numbers from the interval $[0,1]$.

In mathematics, one of the natural generalizations of real numbers are complex numbers. Not surprisingly, complex-valued generalizations of fuzzy sets have been proposed and used; see, e.g., [1], [2], [4], [10].

From the practical viewpoint, complex-valued fuzzy sets are useful. In many practical situations, complex-valued fuzzy sets turned out to be useful; see, e.g., [1], [2], [4], [10].

From the intuitive viewpoint, complex-valued fuzzy sets remain a puzzle. The problem is that fuzzy sets are not just a mathematical theory, it is an intuitively clear way to describe how we humans deal with uncertainty.

- The original idea of describing possible expert's degrees of confidence, degrees ranging all the way from "absolutely false" (0) to "absolutely true" (1) by numbers from the interval $[0,1]$, is very natural.
- In contrast, the idea of using complex numbers is not clear at all.

Why complex-valued fuzzy numbers are useful is thus still largely a mystery.

What we do in this paper. In this paper, we use the latest developments in theory of computation to show that the seemingly mysterious appearance of complex-valued degree of truth has a simple explanation – the need to make fuzzy computations efficient.

II. Our Explanation of Complex-Valued Fuzzy Sets: Fuzzy-Related Computational Problems and How Complex-Valued Degrees Help

Processing fuzzy data: what are the main computational challenges. To analyze why complex-valued fuzzy techniques may be useful let us recall computational challenges related to different uses of fuzzy techniques.

The most widely used fuzzy-related techniques are techniques of fuzzy control and fuzzy modeling. In these techniques:

- we have some rules,
- we combine membership functions corresponding to these rules – getting the membership function $\mu(u)$ for the result – and then
- we apply defuzzification to the resulting membership function to get a control value $\overline{u}$.

The most frequently used defuzzification technique is *centroid* defuzzification, in which we compute $\overline{u}$ as a ratio of two integrals:

$$\overline{u} = \frac{\int u \cdot \mu(u)\,du}{\int \mu(u)\,du}.$$

Combining membership functions is computationally simple: e.g., to find the degree to which the condition $A_1(x_1) \,\&\, A_2(x_2)$ of a fuzzy rule is satisfied for each x_1 and x_2, we simply apply the corresponding t-norm ("and"-operation) $f_\&(a,b)$ to the degrees $\mu_1(x_1)$ and $\mu_2(x_2)$ with which the conditions $A_1(x_1)$ and $A_2(x_2)$ are satisfied. This can be done in parallel and thus, really fast. In contrast, integration cannot be paralellized that easily, and thus, integration is the main time-consuming part of the computations related to fuzzy control.

Similarly, in fuzzy decision making, we:

- first compute a membership function $\mu(u)$ that describes to what degree each alternative u satisfies the requirements,
- and then select an alternative u for which this degree is the largest.

Similarly to fuzzy control, we can compute different values $\mu(u)$ in parallel, and thus – if we have enough processors at our disposal – reasonably fast. The main time-consuming part of the fuzzy optimization algorithm is thus finding the location where a given function $\mu(u)$ attains its global maximum.

Summarizing: in applications of fuzzy techniques, the most time-consuming computational operations are:

- integration and
- global optimization.

Integration and maximization problems are indeed computationally difficult. Both integration and maximization may be easy when we deal with simple functions, e.g., with triangular functions used in many applications of fuzzy techniques. However, for general functions, both integration and global optimization are NP-hard; see, e.g. [8]. This means, crudely speaking, that we cannot feasibly solve all particular cases of these problems.

Integration and maximization problems are feasibly solvable for analytical functions but not for smooth ones. Since we cannot solve *all* particular cases of the integration and maximization problem, a natural question is: for what classes of functions *can* we feasibly solve these problems? We know that these problems are feasible for triangular functions, we know that these problems are not feasible for general continuous functions. Where is the "threshold" separating feasible from non-feasible cases?

Such a threshold have discovered in a recent paper [6]. In this paper, it is shown that the threshold goes between smooth and analytical functions:

- both integration and optimization problems are NP-hard for smooth (differentiable) functions;
- however, these two problems become feasible (i.e., solvable in polynomial time) if we restrict ourselves to analytical functions.

Relation to complex numbers. For real-valued functions of a real variable, an *analytical function* means a function which can be (at least locally) described by a convergent power series

$$f(x) = a_0 + a_1 \cdot (x - x_0) + a_2 \cdot (x - x_0)^2 + a_3 \cdot (x - x_0)^3 + \ldots$$

All such functions can be naturally extended to complex numbers – that is why all standard functions such as $\exp(x)$, $\sin(x)$, etc., can be easily extended to complex values $z = x + \mathrm{i} \cdot y$ of the input.

In the complex domain, analytical functions can be defined much easier: as functions $f(z)$ which are differentiable with respect to z; see, e.g., [12]. From this viewpoint, analytical functions can simply be defined as smooth functions which can be extended to smooth functions of a complex variable. Thus, the above threshold result can be formulated as follows:

- for general smooth functions of a real variable, both integration and optimization problems are NP-hard;
- on the other hand, for functions which can be extended to smooth functions of a complex variable, both integration and optimization become computationally feasible.

Resulting computational explanation of why fuzzy-valued fuzzy. We thus arrive at the computational explanation of why complex-valued fuzzy sets are practically useful: because

- the most computationally intense operations involved in fuzzy techniques are integration and optimization,
- aintegration and optimization are computationally feasible only when the corresponding functions can be extended to smooth functions of complex variables.

III. Why Complex Numbers in General?

Why complex numbers in general: formulation of the problem. Complex numbers were first invented as a mathematical trick. It is not accidental that $\mathrm{i} = \sqrt{-1}$ is called an *imaginary* number: it was invented to represent something that does not exist in real life.

Surprisingly, it turns out the complex numbers are very actively used in modern engineering and in modern physics, to describe real-life phenomena; see, e.g., [5]. For example:

- it is difficult to imagine quantum physics without complex numbers;
- complex numbers are ubiquitous in control theory.

Our explanation. Our explanation is similar to the explanation of why complex numbers appear in fuzzy: that they make the corresponding computational problems computationally feasible.

Indeed, what kind of computational problems do we encounter in *physics*?

- usually, physical theories are described in terms of *differential equations*; so, to solve the corresponding physical problems, we need to solve (integrate) the corresponding differential equations;
- alternatively, physical theories can be describing in terms of *optimization principles*; in this case, to solve the corresponding physical problems, we need to find optima of the given objective functions.

Similar, in *control*:

- we have *differential equations* that describe the system's dynamics; so, to predict the system's behavior, we need to solve (integrate) the corresponding equations;
- the main objective of control is to find the control strategy that *optimizes* the appropriate objective function, so we need to solve the corresponding optimization problem as well.

In both types of applications, we need to solve integration and optimization problems. And we know that both problems become feasible only if the corresponding functions allow extension to the complex case.

Thus, it is natural that physical theories which allow feasible predictions and control theories that lead to efficient control strategies are often formulated in complex terms: the use of complex numbers guarantees that we have feasibility.

IV. Complex Numbers Not Only Make Computations Feasible, They Often Make Feasible Computations Faster

Why complex numbers: an answer that we gave so far. In the previous sections, we explained the effectiveness of complex numbers by citing a result that operations such as integration and optimization are only feasible when the corresponding functions can be extended to smooth functions in a complex domain. In this result:

- in the general case, the problem is NP-hard, so no feasible algorithm is possible;
- on the other, when functions can be extended to the complex domain, feasible algorithms become possible.

Additional reasons why complex numbers: they lead to faster feasible algorithms. It turns out that sometimes, even in the situations when a real-valued feasible algorithm is possible, the use of complex numbers can speed up computations. A typical example of such a situation is the use of *Fast Fourier Transform* (FFT), an efficient algorithm for transforming a real function $x(t)$ into its complex-valued Fourier transform $\hat{X}(\omega)$. The use of FFT leads to most efficient algorithms for multiplying polynomials, for multiplying large integers, for solving linear differential equations with constant coefficients, etc.; see, e.g., [3].

This is also true for fuzzy-related computations: complex numbers speed up feasible fuzzy computations. For example, in [9], it is shown that the use of FFT speeds up the computation of fuzzy arithmetic operations – e.g., when we use Zadeh's extension principle to compute the sum or the product of two fuzzy numbers.

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